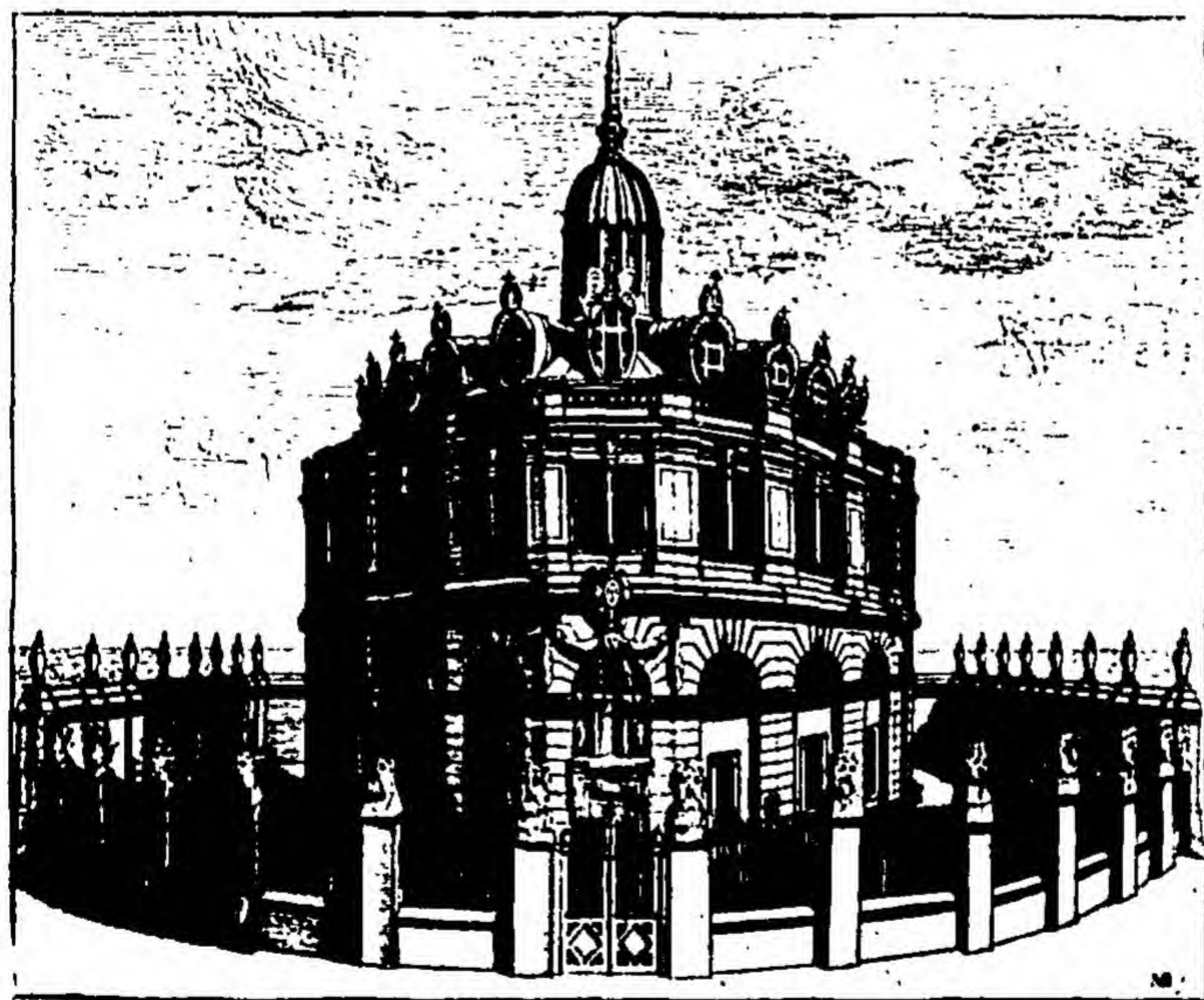


THE
ENGLISH EUCLIDE,
BEING
The First SIX ELEMENTS
OF
G E O M E T R Y.

THE
ENGLISH EUCLIDE,
BEING
The First SIX ELEMENTS
OF
G E O M E T R Y,

Translated out of the GREEK,
WITH
Annotations and useful Supplements,

By EDMUND SCARBURGH M. A.
Chaplain to his Grace the Duke of BUCKINGHAMSHIRE L^d Privy Seal;
Prebendary of SARUM;
And Rector of UPWEY in the County of DORSETT.



O X F O R D,
Printed at the THEATER. 1705.

Imprimatur,

GUIL. DELAUNE,

Vice-Can. Oxon.

Martii 2. 170⁴.

TO HIS ROYAL HIGHNESS
G E O R G E
HEREDITARY PRINCE OF
DENMARK, NORWAY, *and of the* GOTHs *and* VANDALS;
DUKE OF
Schleswick, Holstein, Stormar, Dickmarsh and Cumberland;
EARL OF
Oldenburgh, Delmanhorst, and Kendal;
Baron of *Wokingham;*
Generalissimo of All Her Majesties Forces;
Lord High Admiral of *ENGLAND,*
AND
Knight of the most Noble Order of the GARTER.

May it please Your ROYAL HIGHNESS,

THIS Piece was once intended for the Service of Her Majesty, and Your Royal Highness in the Education of A PRINCE, which was then the Hope, and Joy, and Glory of the *British* Nation; on whose Character I would enlarge, but that I fear to touch
a upon

THE DEDICATION.

upon so Soft and Tender a Part, least whilst I endeavour to adorn His Memory, I renew Your Grief, and revive again the sorrowful Image of THAT INEXPRESSIBLE LOSS.

After which, the greatest satisfaction I have taken in Composing, and Finishing this Work, hath been from my hopes of having a proper Opportunity of expressing to the World, the high Esteem, Respect, and Veneration I have for *Your Royal Highness*.

Your glorious Deeds of War, and mighty Atchievements in the Field of Battle; a Brother Rescued, and a Kingdom Sav'd, are Glories proper for another Pen. But Your wonderful Zeal and Courage in preserving these Kingdoms, Your entire Love to our Countrey, and unparallel'd Affection for our most Excellent, and Glorious Queen, are Vertues, and Merits, which none, that Love our Crown or Countrey, can conceal; and I am proud to have an Opportunity to joyn in the general Applause and Celebration of Them.

By these Vertuous, and engaging Arts, GREAT PRINCE! You have won the Common and Universal Love and Esteem of the whole Nation; which, however Divided in other Matters, are yet Entirely *One*, and firmly *United* in their just *Esteem*, and *Honour* for Your Princely Vertues, and Merits to their Countrey.

It speaks a mighty, and powerful Charm, GREAT SIR, to Unite such Divided Hearts; and nothing but Your incomparable Vertue, and Goodness, and those infinite Obligations You are still laying upon our
Country,

THE DEDICATION.

Countrey, could ever have established such an Universal Consent and Agreement in the Hearts of a *People*, so little acquainted with the pleasures of *Union*, either in *Interest*, or *Affection*.

Long may You live, GREAT PRINCE! the common Object of all our Praises, and of our Prayers to Almighty God for the Preservation of so valuable a Life. How dear it is to us hath already publickly appeared in the Solemn Prayers and Supplications of Our Church, and is daily acknowledged in secret by the private Intercessions of Many, who earnestly beg of God to continue so Bright and Illustrious an Example of Vertue and Goodness amongst us; and preserve to an Age, that hath so few of them, the Incomparable Pattern of THE BEST OF HUSBANDS, THE BEST OF MASTERS, and THE BEST OF FRIENDS.

A Character of Your *Royal Highness* which all Men do professedly agree in, and of which our Family in particular have had the Clearest and the Noblest Demonstrations, having long had the Honour and Happiness to attend Your *Royal Highness*, and be Eye-witnesses of those Vertues, which others Admire and Celebrate at a distance.

How unable I am, GREAT SIR! to do Justice to Your Illustrious Character is sufficiently seen in the poor Attempt I have already made: but I humbly beg leave to assure Your *Royal Highness* that no Man hath a truer, or greater Zeal for Your *Glory* and *Honour*; no Man hath a juster Respect, and Veneration for Your
a 2 *Person,*

THE DEDICATION.

Person, nor does any Man send up more Ardent Supplications to Heaven for the Encrease of Your *Health*, and long Continuance of THAT IMPORTANT, and BELOVED LIFE, on which the Joy, and Glory of THE BEST OF QUEENS, The Happiness of Her Reign, and the Prosperity of Her People do so much Depend. In these things no Man exceedeth him, who is, GREAT PRINCE,

Your Royal Highness's

Most Obedient,

Most Faithful,

And most humble Servant,

EDMUND SCARBURGH.

THE P R E F A C E.

LONG Prefaces how pleasing soever to the Writer's own humour, who is very apt to be favourable to his own Productions, yet are seldom agreeable to his Readers: nor can all his Courtesy, Insinuation, and affected study to please and invite, make any tolerable amends for the tedious and ungrateful Fatigue.

If the Piece be well perform'd, and answers the design and intention of it, the Reader is uneasy to be detain'd from its perusal; if it be not, all the plausible Pretences and Excuses in the World will never recommend it to the Approbation of a Man of Taste and Judgement.

It were very easy to run out, and Harangue the Reader in commendation of this Excellent Study, so highly celebrated by the Ancients, and so much in the Esteem and Fashion of the Age we live in: But this is a design that the best Wits of every Age have perform'd to admiration, and have left no colours for any new Pretender to adorn it with; therefore waving any attempt of that Nature, and all the customary Modes of Formal and Ceremonious Apologies, I shall apply my self to inform the Reader, that shall please to peruse it, What he may expect, and What he will find to his satisfaction in this following Work.

First, A plain, but I hope a just and exact Translation from the Original into Our Mother Tongue, without neglecting too much the Turn and Idiom of the Language it was at first written in.

Secondly, He will find Such Illustrations for the benefit of Younger Students, and Such Annotations annex to the most difficult Places, as may serve to clear the Author's Sense, and explain it to the Capacity of the meanest Reader, that is never so little conversant in these Studies.

Thirdly, and lastly, He will see Our Author, The Great and Noble Elementator himself, Vindicated from the many captions

THE PREFACE.

and unreasonable Objections brought against Him by some Severe, and over Critical Commentators.

If I have perform'd THIS, I have my desire in Compleating and Publishing a WORK, which was design'd, and begun many years ago at the Command of my much HONOUR'D FATHER, and THAT ILLUSTRIOUS PRELATE, the late Lord BISHOP OF SARUM. A Prelate whose Piety, Charity, Hospitality, Friendship, and Wonderful Attainments as well in the Politer Arts, as Deeper Searches in Divinity, made HIM ONE of the Glories of his time, who hath deriv'd a lasting Honour on THAT SEE, and left a most incomparable Example of Charity and Munificence, to all his Reverend Successors in That Church. A PRELATE whose Memory will be for ever BLESSED, not only in his own Diocess, but wherever Piety, and Learning, and sweetness of Conversation have any Name, and whose Authority, I dare promise my self, will not a little recommend this Undertaking to the World. For tho' his Lordship did not live to see it finish'd, yet He, in his perfect Health, highly approv'd the Design, and laid his earnest Commands upon me to Compleat it.

*MY FATHER, whose name (I presume) is not the least in the Register of Men Learned, and Famous in these Studies, liv'd to have the perusal, and Correction of the greatest part of this Work, which may in some measure recommend it to the Judicious Reader, and vindicate the Piece from the imputation of being a Common, and Useless Performance. He had the honour in his Life time to have the Acquaintance, and Conversation of the most Celebrated Masters of these Sciences, and had made so Large, and early an Advance, and Progress in these Studies, as to deserve that kind and honourable Character from the Learned and Judicious M^r Oughtred in his Preface to his Clavis Math. "A Man, says He, of a
"pleasant and obliging Temper and Conversation, of a piercing
"Wit, and penetrating Judgement; so admirably versed in Ma-
"thematical Studies, and of so happy and strong a Memory withal,
"that he was able upon any occasion to Repeat, and Apply every
"Proposition in Euclide, Archimedes, and several other Ancient
"Masters in these Studies." A Character, which not only speaks
the high Esteem that excellent Author had of Him; but shews*

HIM

THE PREFACE.

HIM likewise to be no improper Person, upon whose Authority, and Direction, a performance of this nature might be undertaken.

But neither did He live to see it finish'd; For my many concerns, and unavoidable Avocations, kept me many years from pursuing This Work: till at last I had a very fair Prospect of making it serviceable to the Ever to be Lamented DUKE of GLOCESTER, Whose Death put an Other, and almost Final stop to this Work; till being more at leisure, and continually stirr'd up by the remembrance of those Worthy and Excellent Persons, that had recommended the Compleating of it, I resolv'd to go thro', and Publish it to the World, having at the same time the Encouragement, and Recommendation of the The GREAT D' WALLIS, and his Learned Friend the WORTHY D' GREGORY; Men whose very names are of Virtue to keep the Work from blushing, and not only shelter it from Censure, but Recommend it to the Approbation of Men of Judgement.

For the First was undoubtedly THE GREATEST MASTER of THIS SCIENCE, that hath appear'd in any of these later Ages; The honour of Our Countrey, and Admiration of Others, whose Character can never be more fully, or lively express'd than in that just and excellent description of Him by the Learned and Judicious M' Oughtred; "A Person (says He) adorn'd with all ingenious, and excellent Arts, and Sciences; Pious and Industrious, of
"a deep and diffusive Learning, and an accurate Judgement in all
"Mathematical Studies, and Happy and Successful to Admiration
"in Decyphering the most difficult and intricate Writings." Which was indeed his Peculiar Honour, and the greatest Argument of a most subtle and searching Wit and Judgement.

As for the Latter, The Learned Professor of Astronomy, as He wants no Commendation to the Present, so will He not fail to leave a Noble and Lasting Character to future Ages, and Live for ever in his many learned Discoveries, and incomparable Performances in ASTRONOMY.

An Index of the Authors mention'd in the Annotations.

Apollonius Pergæus.	Marinus Gethaldus.	Plato.
Archimedes.	Metius Adrian.	Possidonius.
Benedictus Joh.	Mydorgius.	Proclus.
Borellus.	Nazaradinus.	Pythagoras.
Bovillus.	Nicomedes.	Tacquet.
Campanus.	Orontius.	Theon.
Cicero.	Oughtred.	Vitellio.
Clavius.	Pappus.	Wallis.
Commandinus.	Peletarius.	Zambertus.
Eutocius.		

The Reader is desir'd to Correct these few Escapes of the Printer,
according to the following ERRATA.

Page 5. *line* 7. and elsewhere, for Extream, or Extreains read Extreme, or Extremes. p. 11. l. 10. r. Superficies. p. 45. l. 27. r. ABC. p. 48. l. 30. for Propolition r. Proportion. p. 53. l. 11. for DH r. DG. p. 63. l. 38. r. *diuina*. p. 66. l. 21. r. to DA. p. 68. l. 42. for it r. is. p. 81. l. 32. to strait add line. p. 87. l. 1. to point add A. p. 88. l. 22. for in r. is. p. 96. l. 13. for known r. know. *ib.* l. 39. r. to a Rect. p. 108. l. 12. r. Parall. p. 110. l. 5. after sub-
tending *del* r. p. 181. l. 29. for diuing r. dividing. p. 184. in hends *del* s. p. 186. l. 3. r. lyable. p. 204. l. 12. r. of D.
p. 214. l. 12. for *r.* $\frac{2}{3}$. p. 219. l. 21. *del* and. p. 224. l. 2. to hath add to.

THE

THE FIRST
ELEMENT
OF
GEOMETRY.

DEFINITIONS,

OR

Expositions of Geometrical Names, and Terms of Art.

DEFINITION I.

A *Point is That of which there is no part.*

ANNOTATIONS.

EUCLIDE begins the Elements of Geometry in a true Elementary method of Doctrine from the most simple Things, and Notions in Geometry to the more and more Compound.

And because Magnitude the subject of Geometry hath not a *Minimum* to begin with, as Number hath an *Unit*, therefore nothing else could be more simple in these Mathematical Speculations of Magnitudes, than first to put a Point as void of Magnitude. Which notwithstanding is a proper beginning to enter into the present matter, as the first step towards Geometry: and It is in some manner like to a Cypher in Arithmetic, which may be call'd an Arithmetical Point.

The definition of a Point is plainly negative, and no otherwise informs us what a Point is, than by telling us what 'tis not. Yet in many Things that are in nature most simple These kinds of Negative Definitions are sufficiently instructive; tho' not to the Essence of the Thing defined, yet very well to the Use that is to be made thereof.

So here a Point is defined by a negation of Parts. Which Definition in respect of Magnitude, That was next to be considered as divisible into parts, is Instructive, or Preparatory to the right understanding of the Doctrine of Magnitudes; and lays down what conception of a Point is hereafter used, or useful in Geometry, namely, *To have no parts*. Which is sufficient for the present to a Geometrician. Let the Philosophers dispute further, as they please, about the reality of a Point, or the nature of an Atome.

But now in the first place, we are to understand, that A Point made with a Pen or any other Instrument is but an imperfect, and gross Notation of a Mathematical Point here defin'd by *Euclide*. Which is abstracted from Matter, and Quantity, and only to be conceiv'd in the Mind. Yet those kinds of material, and visible Points being put as the least object of our Sight, may be allowed for a Note of

A

a ma-

a Mathematical point, which is the least object of our Understanding in these Geometrical Contemplations.

And to go on with the name. PUNCTUM, A POINT is among the Greeks vulgarly *σπυγή* & *σπυγή*. A Prick, Note, or Mark. By Plato, Aristotle and other Philosophers, *σπυγή* is taken in a strict Geometrical sense for an Indivisible Mark, or a Notation of an Indivisible Thing. Which is the same with a Mathematical Point. But Euclide names it most properly *σημεῖον*, A Signe. That is, such a Signe, as denotes in space An Impartible Here, or There.

Now space is an Infinite, and Unmoveable Diffusion every way: such as can afford a Locality to any one Thing without the Resistency, or Cession of any other. *πάντων ἀποδέχεται*, the Receptacle of all whatsoever is, or can be. Thus Virgil makes old Silenus sing the Beginnings of Things.

*Namque canebat uti Magnum per Inane coacta
Semina terrarumque, animæque, marisque fuissent.*

Virgil's *Magnum Inane*, that Immense, and Empty space both the Philosophers, and Mathematicians, put as the *Primum Conceptibile*, The First Conceivable in the Being of Things. Call it space Physical, or Mathematical. For we dispute not here with Democritus, and Epicurus, whether Space be a Thing in Nature distinct from Body posited in Space. Only the Mathematician gives a Being to It, as in general abstracted from any kind of Body, that may fill, or possess a Space.

In this Universal Space A Point, or Signe is a certain Position without any Quantity. It is An Indivisible Ubi to be put at pleasure any where. An Ubi Ubilubet. And because wheresoever a Point be put, the same is conceived to be *ἀμερές, καὶ ὀδιάσπεν*, void of Parts, and Interval; therefore it is the most Simple, the First, and Least Thing imaginable in Space. *μιμνήσκω τὴν ἀμεράτην ἢ ὅτι οὐ φέρει*, says Proclus most acutely. A Point represents the Utmost nature of Things. That is, a Point is of a nature so Subtile, that it has the very Extremity of Being, or the Next to Nothing. So Lucretius,

*Punctum sine partibus exstat,
Et minimâ constat naturâ.*

But yet a Point hath such a Being in Nature, how little soever it may seem, that from very many Instances it evidently shews it self to Be.

In the ordinary use of Burning Glasses there is vulgarly taken notice of a certain Burning point. And such a point It is, as proves it self really to be. In Loadstones it is commonly known that there are Polar Points, called North and South. In the descent of Bodies towards the Earth, and in all Parts of the Earth, while they every way pressing together do fall into an Orb, or Globe, there must arise a respect to a certain Point; Insomuch that upon this very Point the Mass of the Earth,

*Moles Telluris
Ponderibus librata suis immobilis hæret.*

And in the like manner, what is more manifest to be, than the middle point in the Balance? Than the point of *Æquilibrium*, or Equal-poise in every Body? And it is no part of the Body. For that every part of a Body is a Body: and whatsoever is Body, has a point of *Æquilibrium*. So then in every Body such a Point there is. Which Point being no part of that Body, must be a meer Mathematical Point.

A Point therefore has a Being, tho' Indivisible. Yet it is not the only Thing to be conceived indivisible in Nature, But in Geometry It is the only indivisible Thing.

And in respect of this its Indivisibility, for Illustration sake, a Point in Geometry is compared to an Unite in number, and to an Instant in Time. Both These being alike conceived under the same notion of Indivisibility with a Point, tho' in other respects all the Three be much different from one another. But because the Indivisible natures of Unity, and Instant are more obvious to common Apprehensions; These do well enough illustrate the Indivisible Being of a Geometrical Point.

By the Pythagoreans, who bring all things into the Mystery of Numbers, A Point is said to be *Μονὰς ἵστω ἔχουσα*. A Monade, or Unity having position. Indeed to have Position, or Situation is the only positive conception to be made of a Point: Whose Existence is in its Locality: As Proclus says, *ὡς ἐν τόπῳ ᾔστανται*. Existit tanquam in loco, Here,

Here, or, There. But now Unity properly taken, as the principle of Number, has nothing of Position. As *Proclus* goes on, ἄθετος ἡ Μονὰς, ἐκ πέντε ἐξω τῶν π. And upon this consideration Unity is *πρῶτης ἀσπλάγξετα*, more Simple than an Indivisible Point, for that a Point exceeds the Indivisible simplicity of Unity by the addition of Position. And therefore in this metaphorical Definition, by Unity is only to be understood Indivisibility, or Vacancy of parts. So that, *A Point is to be conceived such a kind of Unity, such a Monade, or Indivisible Being, as in Space to possess an impartible Place, or Position.*

And to proceed in a familiar comparison between a Point in Geometry, and an Instant in Time, there is such an Agreement in their Indivisibility, that even Life it self, our very Being is but a Point. For only τὸ νῦν, This Point of Time, The Indivisible Instant, The present Moment is. And what was, and what shall be, is not.

Thus much of a Point. πρὸς τε τῇ ὑπαρξίν αὐτῆς, ἐκ τῇ ἀνυπαρξίαν. Whether of Something, or of Nothing.

The Geometricians do commonly note a Point by some one Letter of the Alphabet. As the Point A, or the Point B.

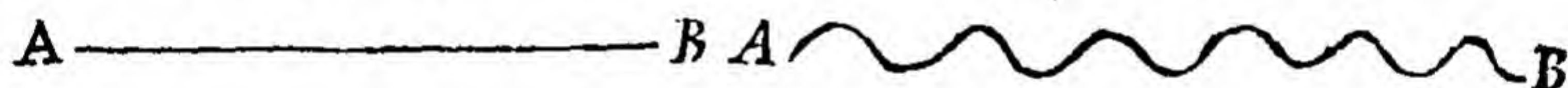
DEFINITION II.

A *Line is Length without Breadth.*

ANNOTATIONS.

A Point was the first thing towards Elementary Geometry posited in Space, and That without any Quantity. The next and first most simple Thing with Quantity is a Line, That is, *A meer and simple Extense; or solely Length.* In which Definition there is a positive notion of Length, restrained by a negation of the next immediate composition made of Length and Breadth. And herein for the present It confines our Imagination to one single consideration of the mensuration of Magnitudes, and That according to *Length alone.* whereby we understand what conception of a *Line* is ever and only to be made use of in Geometry, **LENGTH ALONE.**

We may help our imagination in the conception of a Mathematical Line after this manner. Put a Point (I mean here a Thing, whose quantity is not considered) and let it be supposed to move from one place to another. Then shall the same leave an imaginary Track only Long, which is called a Line. As if the Point A be imagined to move from A to B, It shall trace forth the Line, named A B.



Not that hereby a Line is *A Flux of a Point*, as some define It: (for motion creates not magnitude; tho' the nature of the *several Dimensions* of magnitude, as abstracted from Matter, may be well conceived by motion) but only by this Instance we do give *some Idea of Length*, or of that kind of magnitude, which is called a *Line*, not considering the subject wherein It is.

And therefore, to explain further this present matter, we are not to conceive that a Line described from the motion of a Point does consist of Points. For the motion, or flux of a Point adds not Point to Point in its progression, and thereby constitutes a Line, as the links of a Chain make up a Chain: but only does *by the continuity of its motion* represent to our imagination a *Continued Line*, or Length without Breadth.

The Conception of Length arising from the motion, and the negation of Breadth from the nature of the moving Point, which is conceived such, as to be void of Length, and Breadth, or any thing of Magnitude; and therefore in its imaginary flux cannot form any representation of Breadth, but only of a pure Mathematical Length.

Again, forasmuch as a Thing cannot be made of Nothing; and therefore ten thousand Nothings cannot make One Thing: so a magnitude cannot be made of that Thing, which in it self is no magnitude; tho' never so many of Them were

put together. If therefore a Point be supposed to be put to a Point, there cannot be any thing else conceived, then that They both must wholly enter into one another, and so still be put as one Point. For otherwise if they join to one another, they must join by some common Extreme: but a Point having no part, can have no Extreme, no Beginning, Middle, or End. Wherefore upon the supposition of Two, and so by consequence of more points put together, there can still nothing more result or be imagined, than a Point. Therefore whensoever we suppose two distinct points some kind of interval must be also supposed to be between them. So that the flux of a point proceeds not from point to point; but immediately passeth into Length. And therefore from the common Instance of a fluent point we are not to conceive, that a Line does consist of points; or that a point is any part of a Line. But as every motion begins and ends where there is no motion, so every Length begins and ends where there is no Length.

But tho' there be many irrefragable Demonstrations that a Line cannot be constituted of Points; yet *Theodosius* hath as clearly demonstrated, that a Sphere toucheth a plain superficies only in a point, and how then It doth not in its motion on a Plain trace forth a line of points, *Clavius* acknowledges that he hath not met with a satisfactory Explication. And therefore *Fromondus* his Book *De Compositione Continui* is by him not amiss entitled *Labyrinthus*. Thus the Reason of Man must submit to the Incomprehensible Secrets, which by an Inscrutable, Infinite Wisdom, are planted in the Nature and Frame of Things.

OF the Application of Number to Magnitude, and the Use Thereof in the Mensuration of Magnitudes.

But now in Discrete Quantity the matter is far otherwise. Where *Μονάς*, *Monas*, an UNITE, or *Monade*, tho' It be as indivisible and partless as a point; yet It is a constituent part of every number: and therefore every *Number is a Multitude consisting of certain Units*, or *Monades*. Wherefore it ought to be observed, that in comparing Magnitudes with Numbers a point in Magnitudes does not answer to an Unite in Numbers; but rather to a Cypher. Which is no more a part of Number, than a point is a part of Magnitude. Whereas an Unite is not only a constituent part, but also the *Least Part* of every number, and therefore It is the natural measure of all numbers. *A Measure being the Minimum omnium in eodem genere*. But because in Magnitudes there is not a *Minimum*; therefore Magnitude has no *Natural* measure. Yet here we supply a Natural measure sufficiently for our use in constituting, by common consent, some certain known magnitude for a measure. *Τὸ ἐπὶ μέτρον μετρεῖν*, *A Stated Measure* the ancient Geometricians call It. And upon such a Stated Measure there follows a just agreement between Numbers and Magnitudes. For in the mensuration of any magnitude, what part soever of a magnitude is taken by consent for a Measure, the same truly answers to an Unite in Numbers. For it is in this case a *supposed Minimum* and put like an Unite for the Least Part to be considered in That magnitude; by which supposed Least Part we do agree to estimate the quantity of the Whole. As in the mensuration of any Length, let a *Yard*, or a *Foot*, or an *Inch*, &c. be agreed upon as a known magnitude for a Certain and a Standing Measure. And for Instance, let an *Inch* be put for the measure of Length. The *Inch* is now a supposed *Minimum*, and becomes to be of the like nature and use, that an Unite is in Numbers. For as a number has its value from the multitude of Unites which are numerated to be collected in It: so a Line or Length shall have its value from the multitude of Inches, which are numerated to be contained in the same. As if a Line contain twelve Inches, or eight Inches, or four Inches, The same is valued according to 12, 8, or 4. The Line in a respect had to an Inch, as the number in a respect had to an Unite. So after this manner *The Mensuration and Estimation of all magnitudes is made by the numeration of the Stated measure, as a number is estimated by the numeration of its Units*.

And This is the first and real ground of correspondence between Numbers, and their mutual application to one another. Which ought to be perfectly understood, and

and remembred for the common use that is hereafter to be made thereof throughout all Geometry.

Therefore speaking upon this matter in general, Let the Stated Measure of any kind of Magnitude be called THE MEASURING UNITE, OR THE GEOMETRICAL UNITE. For very useful it is, and most requisite, that It should have some settled Name.

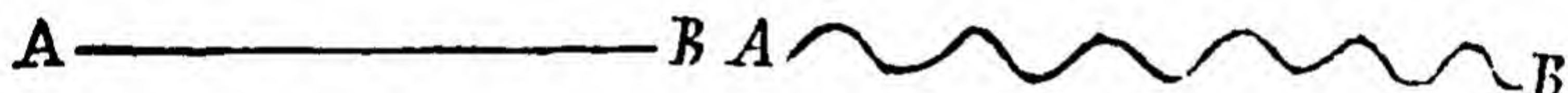
DEFINITION III.

The Extrems, or Ends of a Line are Points.

ANNOTATIONS.

A Line was defined to be Length without Breadth, that is, *Length alone*: and so taken absolutely without any respect to Termination; which is a Secondary notion, and a Supervenient Mode, or Qualification of Magnitude.

In the first place therefore *Euclide* defined a line without limitation. But afterwards if a line be any ways determined, as *Here*, or *There*, or *Any where*, He tells us now that in the termination of a line the Extrems, or Ends thereof are understood to be points; such as he had before defined to be void of parts. For in common sense Length cannot be terminated by any thing having length, because it will again be demanded, What limits This Length, and so forth infinitely. Therefore a line must be conceived to be limited by something void of Length, and by consequence of Magnitude, which Thing is called A Point. As for example, of a limited line AB the Extrems, or Ends are the imaginary points named A, and B, that is as much as to say, that the line AB begins *Here* at A, and ends *There* at B.



So that a Finite line is a continued Length determined by two points, *Here* and *There*.

DEFINITION IV.

A Strait Line is That * which lies evenly to the Points within It self.

Ἡπερ ἕξις τῆς ἐν ἑαυτῇ σημείοις κείται.

ANNOTATIONS.

As for Instance. In a line let there be taken points at pleasure how many soever. A, B, C, D, E, F, &c.

If the line be such as *Euclide* sets forth by the name of a Strait line, then we



are to conceive, that the whole line *κατα ἕξιν*, or *ὁμαλῶς*. *Jacet equabiliter*, *Lies evenly*, that is, just in one and the same position to all those points: so that from any point to any point there is no where made in the tract of the line any change of Position on one side or other, upward or downward, or any various way. But that every imaginable part of the line does bear a like Site and Respect to all points imaginable in the same line.

Whereas in a crooked line Every assigned part has a different respect of Situation to all assignable points in the crooked line. *Here* one way, *There* another; in an infinite variety of positions of the One to the Other, of the Parts of the line to the Points, and of the Points to the Parts of the line. But in That, which *Euclide* calls a Strait line, there is to be conceived One and the same Equability throughout in the position of the whole, and of all its parts to every point in the same line.

The Use of This Definition.

So that *first*, if any strait line be suppos'd to lye upon another strait line, It shall

shall necessarily follow upon this uniform constitution of all strait lines, that their intermedial parts must be congruous, that is, coincident, and every where agreeing exactly with one another, so that They be as one strait line.

And *secondly*, by a natural consequence from this Definition of a strait line *Archimedes* before his Books *De Sphæra & Cylindro* does assume as a Common Notion, That *A Strait line is the Least of all Lines having the same Extrems.*

And these are the two special uses made of *Euclid's* Definition in several Geometrical Demonstrations. As for the *First*, in Prop. 4th. and 8th. El. I. in Prop. 24. El. III. &c. Of the *second*, in those admirable Books of *Archimedes* concerning the *Sphere* and *Cylinder*.

But This Proposition of *Archimedes* some Commentators cite for Another Definition of a Strait Line, wherein they are much mistaken. That Great Geometrician so well understood the ELEMENTATOR in this his most accurate Definition, that he went not about to mend It, and to give a new Elementary Definition, which was much below his thoughts, and present Matter. But from thence, as his Subject requir'd, did assume this Proposition, as a natural and immediate Confectary, and puts It for a granted Maxim, not a Definition. Hereupon therefore it will not be unuseful to consider farther of Mathematical Definitions.

Of Definitions Mathematical and Philosophical.

Definitions may be taken two several ways. First, there may be an Idea, Image or Conception in our Mind of a Thing, which we cannot express to others, but in many words. Now to this Thing expressed at large we would give in brief a certain Appellation or a settled Name; so that whensoever this Name is mentioned, we intend thereby that the Thing be in such manner conceived, as It was at first expressed, or as we say, defined. Mathematical Definitions are to be taken in this sense. Which indeed admits of no dispute; for that it is free for every man to give what Name he pleases to his own Conceptions. He is only afterwards bound to use the Name always in the same signification, which at first he gave unto It.

Again there are many Things of which Men have a common Idea, and also a Name commonly received to signify every such Thing. But yet upon searching more curiously into the nature of Things we do often require a more perfect Explanation of the Essence and Intrinsic constitution of those Things. This kind of Explanation is call'd a Philosophical Definition. Which ought to be an Analysis of the Thing into the Essential principles of which It is compounded. As when we demand what is Water? What is Fire? Of these Things there is among men a common Idea, I mean, of *The Whole* or *Totum Physicum*, and a like use; Also in every Language a certain Name; Notwithstanding which knowledge of Ours, we do by these questions require something more to be instructed in, concerning their original constitution. That is, we would resolve the Whole into its constituent parts, and lay open the secret Composition of their Natures. But whether the Intellect of Man can pierce so deep into the Intrinsic State of Things, as to give an Essential Definition of any Thing; is to me so much unknown, that altho' *Aristotle* rightly teaches what That Demonstration, which he calls *Διότι* ought to be, yet I believe the most subtile wits in the world never fully discern'd the Essential frame of any Thing, or the Natural progress of Causes and Effects, whereby to be enabled to give just Definitions of Things, and accordingly Demonstrations *από* in the Course of Nature. Our Definitions being only the Analysis of our own imperfect Conceptions of Things, rather than of the Things themselves.

For to Know the Divine Mechanism of this material World belongs only to THAT Eternal, Almighty Self-Being, *το ὄν, τὸ εἶναι, καὶ ἐνὶς ὄν.* The *very* Being, The One, and of *one only* Thought, Which (The same with Himself) together Knows and makes all Beings.

What Intuition into the Essence of Things, Spiritual Creatures may have is altogether unconceivable by Man. Certain it is, that Humane Understanding proceeds

proceeds upon Conceptions inadequate to Nature, taking Things only by Parts; and from their outward Appearances. λογισμὸν εἶδα πραγμάτων διαφεσιν. Is the pure Iambic of *S. Greg. Nazianzen*, Reasoning I know to be the Division of Things. Our Ratiocination indeed is made after that manner, step by step, to lead us into some kind of knowledge. And in observing what appears the First, most Simple, and immediate Emanation of a Thing, we do from Thence determine the nature of the Thing in It self, and distinguish It from all Others. And hereupon we form in words a Proposition, commonly called an Essential Definition. It serves indeed as well for our use in Reasoning; if we can from thence as *A Cause* gradually deduce all other Affections and Properties observable in the same Thing.

Thus there are these two kinds of Definitions. The first is by Logicians called *Definitio Nominis*. Which is of most use among the Mathematicians, who giving Names to their Conceptions, do in their Definitions put *A Name* for the *Subject* of the Proposition, and *The Thing* which is to be understood by that Name is made the *Predicate*. And in This kind of Definition there is only imply'd that *such a Thing* is so Named.

The second is called *Definitio Rei*, used generally by Philosophers, in which the *Subject* of the Proposition is the *Totum Physicum*, or a Thing conceived in gross under a natural composition, and signified by a certain Name: And the *Predicate* is, or ought to be the Essential constituent Parts of that Physical Compound.

Thus a Mathematical Definition consists of *the Name and the Thing*, and a Philosophical Definition of *the Thing named and the Essential parts* of the same Thing.

And now to our present matter. First, as the Mathematicians understand their Definitions, This of a strait line is with all the Rest to be received alike by Geometricians without exception. For that with Them, as we have said, there is only put or supposed such or such a Thing, and a Name is given to It. As here *A Length* is put, which is to be conceived to lye evenly to all its Points; and such a Length is called *a Strait Line*. Now first against this Notion of Evenness in Length or an Equable interjacency of every part in a certain Length, there can be urged nothing as impossible or incomprehensible; and therefore It is at present to pass as a Legitimate Supposition. And next in calling this kind of Length a Strait Line, The Name is free and arbitrary as all Names are.

But again, if as Philosophers we take the Subject of the Proposition for a Thing, and here intend to define a Strait Line; as a Thing commonly known in It self, and also by That Name; yet to clear our Conception therein, we would Analyse, or Resolve a Strait Line into the *Essential Grounds of Rectitude*, I say, even in this Philosophical acceptation *Euclid's* Definition will appear most accurate upon this very reason, that whatsoever Notion put by any One Philosopher, or Mathematician for the Definition of a strait line, or whatsoever Properties and Affections are attributed to a strait line, They do All evidently arise from *Euclid's Definition*, as from the *Nature of Rectitude*, and the Essential Constitution of a strait line. For *Euclid's* Notion of a strait line does consist in the *Equability of its position to all imaginable points in the same line*. And there arises from This conception such a Community, or rather Identity of Constitution in all strait lines, that they being considered absolutely in themselves, *As Strait*, do differ from one another only in Situation, and variety of Place. So that changing in our imagination the place of one strait line into the place of another (which is called *Εφαρμογή*, *Epharmosis*, or an Adaptation of one line upon an other) there follows,

First, that *All strait lines are Congruous to one another*.

And This is, as we have noted before, the Primary use of *Euclid's* Definition. Or rather, This is not so much to be accounted a Confectary, as rather the same notion with *Euclid's* Definition, tho' in different words. For to conceive strait lines to be such as to have every where an equable interjacency of all their parts to all their points; Or strait lines to be such as to have every where a mutual Congruability of Themselves, and all their parts to one another is in effect the same Thing.

Secondly, That *A strait line is the least of all lines between the same points*.

Which notion informs not what a strait line is in It self; but only what It is in comparison to lines not strait. *Archimedes* makes use of This as a natural Confectary from *Euclid's* Definition. Διόν, says *Proclus*, ἐξ ἵτις κέεται τῆς ἐφ' ἑαυτῆς σημείois; διὰ τὴν ἐναγίαν ἐν τῇ αὐτῇ περὶ αὐτῶν ἐχέτων. *Because a strait line lyes evenly to all its points, for that very reason It is the least of all lines having the same Ends. For if any other were less, The first did not lye evenly between its Ends.* εἰ γὰρ ἦν τις ἐλάττωον οὗ ἐξ ἵτις κέεται τῆς περὶ αὐτῆς.

Thirdly, From *Euclid's* Definition says *Proclus* 'tis manifest, that

Only The strait line does possess a Space equal to That, which is between Its points. ὅτι τὸ ἐν μέσῳ τῶν σημείων διάστημα τὸ μὲν αὐτῶν ἐπ' αὐτῆς σημείων. And *Proclus* gives this reason; For how much One point is distant from the Other, so much is the magnitude of the strait line terminated by the same points. ὅτι γὰρ ἀπέχουσιν ὅσπερ τῶν σημείων ὅσον τὸ μέγεθος τῆς ἐν μέσῳ αὐτῶν περὶ αὐτῶν. This is, says *Proclus*, ἐξ ἵτις κέεται. That is, A line to lye Evenly between two points, is the same Notion as a line to be equal to the Intermedial space between two points. Therefore

Fourthly, A strait line is the natural measure of Distance between Point and Point, or Here, and There.

Fifthly, A strait line is Ordinate between its Extrems, ἐπ' ἀχρῶν ἐν πλάγμηνι, says *Proclus*.

Or if we take it as *S^t Henry Savile* corrects it, πλάγμηνι. Then thus it is. A strait line is stretched to the utmost between its Extrems. And therefore

Sixthly, A strait line is such, whose Extrems cannot in our Imagination be moveable further from each other, preserving the Quantity of the same line.

Whereas the Extrems of any crooked line may without change of its quantity be further and further diduced, till the crooked line be stretched to a strait line.

Seventhly, A strait line is The only singular between the same Extrems.

That is, there can be but one strait line between the same points, whereas of crooked lines there may be infinite. Lastly from *Euclid's* Definition there is observed an other Property of a strait line relating to Vision: To wit,

Eighthly, A strait line is That, All whose Intermedial Parts do obviate the Extrems. ἡ αὐτὴ μὲν αὐτῶν τῶν ἀχρῶν ἀποκρίνεται. As *Proclus* speaks from *Plato*. And the meaning is, that if One Extream be supposed a Lucid point, and the other Extream an Eye, all the interjacent Parts of a strait line shall obviate, or stand in the way, and obstruct the Radiation of the Lucid point unto the other Extream; so that It cannot be visible to the Eye in that place. As for instance, we find the Eclipse of the Sun to be made by the direct interposition of the Moon between the Sun and our Sight, All Three then lying in a strait line.

These Notions, or Conceptions, and whatsoever other Attributes are by any One given to a strait line, They are only Confectaries from *Euclid's* Definition. Which for this very reason shows It self to be the Primary Conception of Rectitude in lines, for that It comprehends all strait lines in general, whether taken finite, or infinite. Whereas Those Other here now mentioned are Secondary, relating only to a finite strait line as determined between two Extrems. And most or all of Them are useless in Geometry.

Thus have we fully set forth the several notions of Mathematicians and Philosophers concerning a strait line, because Many of Them have thought It worth their While to busy Themselves therein, and especially two Great Men of our Age, *M^r Heib* and the incomparably Learned *D^r Wallis*. And besides we have the rather insisted upon this Argument, for that It gave us a just occasion to Expound the Nature, and Difference of Definitions Mathematical and Philosophical. So that in the right understanding of Them Both, our younger Students might be provided against the Cavils made upon some of *Euclid's* Definitions. Lastly, It is observable, that Things, the more common They are and seem most known, are the most troublesome to be defined. For that there is in every Man One and the same anticipated Idea of Familiar Things, whereby they are better known within Us, than Words can make them known unto Us. The Things themselves stamping a clearer Image of Themselves into our Imagination, than any words

words can imprint. *Adeo difficile est* (to speak with *S^r Henry Savile*) *rem maxime perspicuam perspicue definire.*

A strait line altho' it be no where existing by it self, yet there is nothing of a more common conception, and in more frequent practice. For the Distance of Places from one another, Land, and Sea, Heavens, and Earth, the height of Buildings, Mountains, Clouds, Comets, Planets, &c. is only considered and measured *Directly by Length*: which is nothing else, but what *Euclide* means by a Mathematical strait line. And thus in very many cases of Humane Affairs this notion is necessary, and applyed to a real use.

A Thread stretched by a Plummer, The morning Rays, and Beams of the Sun (tho' in themselves refracted) do in some manner represent That, which we call Rectitude in lines.

DEFINITION V.

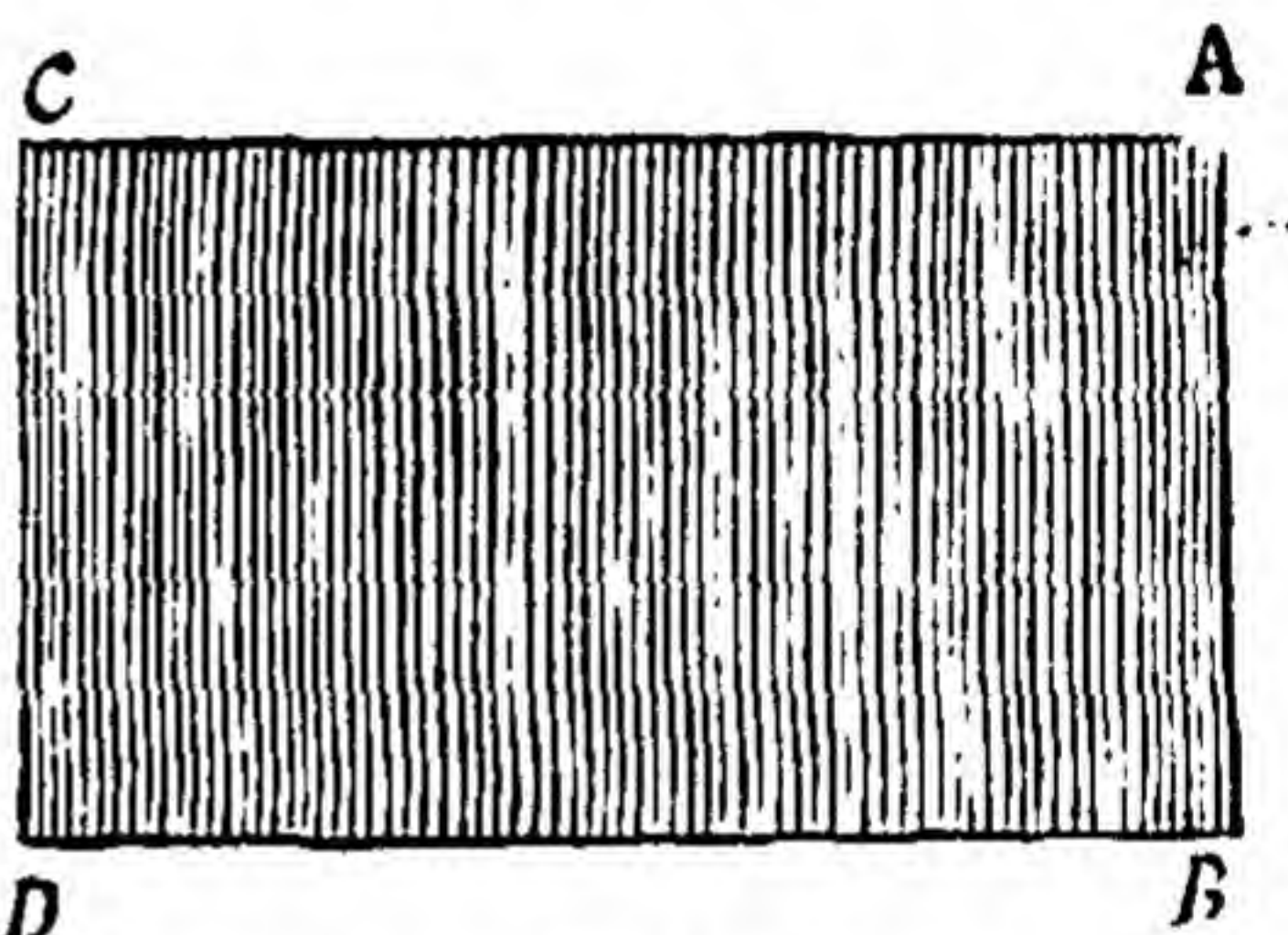
A *Superficies is That, which has length and breadth only.*

ANNOTATIONS.

The Mathematical conception of a Superficies, or Surface, may be thus explained by the motion of a Line. For if a line be supposed to move transversely, it shall trace forth a certain Length, and Breadth called a Superficies; *Long* by reason of the Length of the moving line; and *Broad* by reason of the *Motion* of the same line *sideways*.

As if the line AB, be imagined to move from AB to CD, it shall trace forth the Superficies ABCD.

But now from this illustration of the nature of a Mathematical Superficies, by the transvers motion of a line, we are not to conceive, That a Superficies consists of an Aggregation of lines, for the like reason as were before given, That a line was not made of an Aggregation of Points. But if



any Ones curiosity leads him towards an inquisition into this subtile Argument, let him read That Book under *Aristotle's* name *De lineis Infecabilibus*, with his Commentator *Pachemerius*. Also *Sextus Empiricus adversus Mathematicos*, a Greek Sceptic Philosopher, and of late Writers *Libertus Fromundus De Compositioni Continui*, which Treatise he justly entitles *Labyrinthus*.

Thus *Euclide* proceeds from a Point to a Line, and from a Line to a Superficies. And as a Line was said to be an *Extense One way only*, and Therefore can have but One Dimension, or One way of Mensuration according to Length; so a Superficies is an *Extense Two Ways*, which Ways are distinguished by the names of Length and Breadth, and therefore a Superficies has Two Dimensions, or Two Ways of Mensuration according to Length and Breadth. And this in one word may be called AN EXPANS; as a line was called Simply an EXTENSE. Moreover a Body, or Solid, is an Extense three ways, (that is every way) and therefore It has three Dimensions, or three ways of Mensuration by the names of Length, Breadth, and Depth, or Thickness. And that there are no more, than three Dimensions (as in this matter *Aristotle* says, *τὰ τρία πάντα*, *Three are All*) *Galileus* does demonstrate in the beginning of his first Dialogue of the System of the World. A Book, that deserves to live for ever with the World.

Yet in this matter the Geometrician does acknowledge with the Philosopher, that there is in Nature nothing existing of real magnitude but Body. Only there are seperate considerations of the *Mensuration* of Body, which we make to our selves for our own use, and do, as we have said, signifie by the names of *Length*, *Breadth*, and *Depth*. Of which three varieties of Dimensions every Man has naturally a clear, and distinct Idea.

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Now

Now the most obvious and sensible representation of a Mathematical Superficies is a Shadow ; And the common Extream of a Superficies partly shaded, and partly enlightened, represents a Mathematical Line. Moreover the general Object of our Sight is only a very Mathematical Superficies Illuminated. For we see not into the Body, or substance of Things. And to this Consideration aptly answers the Etymologie of the Greek name *ἐπιφανεια*, *Apparentia*, the Apparence and Surface of a thing.

But farther, the use of this Mathematical Superficies is made most manifest in the Estimation, and Mensuration of Lands, wherein the Surface only is considered, and valued, and nothing of the Depth. Which shews, That the Mathematical conception of a Superficies, is also a real Notion, and of general use, in like manner as That is of a Mathematical Line.

DEFINITION VI.

The Extrems of a Superficies are Lines.

ANNOTATIONS.

As a Line cannot be limited by a Line ; so upon the same reasons a Superficies cannot be limited by a Superficies. A point puts a stop to Length ; but It cannot to Length, and Breadth. That therefore must limit Length and Breadth, that is, an Expans, or Superficies, which is something more then a Point, yet less than the least Expans. And This can be nothing else but a Line, which only Magnitude is void of Expansion.

In the fifth Definition, a Superficies was first considered in its own nature, and simply in it self, as meerly Length and Breadth, but undetermined and as infinitely diffused. Yet now if in a superficies thus at large, there be a limitation any where supposed, the same, says *Euclide*, is conceived to be a Line. As if *Here*, by the line AB, If *There*, by CD, If *Elsewhere* by EF, GH, &c. These limitations being allways Lines, whether strait or crooked, conjoin'd, or not conjoin'd, one line, or many ; for these conditions come not as yet to be considered.

Only we are to conceive *A Superficies to be a Continuity of Length and Breadth, or an Expans determinable by Lines. As a Line is a Continuity of Length determinable by Points.*

Lastly, we are to observe, That This Definition of the Extrems of a Superficies; and the Third before of the Extrems of a Line are not properly to be accounted Definitions; altho' They be commonly numbred among Them. For in neither of them is any new Geometrical Term defined: but they are only necessary consequences, and common notions, Resulting from the Definitions of a Line and a Superficies. For Length or a Magnitude of one Dimension must be terminated by something void of Length or any Dimension: which is a Point. And Length and Breadth or a Magnitude of two Dimensions, must be terminated by something wanting one of those Two. Which is a Line, or Length without Breadth. As *Proclus* well says, *πάν τὸ μερίστο διὰ τὸ ἀμερίστο πλεονέχει.* Whatsoever is partible, the same is terminated by That, which is impartible, to wit, as It is used for a term or limit, so It has no magnitude, but is impartible, tho' in another respect the same may be a partible magnitude. As a line in respect of a Superficies, and a Superficies in respect of a Solid. And briefly, Whatsoever terminates an Other the same is less compound than the terminated magnitude by one dimension. So *Proclus* in general states this matter. *τὸ παρατὲν τὸ πλεονέκον μία λεπτοτα διατέσθ.* *Terminans a Terminato superat uno intervallo.*

DEFINITION VII.

A Plain Superficies, or A Plane is That * which lies Evenly to the strait lines within It self. * Ητις ἔχει τὰς ἐν αὐτῇ εὐθείαις καίται.

ANNOTATIONS.

After the definition of a Line, *Euclide* defines a Strait Line, and here accordingly, after the definition of a Superficies, he defines a Plain Superficies, which answers to a Superficies in general, as a strait Line to a Line in general.

And as a strait line was defined to lye evenly to the points which are in the same line; so a Plain Superficies is defined to lye evenly to the strait lines which are in the Superficies. For that a Plain Superficies has a like equable respect to all possible strait lines in the same Plain, as a strait line has to all possible points in the same Line.

Whereas curved surfaces, as a *Conical*, and a *Cylindrical* superficies, have in every part a different situation in a respect to the strait lines which may be seated in them: and therefore they do not lye *εῖρος* (as *Euclide* expresses) in an even position to their strait lines, as all plain superficies do. And we are therefore to conceive from *Euclid's* Definition, that a Plain Superficies is such as lyes every where so just, and even, that if we imagine strait lines to be every way seated in a plain superficies the strait lines shall wholly, and in every part touch the superficies, so as to lye just in it with a mutual agreement to one another. As *Sextus*, the Sceptic cites p. 101. lib. 3. *adversus Geometras*, *Επιπῶδες τυγχάνει, ὃ ἡ καταγομένη εὐθεῖα παύει τοῖς μέτροις ἀλλήλων. Planum id esse per quod circumacta linea recta omni ex parte eidem congruit.* And upon this natural conception of a plain Superficies, Mechanics use to apply the edge of a strait Ruler to a superficies, thereby to examine, whether that strait line does in all its parts agree with, and every where touch the superficies; and accordingly they judge of the exactness of the Plane.

If we conceive a strait line to move transversely It traces forth a Superficies. And if It move in such manner transversely as that every point thereof describes a strait line; It truly represents to our imagination an *Exact Plane*.

Now the PLANE, which *Euclide* has here defined, is the *ἑστία γεωμετρικὴ*. That Geometrical Seat, and noble Table, wherein the whole matter of the First Six Elements, and all the admirable speculations of Plain Geometry are placed.

Of Plain Angles.

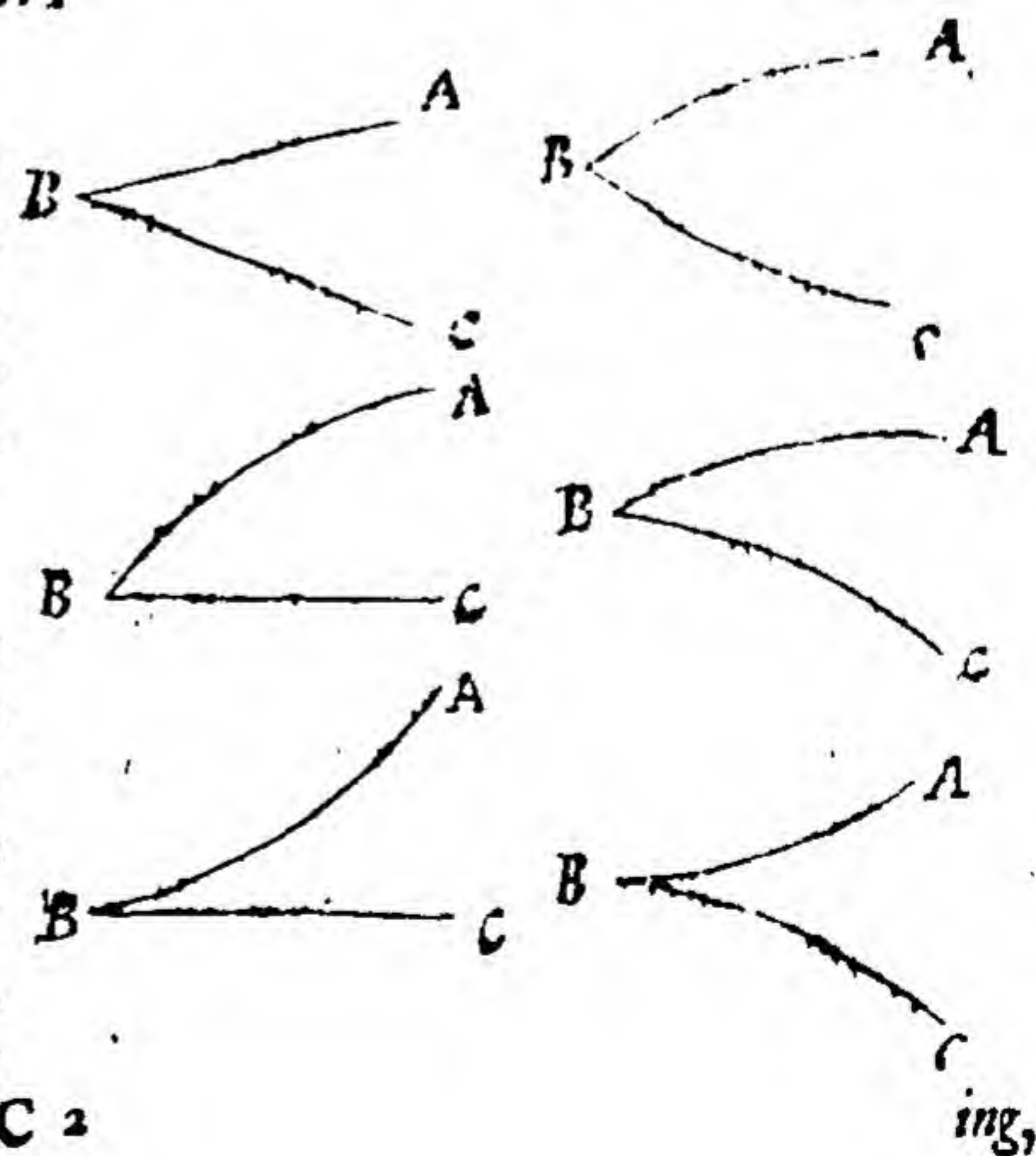
DEFINITION VIII.

A Plain Angle is in a plain Superficies an Inclination of two Lines to one another, *meeting together, and not directly situated One to the Other. * Ἀπομένον ἀλλήλων.

ANNOTATIONS.

Let there be two lines not directly situated one to the other; as AB, CB, Touching one another, or meeting together in the point B. Of these Two Concurring lines AB, CB, Their Inclination to one another is called an Angle.

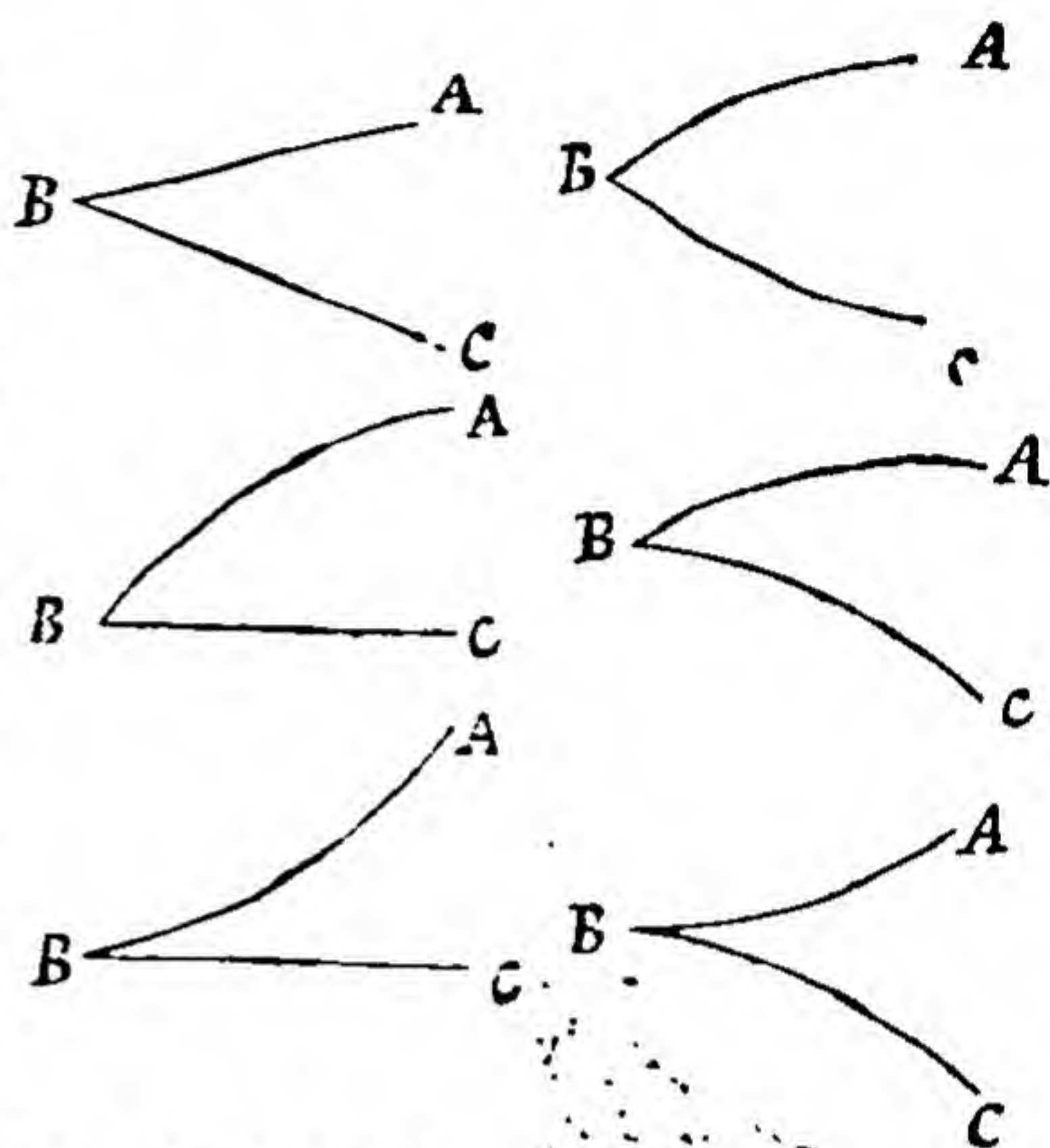
But an Angle is not to be said to be The Concours of two Inclining lines. As the late and much famed French Logician, transposing *Euclid's* words; makes Him to say, that an Angle is *la Rencontre de deux lignes inclinées*. Part. 4. Chap. 4. *De L' Art de penser, Of the Art of Think-*



ing, as the Author entitles his Logic. Whereas if he had rendred *Euclid*, he should have said, that an Angle is *la Inclination de deux lignes Rencontrées*. For *Euclid's Inclination* of two Concurring lines is divisible, and mensurable; but *la Rencontre*, the *Concurs* of two inclining lines, being in a point is indivisible: as the French Logician rightly observes; but hereupon he unjustly taxes *Euclide* with a fault of his own making, by changing *Euclid's* words, and sense. And it seems strange to me That the Author of a most Excellent Logical Institution should not advert the difference in Expression and Signification, between the Inclination of two Concurring Lines, and the Concurs of two Inclining Lines: for that Inclination manifestly denotes a divisible Thing; and Concurs an Indivisible Point.

Now lines are said to *Incline* to one another, when in Each line Part after Part comes nearer and nearer to the Other line, that is, when The lines do continuedly approach toward one another.

An Angle is commonly signified by three letters, as the Angle ABC . Where the middle letter ever denotes the point of Concurs of the two lines containing the Angle. Yet in the Notation of an Angle, we are to observe that The *Greeks* say not γωνία ἢ αβγ. The Angle ABC , but more properly ἡ ὑπὸ αβγ, that is, ἡ ὑπὸ αβ, ὅ γ περιέχεται γωνία. The Angle contained by the lines AB , BC . But the repetition of β the *Greeks* omitt for brevity sake. And for the more brevity it is translated the Angle ABC , instead of the angle contained by ABC , or AB , BC .



In the definition of an angle *Euclide* puts the *Simple* word κλίσις, *Clistis*, with which the *Latins* were not acquainted; never saying *Clinatio*; but always using some compound word, as *Acclinatio*, *Inclinatio*, *Declinatio*, &c. *Euclid's* word κλίσις, is generally translated *Inclinatio*. But I should rather have chosen *Acclinatio*, as best answering to *Euclid's* word πρὸς ἀλλή-
 λας ἢ γαμμῶν κλίσις. For πρὸς ἀλλήλας κλίσις, is more properly to be rendred, *ad se invicem Acclinatio*, than *Inclinatio*. *sefegue acclinat ad illam* is *Latine*; But *inclinat* is *Barbarism*. Yet in this matter *Inclinatio* is the word in use, and to the power of custom in Speech we must submit.

Euclide having laid down a Plain Superficies as the Platform for his Elementary Geometry, a *Rasa Tabula* aptly disposed to receive any impression, does begin with a Plain Angle, It being the *most Simple* of all other delineations, whose only and proper seat is to be in a plain Superficies. For tho' a Line be a delineation more simple than an angle, which to its constituton requires Two lines, and also that a line may be in any kind of Superficies; yet a Superficies is not the necessary seat of a line. For if so, it had been immethodical to have defined a Superficies after a Line. But both Line and Superficies were taken singly and without respect to one another, or to any place, other than That of *Universal Space*. In which, first *Euclide* puts a Point. next a Line. Then a strait Line. afterward a Superficies. and then a Plain Superficies. In which Smooth Mathematical field the Elementator begins, and therein first places a plain Angle.

Now in this Geometrical progress *Euclide* has not (as some Commentators impertinently have) enumerated all sorts of Lines, of Superficies, of Angles, into which Each of These might have Logically been divided, and subdivided. That manner of Doctrine is the method of Philosophers, and very proper to Them. Who define first the Science, and the Subject they Treat of; and again, divide the Subject into all its several kinds and species, as They ought to do; for that Philosophers undertake to comprehend the Whole Nature of their Subject in every part thereof, with all Its properties, and affections. Whereas the Mathematicians, passing

passing over those Logical Divisions, do select only what is necessary to their present Purpose. As *Euclide* in this place specifies only those Things, which are most easy in themselves, and best serviceable for an entrance into Geometry. So the Mathematician orders the same subject in one way, and the Philosopher in another: yet both equally right as to their several Ends. But *Ramus*, not justly considering the difference, that ever was, and ought to be had, between the Mathematicians and Philosophers in handling the same subject, endeavours to join both methods together. He therefore first undertakes to define *Geometry* as the *Science* he treats of, and next *Magnitude* as the *Subject*. Then divides Magnitudes into Commensurable and Incommensurable, Rational and Irrational, and so forth in a Logical Form. Which Notions being very subtle, and above the capacities of young Scholars, are in no manner agreeable to an Introduction into Geometry, and the proper way of Teaching and Learning the same. Indeed *Ramus*, makes a very ingenious excuse for himself, when he says, *Magis Logicam in Mathematico themate exercui, quam Mathematicam in suo pulvere seroque usu tractavi*. To which I must return, *Non igitur mirum est, quod tam infelicitur tractavit Geometriam*. For in his Logical ordering of the Elements by new Definitions, by impertinent Divisions, and Subdivisions, and in the general course of his Propositions, He has so disordered *The Elements*, that you scarce meet with a just Mathematical demonstration: But only a perswading face or some semblance of Truth, not a demonstrative Conviction. And of all our late Transformers of *Euclide*, He is the most Ungeometrical in Demonstration, how Exact soever in his Logical, or rather Verbal method, and disposal of his Propositions, He may pretend to be. And therefore to go on with *Euclide*.

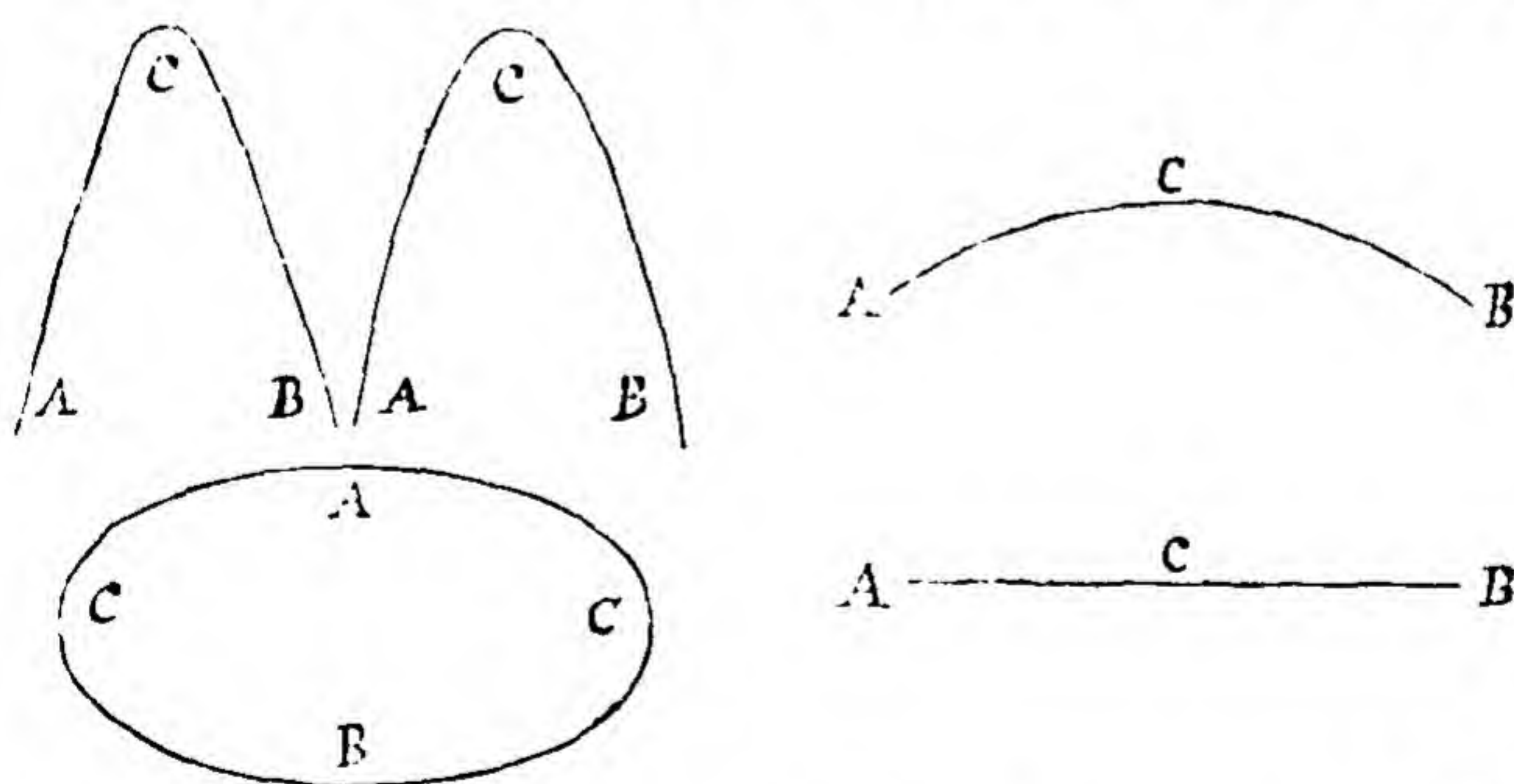
Of the Nature and Constitution of a Plain Angle.

For the Constitution of a plain angle. First, there must be a *κλίσις*, as *Euclide* calls it, an *Inclination, Vergency, Leaning* or *Tendency*, of Two lines one to the other. By which words is meant, that the Two lines do one way continually approach nearer and nearer to one another.

Secondly, upon this approaching there is to follow a concurse, or meeting together of those Two lines.

Thirdly, this concurse is to be in such a manner, as that, upon meeting, The Two lines lye not *ἐπ' εὐθείας*, that is, *ἐπ' εὐθείας ἑξῆς*, in a current or continued way towards one another, so as to join together in *one and the same line*; but that, after the point of concurse if each line be produced in its *Proper course*, They shall still be Two lines, and depart again from one another. Otherwise, it may so be as in these following Examples.

That a line drawn from the point A towards C, and another from the point B



towards C may in this tract, have an Inclination, Vergency, and Tendency to one another, yet in their meeting at the point C make not an angle, but that the line AC is so joined to the line BC that both together make one continued line ACB, and so no angle at all, because that an angle does require Two distinct

D

lines.

lines. If the lines thus meeting be Both strait, then they make one strait line. And This is *εὐθεία κλίσις*, a *strait Inclination* of two strait lines to one another, which upon meeting become One strait line. If both lines be crooked, then may they often make *some One continued crooked line*; as in these Instances, and the Like, whereof there may be infinite varieties. Therefore, in the Inclination, and Approaching of two lines to one another, *Euclide* made This special Caution, for the constitution of an angle, That, at their meeting, They be not situated *ἐπ' εὐθείας*, that is in such a course, as to be coincident, and continued one with the other if they be both farther produced. For this phrase *ἐπ' εὐθείας κείσθαι*, to lye strait-wise, is not only applicable to strait, but also to crooked lines, where, upon meeting, their curvature remains unbroken, *ἀνδ' ὁλίστας* as *Proclus* says, *without fraction*, that is, In such a sort, as that if the Two Concurring Lines be both still continued onwards in their proper course, they do not divert, or deviate again from each other, but do mutually pass into one another, and become one single line *continued in sua specie*. The Two lines meeting in this manner are said to lye *ἐπ' εὐθείας*, strait-wise to one another, and the Curvature is not by Geometricians conceived, to come under the notion of an angle.

Επ' εὐθείας, strait-wise, was a brief, and vulgar manner of speech, for *ἐπ' εὐθείας ἵστῆ* (as is before noted) that is, *in a strait way*; as *recta proficisci*, *To go strait forward*, is for *recta via proficisci*. And in the same sense that the Greeks used *ἐπ' εὐθείας*, we now use to say, *strait a long, strait forward*, tho' there be some flexures and windings in the way.

DEFINITION IX.

And when the Lines containing the Angle are strait, the Angle is called a *strait-lin'd Angle*.

ANNOTATIONS.

That is, when the Lines, which have to one another this mutual Inclination, and Concurse, are strait, then the Inclination of Those Concurring strait Lines is called a *strait-lin'd Angle*.

Euclide did just before define a plain angle in general. Yet here He next takes notice only of *Plain strait-lin'd angles*, passing over, after the manner of Mathematicians, the Logical division of Angles into Plain and Solid. And again omits the enumeration of the several sorts of all Plain Angles: for that the consideration of every One of Them was besides his present Matter, which he had confined to the simple speculation of Plain strait-lin'd Angles.

Of the Variety of strait-lin'd Angles.

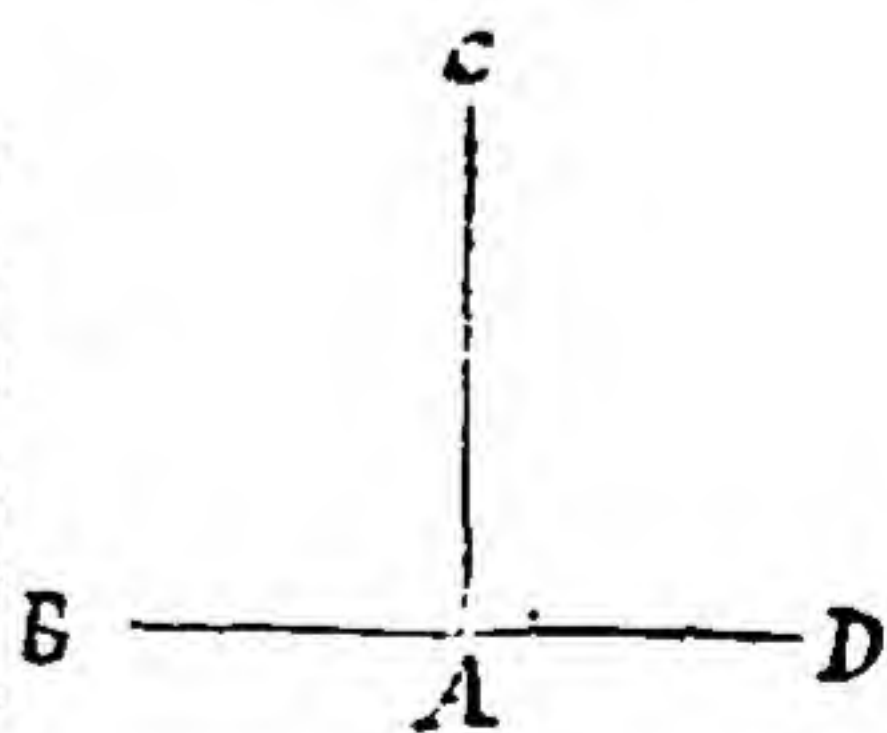
DEFINITION X.

Also when a strait line standing upon a strait line, makes the Consequent Angles equal to one another, each of the equal Angles is a *Right Angle*.

And the standing strait Line is called a *Perpendicular* to that Line upon which It stands.

ANNOTATIONS.

As if the strait line CA, standing upon the strait line BD makes the Consequent Angles, that is, the angles on each side of CA, namely CAB, CAD, equal to one another, Then each of those equal angles CAB, CAD, is said to be a *Right Angle*. And the standing strait Line CA is called a *Perpendicular* to the strait line BD upon which It stands.



These

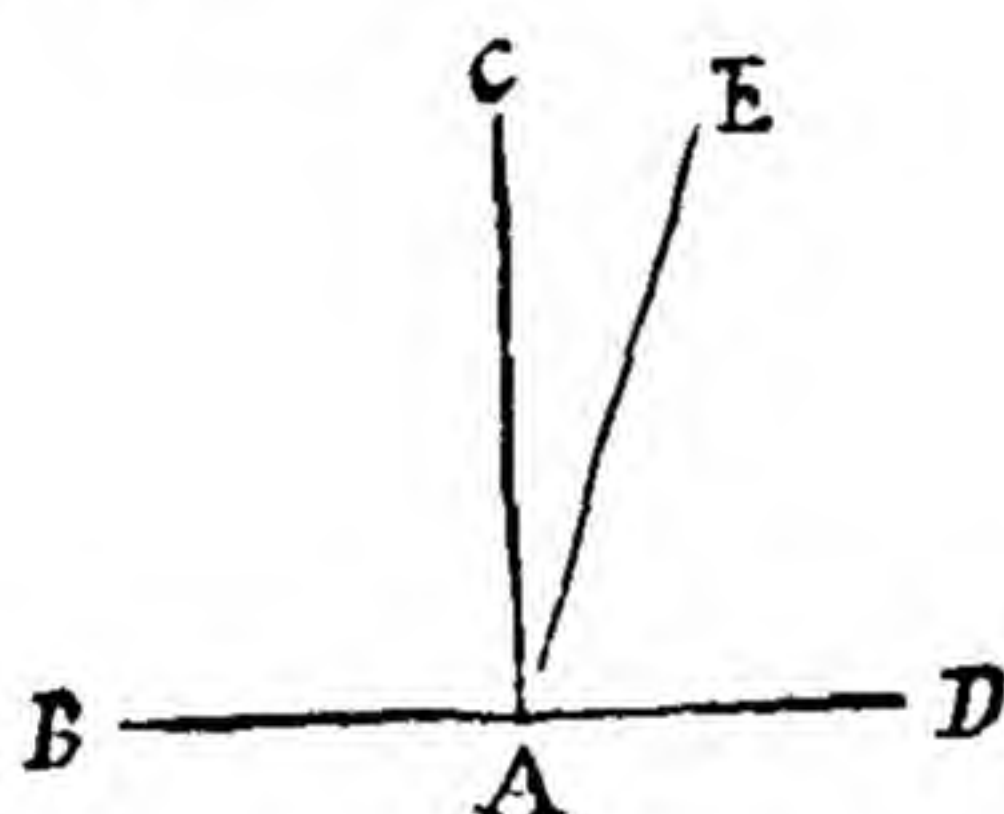
These angles CAB , CAD , which side by side are conjoyn'd to one another; are in general (whether Equal or Unequal) called by the Greeks $\alpha\iota\ \epsilon\phi\epsilon\lambda\lambda\epsilon\varsigma\ \gamma\omega\nu\iota\alpha\iota$, and the Latines *Anguli Deinceps*, Which I think not upaptly rendred, the *Consequent Angles*; for that they do in order mutually follow each other.

And further to consider this matter Philosophically, The Equality of the Consequent angles is the natural and immediate cause of Their Rectitude. That is, because the strait line CA standing upon the strait line BD makes its Inclination both ways, *Here* towards B , *There* towards D equally, and just alike to the line BD , therefore the angles CAB , CAD , are both necessarily to be conceived Right, and the standing line CA to be manifestly Upright to the line BD . Whereupon, It is by the Greeks properly called $\kappa\alpha\theta\eta\tau\epsilon\varsigma$, a *Cathetus*, That is, a line justly seated, or fitted both ways equally in respect of the line upon which It stands: To which therefore It is said to be a Normal Line.

We may here observe, That the English Tongue is as happy as the Greek, by rendring $\epsilon\upsilon\theta\epsilon\iota\alpha\ \gamma\epsilon\gamma\alpha\mu\mu\acute{\epsilon}\nu\eta$ a *strait line*, and $\acute{\omicron}\rho\theta\eta\gamma\omega\nu\iota\alpha$ a *right angle*; whereas the Latines have only one word for Both, *Recta linea*, and *Rectus angulus*, which commonly our English Translators follow, saying a *Right line*, as well as a *Right angle*. But I With that our Mathematical Writers would hereafter use this distinction, according to the exactness of the Greeks; seeing, that our Language affords two such proper words, as *STRAIT*, and *RIGHT*, answering to $\epsilon\upsilon\theta\upsilon\varsigma$ and $\acute{\omicron}\rho\theta\eta\varsigma$; And always say a *Strait Line*, and a *Right Angle*, like to $\epsilon\upsilon\theta\epsilon\iota\alpha\ \gamma\epsilon\gamma\alpha\mu\mu\acute{\epsilon}\nu\eta$, & $\acute{\omicron}\rho\theta\eta\gamma\omega\nu\iota\alpha$.

DEFINITION XI.

AN *Obtuse Angle* is That which is greater than a *Right Angle*.
As EAB is greater than CAB .



DEFINITION XII.

AN *Acute Angle* is That which is less than a *Right Angle*.
As EAD is less than CAD .

ANNOTATIONS.

Of *Euclid's* three sorts of strait-lin'd angles the *Right angle* is made the Rule and Standard to know the Others by. For that the *Right* is always unchangeable, and the same. Whereas the *Obtuse*, and *Acute* are capable of Increasing, and Decreasing infinitely; and only determined by THIS, That all Obtuse are greater, and all Acute angles are less than a Right angle.

Of the Quantity of a Strait-lin'd Angle.

Now whereas *Euclide* defines an Obtuse angle to be greater, and an Acute angle to be less than a Right angle, the Commentators have much labour'd *Wherein to state the Quantity of a strait-lin'd Angle*. For the Quantity of an angle is estimated neither by the Length of the lines which contain the angle, nor by the Superficies which lyes between those lines, nor by the Point of Concourse of the two lines. For the angle ABC is not greater than the angle DBE , but is still the same in quantity; altho' the Line BA be greater than BD , and BC than BE ,

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and

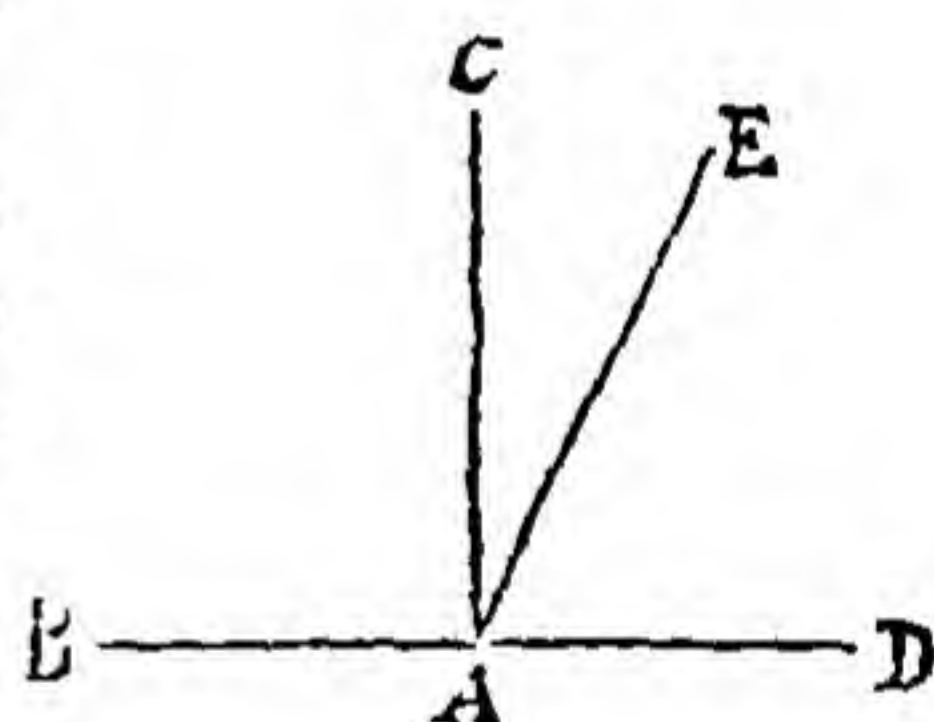
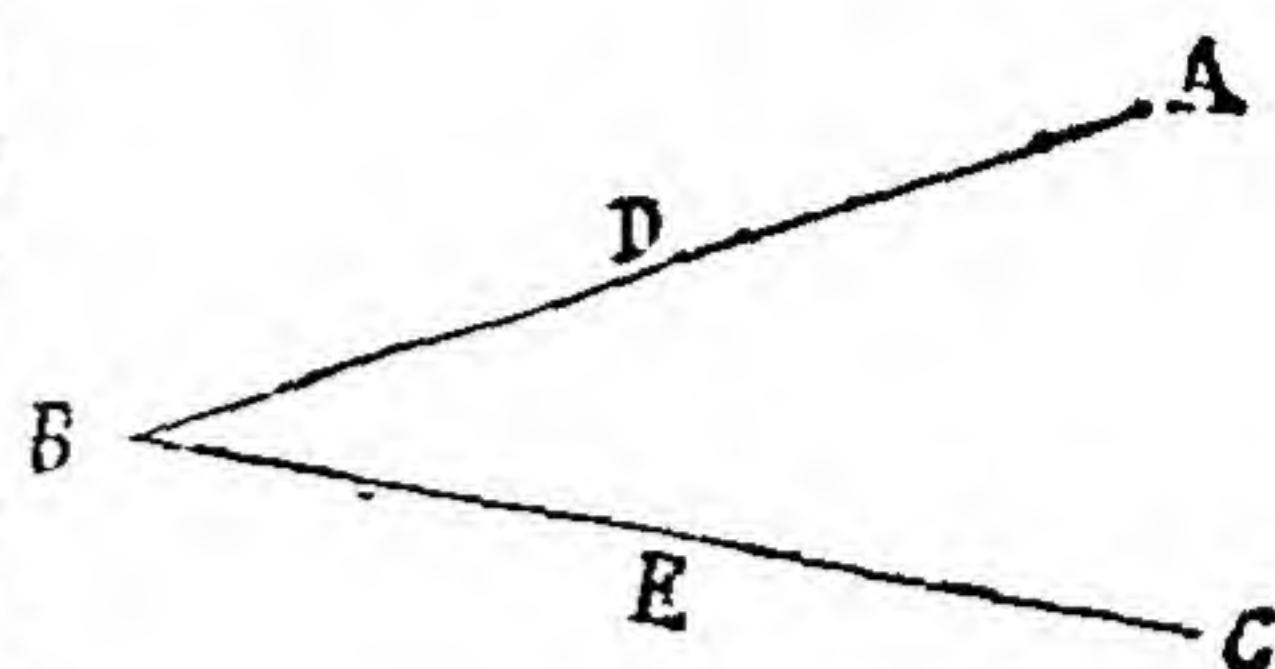
and the Superficies between AB, BC, be larger than the Superficies between DB, BE. So that in any plain strait-lin'd angle the Containing Lines with the Interjacent Superficies may both together be infinitely increased forward, yet the Angle remaineth the very same.

Again, for the Point of Concurs, That cannot come into the consideration of the quantity of an angle, seeing that a Point is in it self no quantity.

There remains then nothing more to be considered in an Angle besides the Inclination of the concurring lines. And thus much in the first place must be acknowledged in common Reason, that *only That can be Essential to the Quantity of an Angle, which being altered, does also alter the Quantity of an Angle.* Now neither the Length of the containing lines, nor the Interjacent space being altered, does alter the angle, as we have already noted. But if the Inclination of the containing lines towards one another be any way altered, the angle is likewise altered in its quantity. As if BC be conceived to move inwardly towards BA, or outwardly from it, the angle ABC will become greater or less. Therefore It necessarily follows, that *The Quantity of a strait-lin'd angle must consist in the Quantity of the Inclination of the two strait lines which contain the angle.* This Quantity of an angle is well said to be, *ἡ ἀπόστασις τῶν ὁρίων.* A Distance of Inclination. As Sextus the Sceptic cites It, Lib. 3. *adversus Geometras*, p. 102.

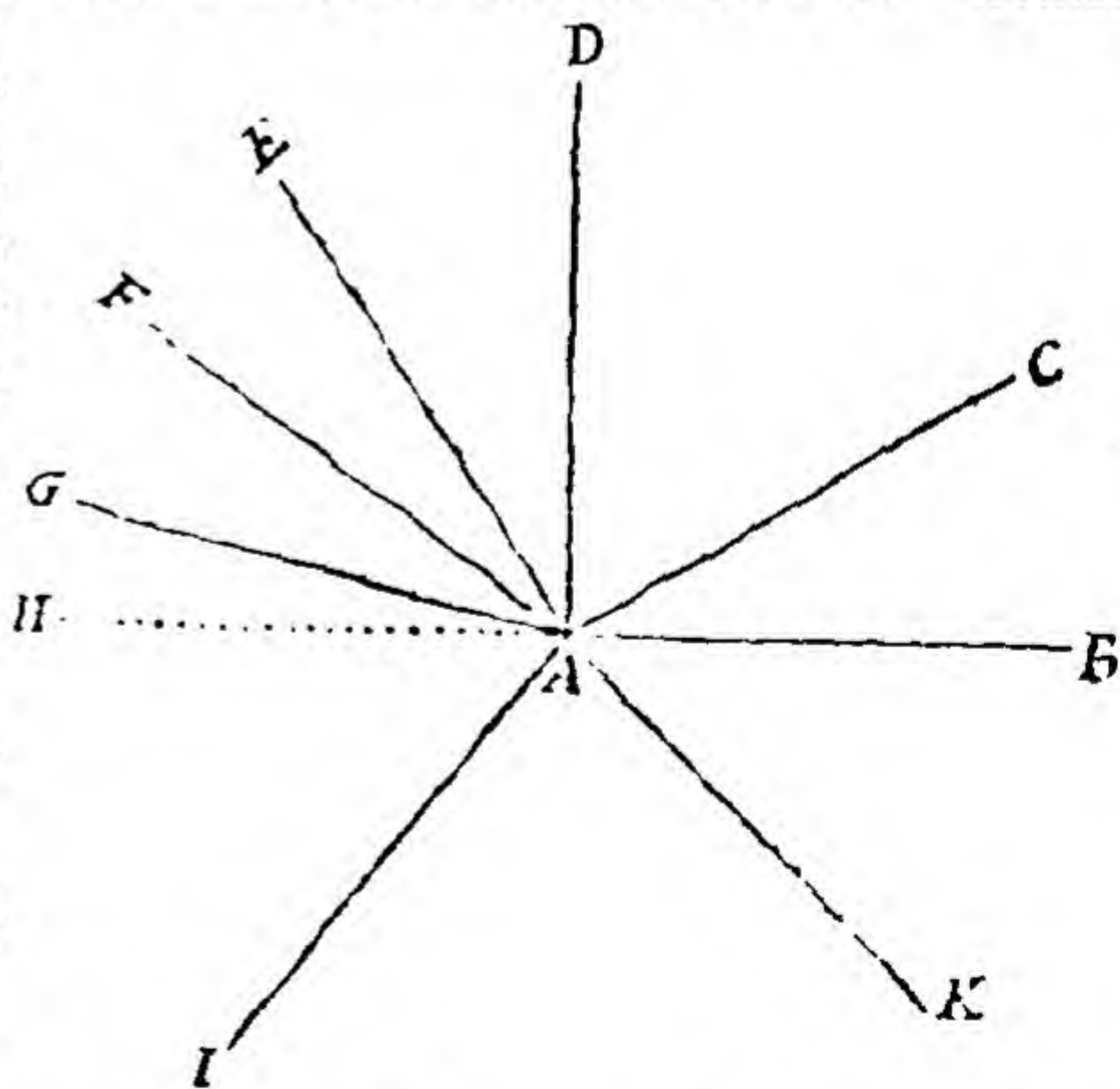
Now therefore when *Euclide* says that an Obtuse angle is greater, and an Acute angle is less than a Right angle, It is to be understood, that the *Amplitude, or Distance of Inclination*, is in an Obtuse angle greater than That, which is in a Right angle, and in an Acute angle It is less.

As the Inclination of the lines BA, EA to one another is greater than the Inclination of BA, CA to one another. And the Inclination of the lines EA, DA to one another is less than the Inclination of CA, DA to one another. And thus from the Dilatation, or Contraction of the Inclination of the containing lines, One angle is said to be greater, or less than another. But now, How *This Distance of Inclination, as an Angular Quantity* may be measured, and estimated, shall be shown hereafter in its due place.



A Second Consideration of a strait-lin'd Angle.

But to illustrate farther, & in somewhat a different manner from the former, the nature, and quantity of a plain strait-lin'd angle, Let us suppose any point, as A in a plain Superficies, and from it to be drawn Two strait lines at any wideness, or Aperture, as the lines AB, AC. Here the angle BAC may be said to be the Aperture or Divarication of the lines AB, AC, which contain the angle BAC, so drawing AD, AE, AF, AG, the angle BAD is the Aperture of the lines AB, AD, which contain the angle BAD. In like manner the several other angles are the Apertures, or Divarications, of their containing lines; Which, may continually be enlarged, and opened more, and more increasing the angles till the line AH comes to lye directly with AB,



AB making one strait line HB. So that this *ὁρθὴ κλίσις*, or *strait Inclination* of the two lines BA, BA, to one another, cannot constitute an angle in their meeting at the point A. But furthermore, if there be drawn the line AI, there is again made the angle BAI, which is the Aperture of the lines AB, AI; and the like is in BAK; and so continually we may approach towards AB making the Apertures (that is, the Angles) less and less, till the approaching line be coincident, and one and the same with AB.

A Second Definition of a strait-lin'd Angle.

Wherefore a *strait-lin'd Angle* may be said to be *The Aperture of two strait lines drawn from a point, and not lying directly to one another.*

And how the Quantity of an angle may be estimated by the Aperture of the containing lines, as well, as by their Inclination to one another shall be shewn hereafter.

A Third Consideration of a strait-lin'd Angle.

Moreover, because these several Angles, or Apertures do together make up the whole space about the point A, from hence may arise a Third Conception of a strait lin'd angle, and its Quantity. For seeing that all angles that can be possibly stated from the point A, do together compleat the whole space about the same point; therefore every particular angle contains some certain portion of the SPACE ABOUT the point A.

A Third Definition of a strait lin'd Angle.

Wherefore we may thus again define a plain strait-lin'd angle. A plain strait-lin'd angle is *A Portion of a plain space about a Point, contained by two strait lines, drawn from the same Point, and not directly situated to one another.*

This AMBIT, or CIRCUMCIRCA, does as justly answer to the nature of an angle, as either Inclination, or Aperture. For as in these Two conceptions, so likewise in this Third the Quantity of an angle is not concern'd either in the Length of the containing lines, or in the space Interjacent between those lines: but only in the space Circumjacent about the angular point, Which Circumcirca being changed, as either amplyfied, or coarcted, the angle accordingly becomes Greater, or Less, as It does upon the Notion of Inclination, or Aperture. And how by a portion of the Circumambient space about a point, the Quantity of an Angle may be also estimated, shall likewise be explained hereafter.

Thus have we Three ways whereby to set forth the Nature, and Quantity of a strait-lin'd Angle. Either from the consideration of the space about a Point, or of the Aperture of two strait lines drawn from a Point, or rather with Euclide, of the Inclination of two strait lines meeting in a Point. Which is the most proper, and the only Notion of an angle useful in all Geometrical demonstrations.

But now after all these considerations, if it be at last questioned in what Predicament, that is, in what Rank, or Classis of Things, whether in Quantity, Quality, or Relation an Angle is to be placed, Proclus well answers to this Logical, and Philosophical Question, that an Angle is not solely in any One of These: but upon the several conceptions, which jointly compleat the Entire Notion of an Angle, It does belong to all the Three. For the Geometrician speculates not the nature of Magnitude in it self, as Magnitude, like to the Philosophers: but only considers some certain kinds of Magnitudes with their demonstrable Properties, and Qualities, and with the mutual Relations that Magnitude may have to Magnitude. And so there is a Compound Idea of all magnitudes Mathematically considered. An angle has a Quantity whereby one angle is greater, or less than another. It has also a Quality by its Form. And in the Inclination of the lines themselves there is a mutual respect to one another.

Of a Term and a Figure.

DEFINITION XIII.

A Term is the Extream of a Thing.

DEFINITION XIV.

A Figure is That, which is comprehended by One, or Many Terms.

ANNOTATIONS.

DEF. 13. In the termination, or limitation of Magnitudes, *Euclide* uses two several words, $\pi\epsilon\rho\alpha\varsigma$ an *End*, *Limit*, or *Extream*; and $\epsilon\upsilon\theta$ a *Term*, or *Bound*. And here he says that $\epsilon\upsilon\theta$, a *Term* is $\pi\epsilon\rho\alpha\varsigma$ an *Extream*. $\pi\epsilon\rho\alpha\varsigma$ he applies to a *Line*, and also to a *Superficies*, when both *Line* and *Superficies* were at first laid down as understood: but $\epsilon\upsilon\theta$ is appropriated to a *Magnitude* every way determined and bounded.

For when *Euclide* defines a *Line* to be *Length* only, he puts there a line as an infinite, or indeterminate *Length*. But then again when a line is considered as to be determined, he says, that the *Ends*, or $\pi\epsilon\rho\alpha\varsigma$ of a *Line* are *Points*.

So likewise in the fifth Definition, *Euclide* sets forth a *Superficies* under the simple conception of *Length*, and *Breadth*, that is an *Expanse*, or *magnitude of Two Dimensions* in general, and Indeterminate. But he afterward adds, That the *Extreams*, or $\pi\epsilon\rho\alpha\iota\alpha$ of a *Superficies* are *Lines*. He does not here mean by the word $\pi\epsilon\rho\alpha\iota\alpha$ a *Comprehension*, or an entire Enclosure of a *Superficies* by one, or many *Lines*; for that had been to make a *Figure* before he had defined It: But only so, as that of a *Superficies* indefinite, an *Extream*, *Limit*, or $\pi\epsilon\rho\alpha\varsigma$, as he calls it, in what part soever it be placed, *Here*, or *There*, is a *Line*. A *Line* being as a *Barr* in an indefinite *Superficies*, like as a *Point* is a stop in an indefinite *Line*.

But $\epsilon\upsilon\theta$ relates to $\tau\omicron\ \epsilon\epsilon\chi\epsilon\iota\mu\epsilon\nu$, to That which is supposed every way enclosed. As *Proclus* says, that $\epsilon\upsilon\theta$ is $\pi\epsilon\chi\epsilon\iota\tau\epsilon\ \tau\omicron\ \chi\omega\rho\acute{o}\nu$, The *Comprehension of a space*. For it is $\epsilon\chi\epsilon\iota\mu\epsilon\nu\ \epsilon\upsilon\theta$ the *Bound*, or *Term of a Figure*; not $\epsilon\upsilon\theta\ \pi\epsilon\pi\epsilon\alpha\iota\epsilon\iota\alpha\varsigma$, the *Bound of a Superficies*; but $\pi\epsilon\rho\alpha\varsigma\ \pi\epsilon\pi\epsilon\alpha\iota\epsilon\iota\alpha\varsigma$, the *Limit of a Superficies*, THIS, or THAT way.

And accordingly, the bounds of *Lands*, or *Territories* are properly called $\epsilon\upsilon\theta$, but not $\pi\epsilon\rho\alpha\iota\alpha$. So that where the subject is $\tau\omicron\ \pi\epsilon\pi\epsilon\alpha\iota\epsilon\iota\mu\epsilon\nu$, or $\tau\omicron\ \pi\acute{\alpha}\nu\tau\omicron\theta\epsilon\nu\ \epsilon\epsilon\chi\epsilon\iota\mu\epsilon\nu$, a *Magnitude comprehended*, or *every way terminated*, as a *Geometrical Figure* is, the word $\epsilon\upsilon\theta$ is properly used; and not $\pi\epsilon\rho\alpha\varsigma$. So that $\epsilon\upsilon\theta$ may be said to be $\pi\epsilon\rho\alpha\varsigma$, tho' every $\pi\epsilon\rho\alpha\varsigma$ ought not to be called $\epsilon\upsilon\theta$. And thus much for the explication of $\pi\epsilon\rho\alpha\varsigma$ and $\epsilon\upsilon\theta$ to make the best of this matter.

For altho' it appears by *Proclus*, that, in his time, This was among the rest of the Definitions; yet I cannot allow it to be *Euclid's*. For first, it is manifest, that the Definition it self is needless, because, if it were wholly omitted, yet nothing that follows, is made thereby either imperfect, or obscure. And besides, it is no more a proper *Geometrical* word of Art, to require a Definition, than it was the common word, in that time, not only among the *Land-meters*, or *Surveyors*, but among all the people who had any *Lands* to distinguish, and appropriate to themselves from those of their *Neighbours*. He might have as well defined, $\pi\epsilon\rho\alpha\varsigma$ to be $\pi\epsilon\lambda\omicron\mu\eta$ an *Ending*, or $\alpha\chi\epsilon\iota\nu$ an *Extream*, as $\epsilon\upsilon\theta$ to be $\pi\epsilon\rho\alpha\varsigma$. Which, as I have said, was no more, at that time than what every *Greek* understood. For both $\epsilon\upsilon\theta$ & $\pi\epsilon\rho\alpha\varsigma$ were words of common use, and equally of known signification, and are in such a sense here taken. It seems most likely, that $\pi\epsilon\rho\alpha\varsigma$ being before used in the Definitions of a *Line* and *Superficies*, and now $\epsilon\upsilon\theta$ in the Definition of a *Figure*, some One noted in the margin, That $\epsilon\upsilon\theta$ was also $\pi\epsilon\rho\alpha\varsigma$. And this afterwards came into the Text for a Definition; as the like has happen'd to many other Authors. For indeed, it is rather the by-note of a *Lexicographer*, than the Definition of a *Geometrician*. And yet This too signifies very little, unless something more were added to $\pi\epsilon\rho\alpha\varsigma$, for the interpretation of $\epsilon\upsilon\theta$. As that $\epsilon\upsilon\theta$ is

is πέρασ συγκλείειν an enclosing Extream; which indeed may pass for a Glossary Exposition of ὅρος. but ought not to be accounted a Geometrical Definition. This also was the Judgement of that most Learned and Renowned Gentleman S^r Henry Savile, the Munificent Founder of the two Famous Mathematical Professors Places, in the University of Oxford, upon this subject in his most excellent Lectures, introductory to the Elements, *That This had not the appearance of a Definition; Non habet Definitionis faciem.* Lect. 5. p. 101.

DEF. 14. Figure is by Aristotle accounted the fourth Species of Quality; and rightly, being there taken abstractedly for the Qualification of Magnitude by Termination, or outward Form. But the Geometrician considers the Quantum Figuratum, the Magnitude Figured. So that, in every Figure, there is jointly taken τὸ περικλειόμενον, & ἡ περιέχουσα, the Inclosed, and the Inclosure, or τὸ ἐκλειόμενον, & ὁ ὅρος, Terminatum, and Terminus, That is, the Magnitude Bounded, and the Magnitude Bounding. Both which together do constitute a Geometrical Figure.

In general, a Figure is χωρίον πανταχοθεν ἐκλειόμενον, A Space every way terminated. So that first, A Space comes to be a Figure by Termination: And next, Every Figure has its specification from the manner of its Termination.

The first distinction here is, of Figures comprehended by One Term only, or by more than One. Which Division being most Simple, Euclid has, with great artifice, couched within his Definition, when he says, That A Figure is a Space comprehended by one, or many Terms. Whereas the Definition were perfect, if he had only said, That A Figure is a Bounded Space. But moreover, he very subtly draws in the primary difference of plain Figures from the number of their Terms, which are lines, One, or more than One, in order to the following Definitions of Figures, under One term, or Two, or Three, or Four, &c.

Thus from a Line Euclid passes to an Angle, and from an Angle to a Figure, for that an angle is in its nature between Them both. An Angle being something more than a Line, as having two concurring lines: yet is something wanting of a Figure, as not having a Compleat termination. And in that respect an Angle is but a Semifigurate. And so is justly placed between a Line and a Figure.

Of a Circle, Circumference, Center, and Raies, Diameter and Semicircle.

DEFINITION XV.

A Circle is a plain Figure comprehended by One Line, which is called a Circumference, unto which all strait lines, falling from One Point of Those lying within the figure are equal to one another.

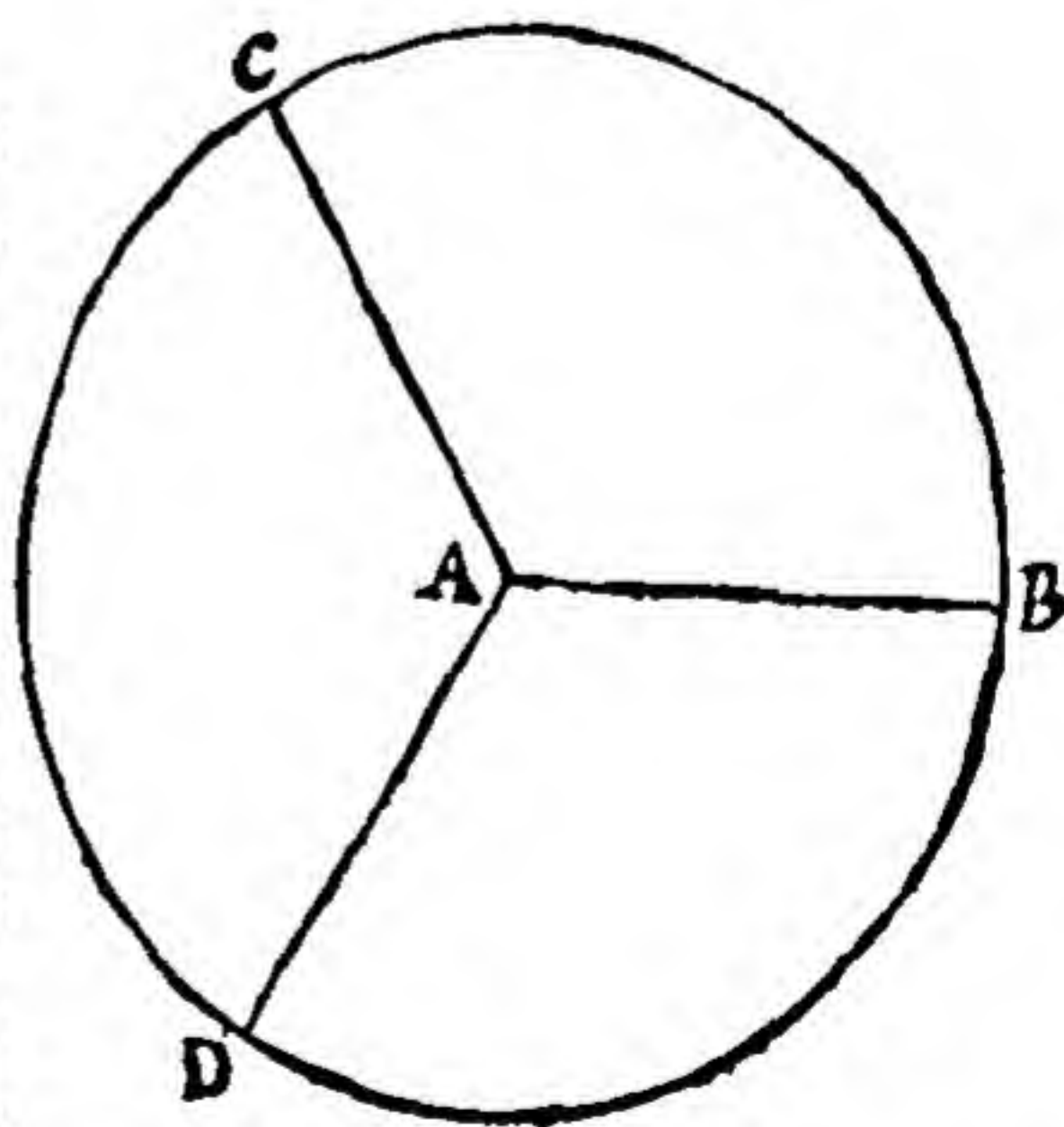
DEFINITION XVI.

And This Point is called the Center of the Circle.

ANNOTATIONS.

By a plain figure, Euclid means a figure situated in a plain superficies, such as he had before defined. Suppose then in a PLANE a Figure comprehended by one line B C D B, which is called the Circumference. And from some One Point seated within this Figure, as suppose from the point A, Let the strait lines how many soever, as A B, A C, A D, &c. falling on the Circumference at the points B, C, D, be all equal to one another, then such a Figure, says Euclid, is a Circle. And the Point A is called the Center of It.

Here also it is observable, that the lines A B, A C, A D, &c. may, in one word of Cicero's, very pro-



perly, be named **RADII**, **RAIES**, which by *Euclide* are always called αἱ ἐκ τοῦ κέντρου, *The lines from the Center*, That is to say, *from the Center to the Circumference*.

And what *Cicero* has said out of *Plato's Timæus*, in describing the Spherical figure of the **World**, does almost word for word agree with *Euclid's* definition of a Circle, *Cujus omnis extremitas paribus à medio radiis attingitur. Cic. Fragment. de Universo.*

The great **VIETA** says, *Radius elegans est verbum Ciceronis*: Which *Ovid* also uses in the same sense, as elegantly in his Description of the Sun's Chariot,

*Aureus axis erat, temo aureus, aurea summa
Curvatura rote, Radiorum argenteus ordo.*

The Circle, Circumference, and Area.

Moreover, in the consideration of this Figure, we are rightly to distinguish between the *Circle*, the *Circumference*, and the *Area*, or the comprehended Space. The Circle is the Whole together, Area, and Circumference.

The Circumference has many properties peculiar to it self, and very distinct from those of the Circle, as hereafter will appear. And This we may first observe, that altho' the Circumference be conceived without any breadth, yet by reason of its curvature, It is *Concave*, and *Convex*, accordingly as it relates to What lies within, and to What is without the Circle. And some peculiar affections of the Concave and Convex Circumference, *Euclide* sets forth in the 8th Proposition of his Third Element.

Of the Circumference of a Circle applyed to the Mensuration of strait-lin'd Angles.

Lastly, it will not be immethodical in this place to take a further consideration of the properties and use of the Circumference of a Circle, in relation to the strait-lin'd Angles before defin'd, and the mensuration of their Quantities.

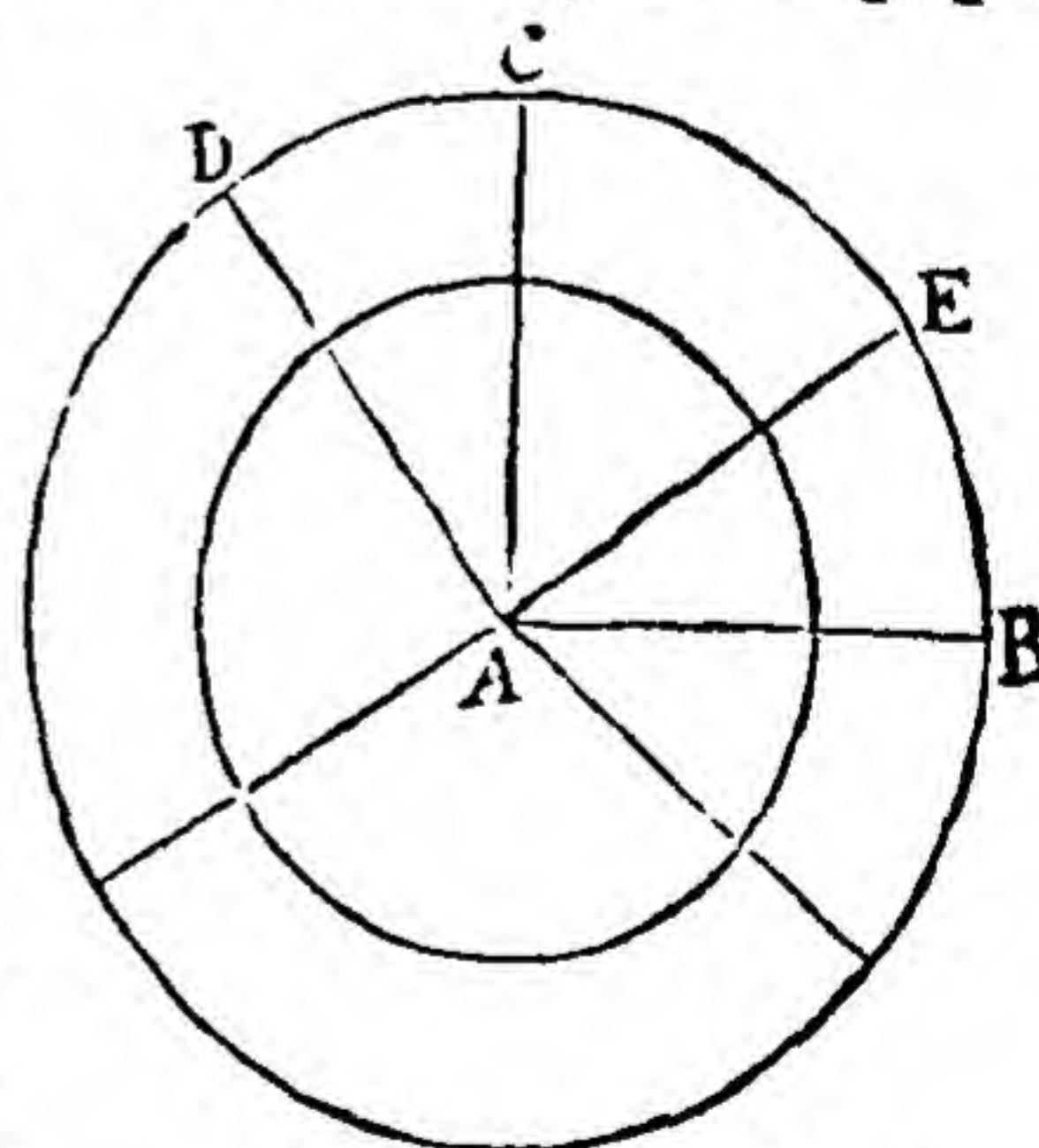
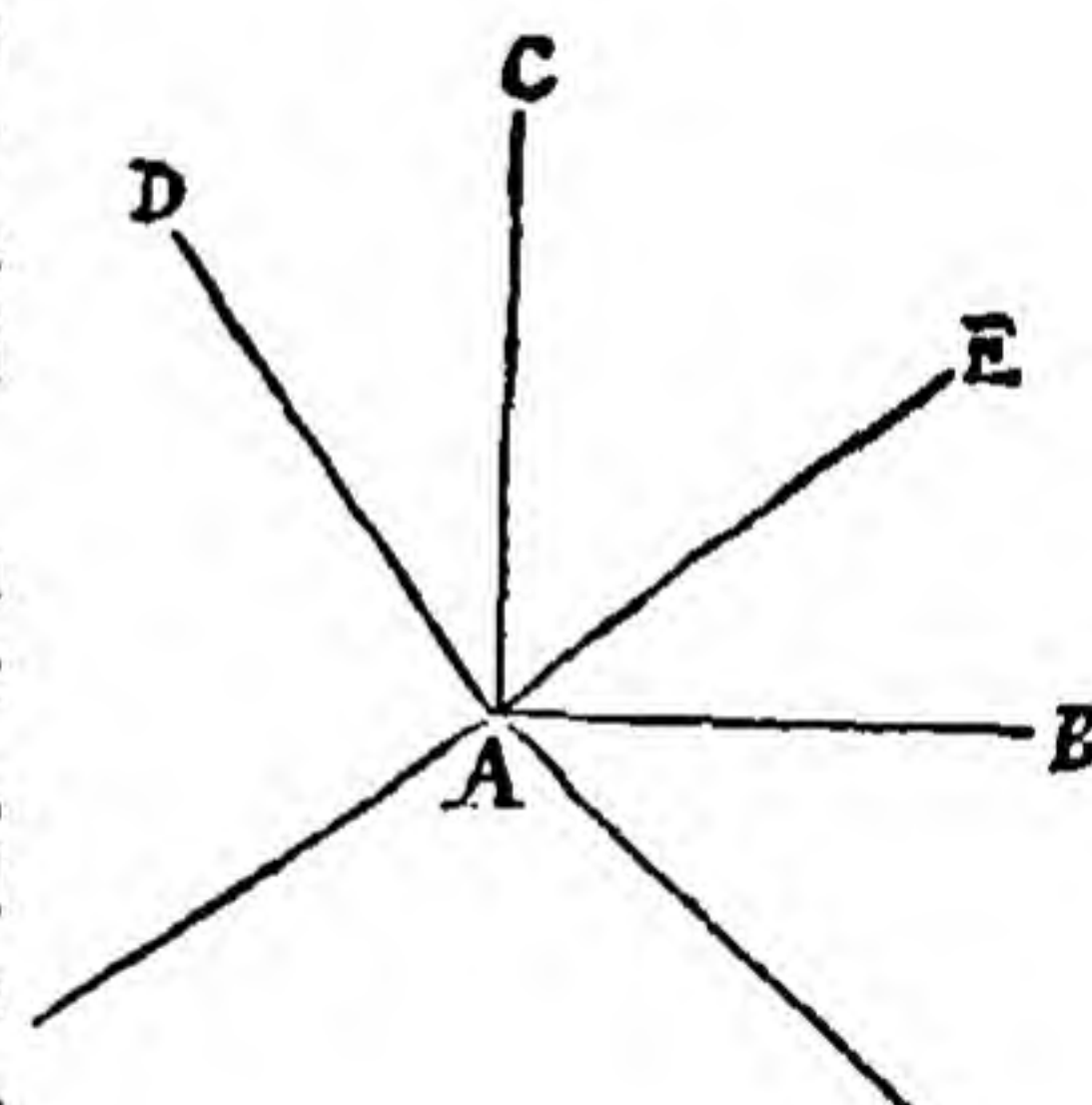
First then, in an infinite plain space (the seat of plain Geometry) if we suppose a Point, as **A**, from whence every way do flow strait lines infinitely, or indeterminately, there shall be contained by them the Three sorts of strait-lin'd angles, Right, Obtuse, and Acute. As for Instance **BAC**, **BAD**, **BAE**, &c.

Here therefore for to estimate by some certain Measure the *Quantity of an Angle*, We are to consider how, and by what means This infinite plain Space, wholly possessed by plain strait-lin'd angles seated ab ut a given Point, may be brought under some Comprehension and Boundary capable of mensuration: and therewithal the strait-lin'd angles, which fill this Space, may be likewise measured.

Now the Definition of a Circle easily leads us unto a discovery of the *proper Measure of an Angular Quantity*: and that It may most fitly be found in a Circular Figure.

For *first* from any Point as a Center, the Circumference of a Circle does comprehend entirely the undetermined circumambient Space about that point: and by This comprehension renders the Ambit capable of mensuration in that the Circumference of a circle is divisible, and so certainly mensurable.

Secondly, at the center of a Circle like as at a Point in an infinite plain Space, may be constituted strait-lin'd angles of any sort, and quantity whatsoever, which are all contained by the Raies of the Circle. As in the foregoing Diagram, Let now the infinite plain Space with all the

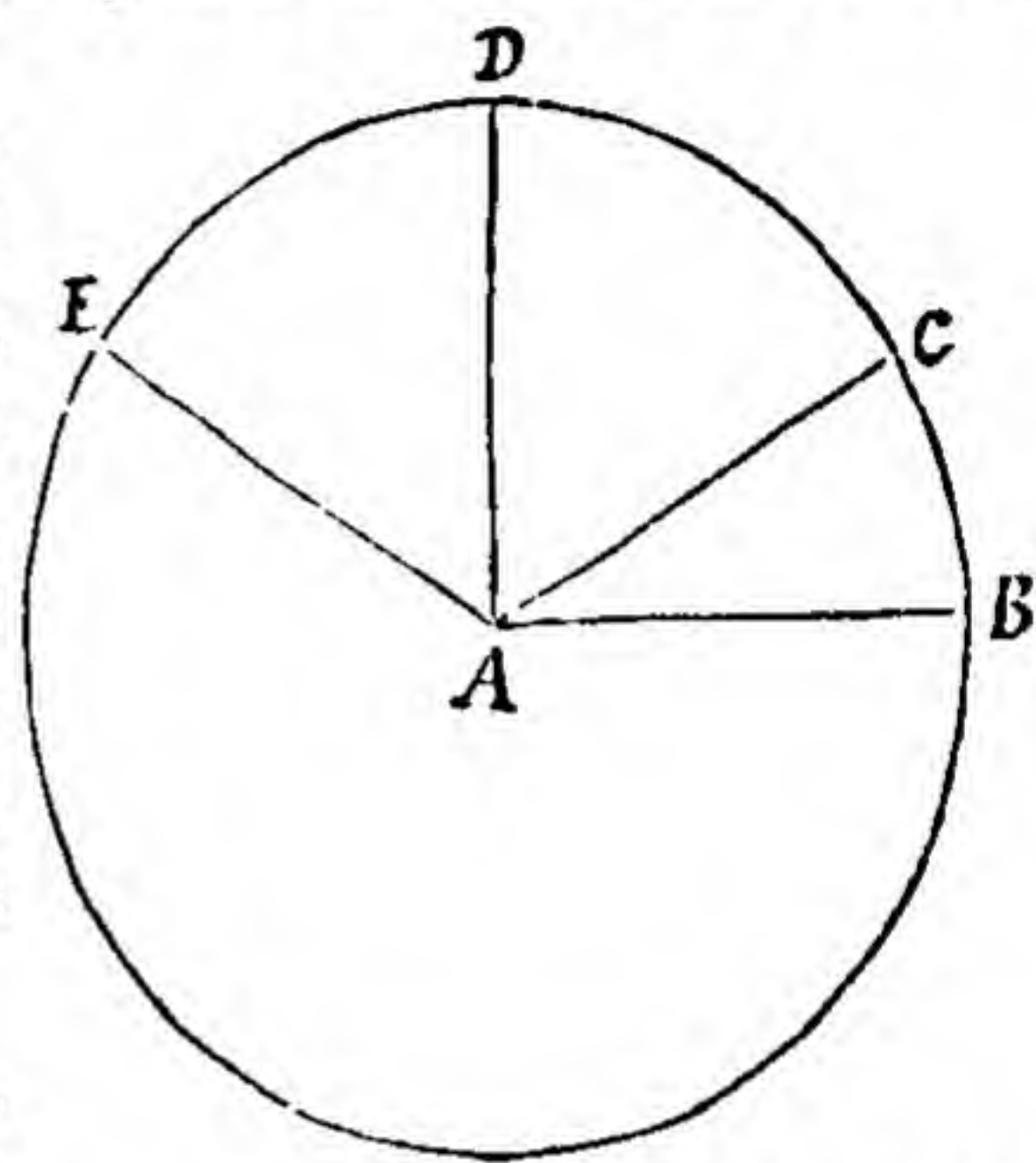


the Angles about the point A be determined by the Circumference of any Circle, whose center is the said point. And therefore forasmuch as the circumference of a Circle may from any point bound an infinite plain Space uniformly round about, and therewithal does encompass all strait-lin'd angles constituted at that point, which taken together compleat the Same Space: and as the angles are greater, or less, so the parts of the whole circumference, which are intercepted by the Strait lines containing those angles, become also greater, or less; It will appear in common Sense, That the Circumference of a Circle may justly be put as the Proper Measure of the quantity of these angles. Thus therefore the Measure of a strait-lin'd angle is defined.

Definition of the Measure of strait-lin'd Angles.

The Measure of a Strait-lin'd angle is an Arch of the circumference of a Circle, described from the angular point as a Center, and intercepted by the strait lines which contain the Angle.

As the Arch BC is the measure of the angle BAC, and the Arch CD of the angle CAD, so BD of BAD, BE of BAE: Every Arch measuring the quantity of its correspondent angle from the center A, at which point all strait-lin'd angles, Right, Obtuse, and Acute, or what quantity soever, may be placed. But this mensuration of an angle is not to be taken in the strictest sense. For whereas it must be acknowledg'd, that *Mensura, & Mensuratum sunt in eodem genere. The Measure, and the Measured magnitude are of the same kind*: yet we cannot say, that an Arch and an Angle are truly of the same kind.



But because 'tis evident, and demonstrated, that Angles and Arches do mutually increase, and decrease alike, so that if the Angle BAD be double, or triple of the Angle BAC, then the Arch BD is double, or triple of the Arch BC. And on the contrary, if BD be double, or triple of BC, then BAD is double, or triple of BAC.

Therefore an Arch of the Circumference of a Circle serves aptly as the Measure of an Angle in all common Uses, and Mathematical Speculations, or Geometrical Practices appertaining to Astronomy, The Mensuration of Distances, Surveying of Lands, Navigation, Fortification, Gunnery, &c.

Of the Division of the Circumference into certain Parts, and the Use Thereof.

Now that these Arches, and by consequence the angles at the center, which are measured by these Arches, might be brought under a certain value, and estimation, therefore the most ancient Geometricians at first divided the Circumference of a Circle into 60 parts. Afterwards It was found more convenient to divide the same into six times 60, that is into 360 parts, commonly called *Degrees*; by which an Arch of the circumference, with its correspondent Angle might be estimated.

So that, for example, we say, That the Angle BAC is 30 Degrees, if the Arch BC be 30 such parts, as of which the whole circumference is divided into 360. Again, if the Arch BD be 60 Degrees, then the Angle BAD is accounted 60 Degrees. In like manner, the Angle BAE is 90 Degrees, if the Arch BE be a Quadrant, or the fourth part of the whole circumference; for 90 is the fourth part of 360. *And This particularly is the measure of the quantity of a Right Angle.*

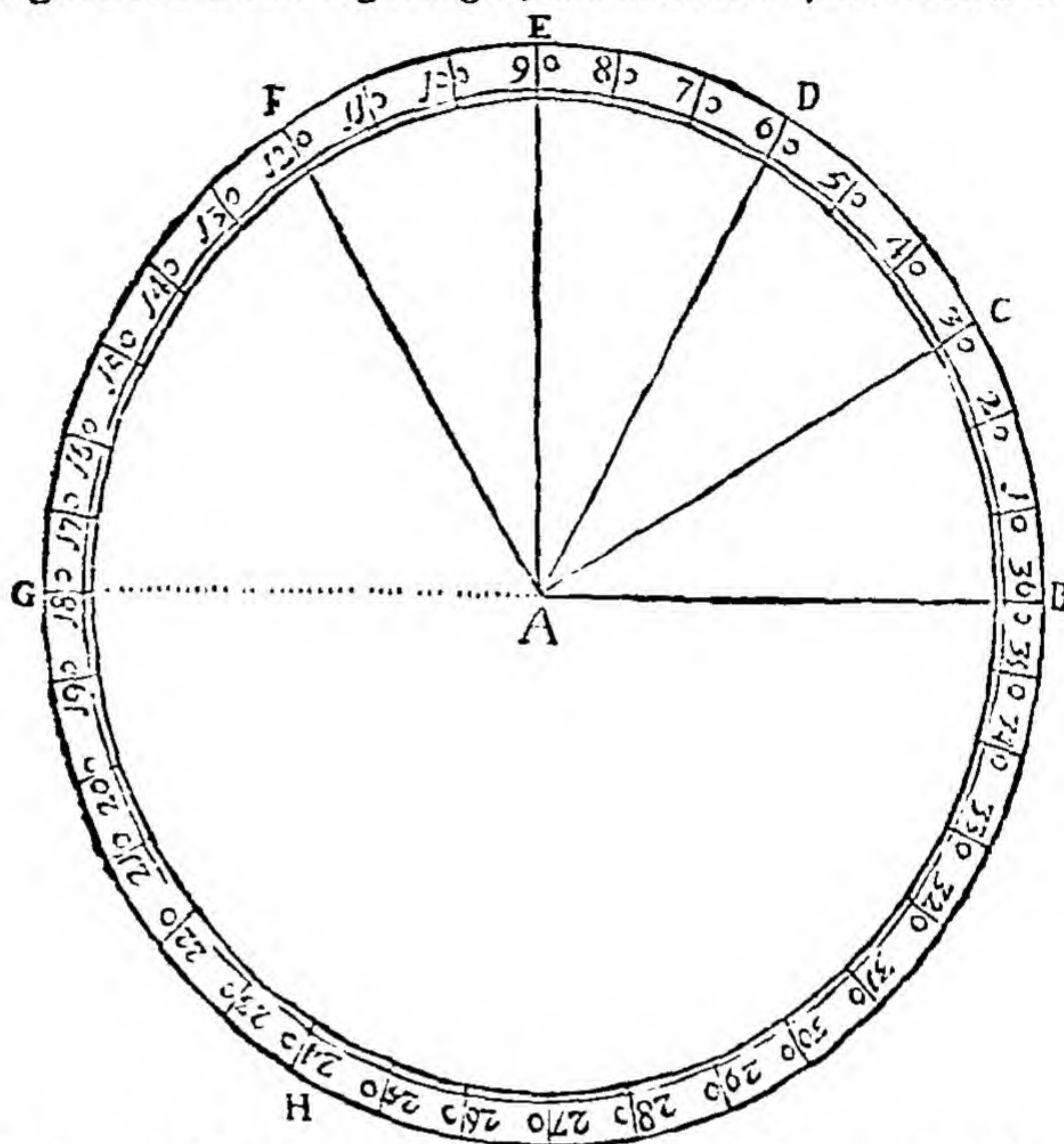
Again, by the same reason, the Obtuse angle BAF is said to be 120 Degrees, if the Arch BF be 120 of the 360 parts of the Circumference, &c.

F

Thus

Thus the Circumference of a Circle is used as a correspondent measure to *Euclid's* three sorts of strait-lined angles, Right, Obtuse, and Acute.

But to take this matter into further consideration. Of the Obtuse angle B A F the measure is the arch B E F. Whose Complement to make up the whole circumference is the arch F G H B. Which arch encompasses the remaining Space about the point A, and answers the contrary way to the adverse, or external Face of the Obtuse angle B A F; and is indeed the measure of the outward angle F A B. For the lines B A, F A, according to *Euclid's* definition, make not only the Obtuse angle B A F, which is greater than one Right angle, and measured by the arch B E F; but also make the contrary way the adverse, or external angle F A B, which is greater than two Right angles, and measured by the arch F G H B.



So the lines B A, E A make the Right angle B A E, measured by the Quadrantal arch B E, and also make the external angle E A B, which is equal to three Right angles, and measured by the arch E G H B. Again, the lines B A, C A make the Acute angle B A C, measured by the arch B C; and also make the external angle C A B, which is greater than three Right angles, and measured by the arch C E G H B. But of these kind of angles, being useless in Geometry, *Euclide* thought not fit to take any notice.

Passing therefore This over, and to return to the partition of the Circumference into 360 Degrees, we must further know, that for the more exact mensuration of an angle, every Degree is again divided into 60 *Prime Minutes*. Every such Minute into 60 *Seconds*; every Second into 60 *Thirds*, and so forth into smaller and smaller particles. For in measuring of Things, the more minute the *Rata Mensura*, the *stated Measure* is made, the more accurate will be the mensuration of the quantity of the thing measured. Therefore in *Essaies* of fine Gold, in the valuation of Pearl and precious Stones, we come to the hundredth part of a Grain, or sometimes nearer. So in Astronomical calculations of the motions of the Planets, we often make a Sexagenary subdivision of the circumference of a Circle into Fourths and Fifths, &c, for the more exact stating the Computation of their Courses, and Periods.

And here observe, That the first division of 360 Degrees into Minutes, cuts the circumference of a Circle into ————— 21600 parts.
The

The next of Seconds into ————— 1296000 parts.

The following of Thirds into ————— 7776000 parts.

And so forth continually we may proceed in subdivisions by 60, which will make an exactness more than sufficient for any common use. The Circumference of a Circle might have been divided into any other Number of parts: As our worthy Countryman M^r Henry Briggs has laid down the Form of a Decimal division, and subdivision. But 360 was thought by the ancient Mathematicians to be the most convenient number, because It admits of many several divisions precisely. As

360. 180. 120. 90. 72. 60. 45. 40. 36. 30. 24. 20.

1. 2. 3. 4. 5. 6. 8. 9. 10. 12. 15. 18.

In like manner the number 60 has these divisions.

60. 30. 20. 15. 12. 10.

1. 2. 3. 4. 5. 6.

And the choice of This number 60 before all Others, was made upon this account, that no number can be divided into 3, 4, and 5, but 60, and the Multiples of Sixty. Which makes Computation more free from Fractions.

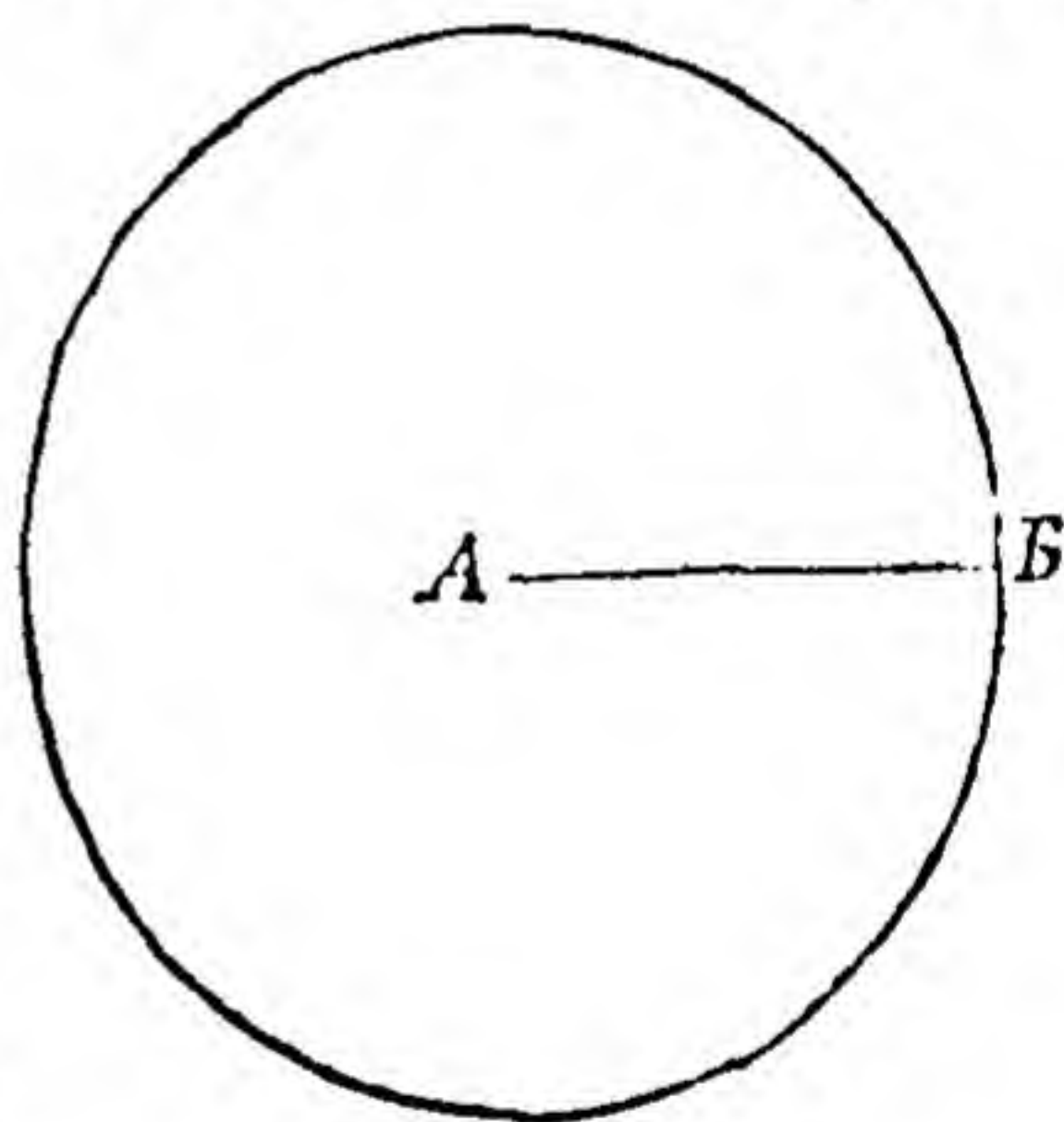
Altho' these animadversions may seem enough, or too much in this place; yet before we pass from the Subject now in hand, it will be requisite to add something more in the behalf of *Euclide*; upon this account, That some of the late Commentators, or rather Transformers of the Elements do cavil at this most accurate Definition of a Circle, for that it does not readily appear, say They, Whether there may possibly be a Figure of such conditions as *Euclide* here lays down. And therefore they would rather define a Circle, from its *Genesis*, or Structure, whereby both the existence, and nature of the Figure may be declared together, after this manner.

Another Definition of a Circle.

A Circle is a Figure described in a Plane by the revolution of a finite strait line (as *AB*) upon one of its extrem points (*A*) being fixt, till the line (*AB*) return to the place from whence it began to move.

The fixt point (*A*) is called the Center of the Circle. And The Curve Line designed by the other extrem point (*B*) of the moving Line (*AB*) is called the Circumference.

Now say They, This Definition makes it manifest, that there is *A Figure having a Point within, from whence all strait lines drawn to the extremity thereof are equal to one another*; because by the construction of a Circle laid down in this Definition, The same strait line circumsolved about the fixt point must necessarily be every where equal to it Self, and therefore all the Raies from that point shall be equal to one another.



This indeed is true, and most evident. But yet it will appear upon better consideration to be very inartificially put in this place instead of *Euclid's* Definition. For first, to except against *Euclid's* Definition, because it may be doubted, whether there may be a figure so qualified as he sets forth, shews, that They do not rightly apprehend the nature of Mathematical definitions. For to add something more to what has been already said concerning Mathematical Definitions, we are to understand that *In every one of these Definitions there is laid down the Notion of some Thing appertaining to Geometry, under a certain Name, and Term of Art.* But whether such a Thing may really be, is not of the present consideration. Only *Euclide* intends, that by such a Name, whensoever he uses it, we are to imagine such a Thing; Let It be, or not be. For indeed *Geometrical Definitions are only Suppositions of Things under a certain Name.* And therefore They are for the present

sent to be allowed, unless That, which is supposed, and laid down, has in the words themselves a manifest contradiction, or in the conception of the Thing an apparent impossibility. This is all that the Mathematicians mean, or require in their Definitions: And These Innovators therefore ought in the first place to have observed, that before *Euclide* makes Use of any Thing by him defined, he first makes evident the existence of the Same.

As before he makes any use of a Right angle, or of a Perpendicular Line (against which they might have had the like exception) he demonstrates in Prop. 11, and 12. El. I. their Being, and how to effect Them. So likewise concerning a Circle, and all other Figures here defined, nay, even a Strait line It self, *Euclide* hath so provided, that first an Assent shall be given to their Being, before any consideration is had of their Uses, and Properties.

Again, to examine farther this Definition. It shews indeed the natural Genesis, but not the nature of a Circle, from which the Properties and Affections of Circles can only be demonstrated. For this Genesis is no ways applicable to all those demonstrations. As in the first place it will appear, if when a Circle is named we put this Definition instead of the *Definitum* a Circle (for the Definition and *Definitum* are always convertible) and then let be examined what Properties of a circle, or what Proposition in the Third Element (where the Affections of circles are Specially handled) can be demonstrated *from the Revolution of a strait line on a fixed point*, Which is the Primary Notion that This Definition imposes on Us. Whereas *Euclid's* Equal lines from the Center are in all those Elementary Propositions serviceable toward their demonstrations. I much wonder therefore at *Borellus*, an excellent Geometrician, to allow in his *Euclides Restitutus* of This Definition, and change *Euclid's* into an Axiom. Which he was forced to do for the constant use thereof in all demonstrations, and Wholly to lay aside his own new framed, and unapplicable Definition. This can be only said for it, that from the Genesis of a circle it doth manifestly follow that all strait lines from the Center are equal to one another. But now to make This a Secondary notion derived from the Genesis of a Circle, and to put the Genesis for the Primary standing Definition, which can never be used as a Definition, is very absurd, and an intolerable Oversight in a Geometrician.

DEFINITION XVII.

A Diameter of a Circle is a strait Line drawn through the Center, and terminated both ways by the Circumference of the Circle.

Which also cuts the Circle into halves.

ANNOTATIONS.

As the strait line BC drawn through the Center A, and terminated both ways by the circumference at the points B, C, is called a Diameter of the Circle.

In this Definition three conditions are laid down to determine That Line, which *Euclide* calls a Diameter of a Circle.

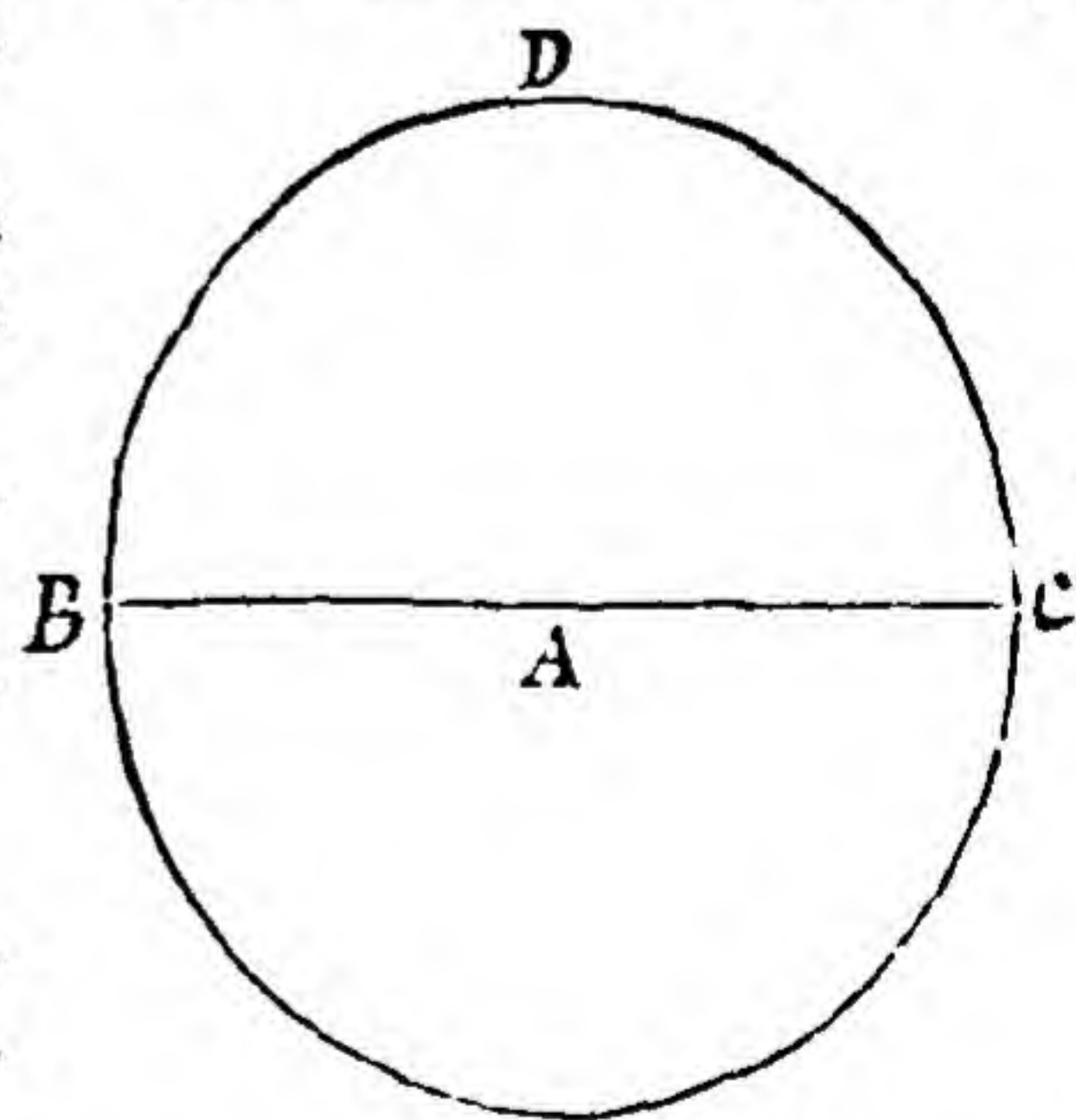
1. That a strait line be drawn through the Center.

2. That it be both ways terminated by the circumference.

3. That it cuts the Circle into halves.

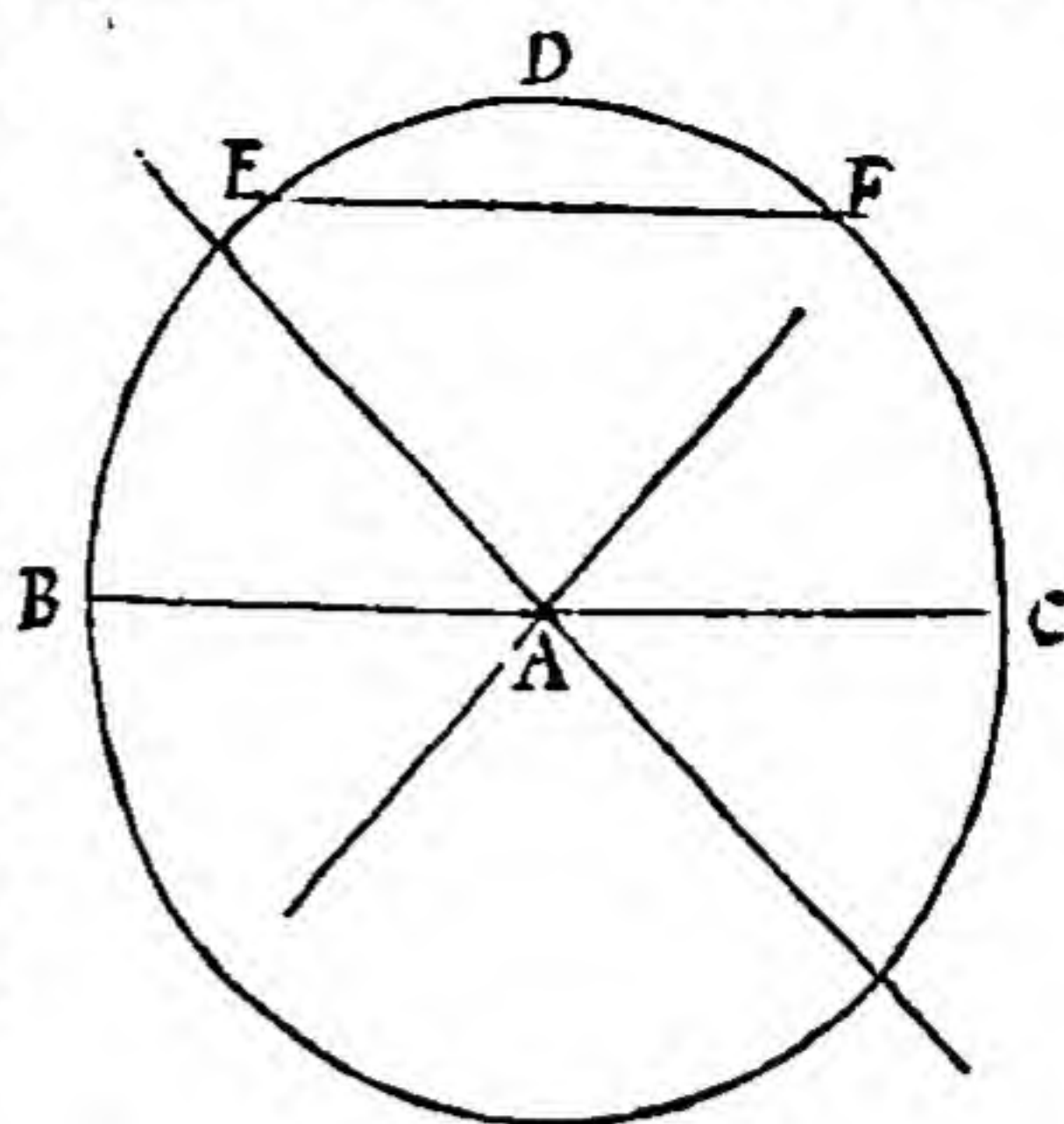
The two former conditions, Each by Themselves, are too general, and insufficient to determine the Diameter.

For as in the Figure it appears there may be finite strait lines passing through the Center, and yet not terminated by the circumference, but all variously ending within



within and without the Circle; So again, there may be infinite strait lines terminated by the circumference which yet pass not through the Center, and are called Chords or commonly SUBTENSES, in respect to the Arches, which they Subtend: as the strait line EF is called the *Subtense* of the Arch EDF.

Not any one therefore of these two conditions can by it self determine a Diameter; But both together, namely the passing through the Center, and the termination in the circumference do fully specificate this Line, and make up a compleat Definition of a Diameter of a Circle.



Note that here is added particularly, *of a Circle*, saying the Diameter of a Circle. Because there are many other figures hereafter to be considered, which have also their *peculiar Diameters*, different in several respects, from This of a Circle.

Now for the third condition, that the Diameter cuts the circle into halves. This truly is altogether needless, except there might be some other line having the two former conditions, which notwithstanding did not cut the circle into halves; And therefore it was necessary to add this third condition, for the just determination of the Diameter of a Circle.

There might as well have been added, That the Diameter is the greatest line in a Circle. For this notion does immediately concern the Diameter it self, whereas the other only declares how it affects the Circle.

The truth is, They are both demonstrable Propositions and in this place alike impertinent. But as This last is demonstrated in the 15th Proposition of the Third Element; so *Thales* has demonstrated the Other here subjoyn'd, as shall be shewn in due place. Moreover, this addition of Bisection of the Circle anticipates the following definition of a Semicircle, contrary to the exact method of the ancient Mathematicians. It is therefore certainly none of *Euclid's*, but some marginal note, how old soever it may be, that happen'd to be Transcribed into the Text.

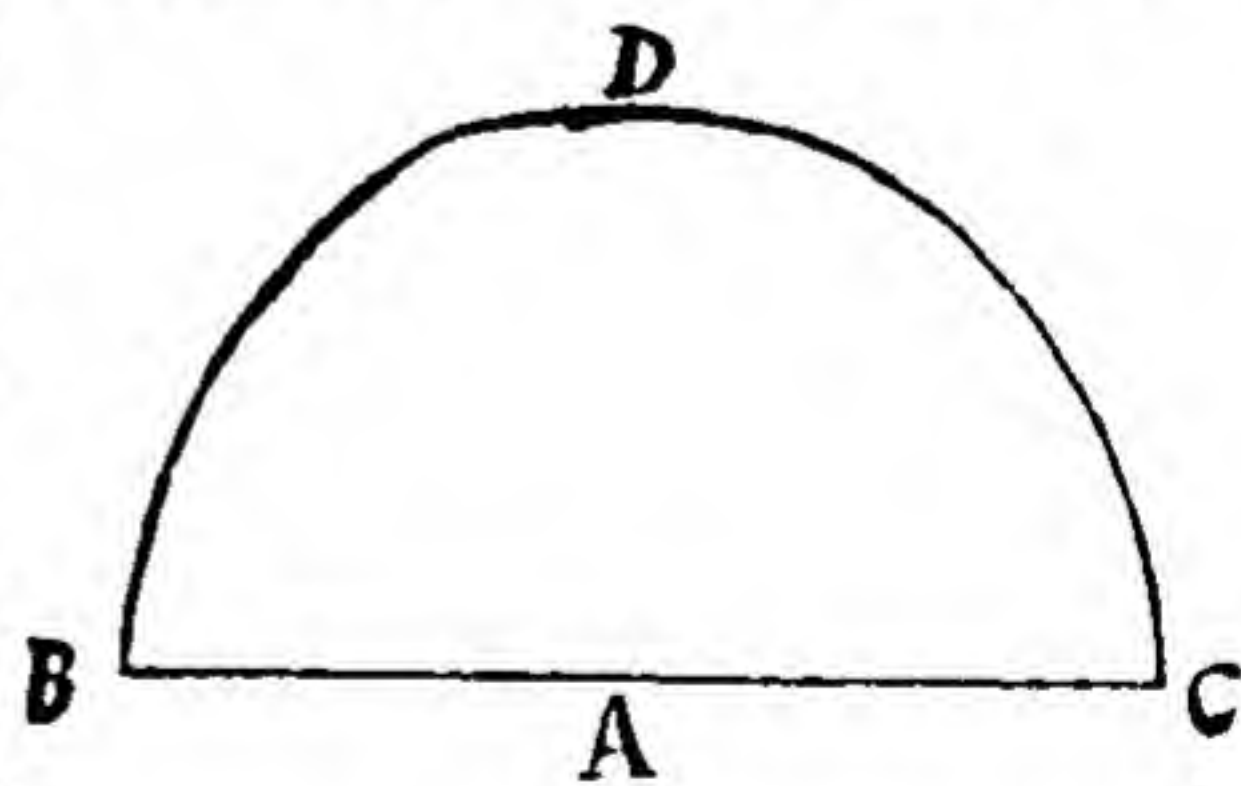
DEFINITION XVIII.

A Semicircle is a Figure comprehended by a Diameter, and That part of the Circumference of the Circle, which is intercepted by the same Diameter.

ANNOTATIONS.

As in the figure BDCB, comprehended by the circumference BDC and the Diameter BC, is named a Semicircle. This is The second of Figures, made indeed by dissimilar and incompatible lines, yet such as are the most simple in their kind, a strait line, and the circumference of a Circle.

Euclide having in the definition of a Circle stated a Center, does next by the position of that point define a Diameter: and then from the Diameter a Semicircle.



The Diameter is the second strait line to be consider'd in a Circle. For the Raies are the primary strait lines, and essential in the notion of a Circle, wherein the Diameter is not at all concern'd. And tho' it happens, that a Radius be half of a Diameter; yet It arises not from the Diameter, as the Semicircle does from the Circle: but the Radius is put absolute in it self, without any respect to the Diameter, or dependence on It: And both Radius and Diameter do immediately relate to the Circle; each distinctly, and upon very different conceptions, without any

any relation to one another. So that tho' from κύκλος a *Circle*, *Euclide* uses the word ἡμικύκλιον a *Semicircle*, yet from διάμετρος a *Diameter*, he never says ἡμιδιάμετρον, or ἡμιδιάμετρος, a *Semidiameter*. Which some modern Writers not well considering have, instead of *Radius*, or in *Euclid's* phrase, *The line from the Center*, preposterously used the word *Semidiameter*, as if It did arise from the bisection of the *Diameter*, as the primary Line; Whereas the *Radius* is before the *Diameter* in the natural conception of a circular figure.

After the definition of a *Semicircle*, *Proclus* inserts for a definition, That *The Center of the Semicircle is the same with the Center of the Circle*. Or rather he should have said, that *The Center of the Semicircumference is the same with the Center of the Circle it self*. For indeed the *Center of the Circle* may be said to be also *the Center of the whole Circumference, or Semicircumference*, or any part thereof, when It is considered meerly by it self, as a Line. But it cannot be so properly said to be the *Center of the Semicircle, or Semicircular Figure*. However, This is not to be receiv'd as a definition of *Euclid's*, but an Annotation, either of *Proclus* himself; Or else it might happen in his particular Copy of *Euclide* to be transferr'd from a marginal Note into the Text. For it is found only in his Commentaries, and not in any other Greek Manuscript, nor extant in the *Basil* Greek Edition of *Euclide*.

But moreover, that *Euclide* is sometimes thus corrupted, by transferring Marginal Notes into the Text, will manifestly appear in this very place, where in the *Basil* Greek Edition (Which as we have said has not *Proclus* his definition of the *Center of a Semicircle*) is put the 6th. Definition of the Third Element, concerning the Segment of a Circle in general, notwithstanding that the same definition is also found in its proper place among the definitions of the third Book. And we may conjecture, that *Euclide* having here defined a *Semicircle*, which is one kind of Segment of a Circle, some One noted what he had elsewhere said of other Segments of a Circle, which by the inadvertency of a Transcriber was afterwards order'd among the definitions of this First Book. For every Book of *Euclide* has its peculiar subject different from the rest, and is therefore accounted a distinct Element, having Definitions proper to the matter It treats of, which are laid down at the entrance into every Book.

Of strait-lin'd Figures.

DEFINITION XIX.

Strait-lin'd Figures are Those, which are comprehended by strait lines.

DEFINITION XX.

Trilateral, by three strait lines.

DEFINITION XXI.

Quadrilateral, by four strait lines.

DEFINITION XXII.

Multilateral are comprehended by more strait lines than four.

ANNOTATIONS.

A plain Superficies is made figurate by certain Bounds, or Terms, which inclose the same. And therefore *Euclide* hath placed the definitions of plain Figures in an order answering to the simplicity of their Terms; first defining a Circle, being

ing a Figure under One Term; next a Semicircle, a Figure under Two Terms; then, in order, Figures of Three Terms, Four Terms, &c.

We are here also to observe the property of several words used in this place, which by degrees are properly changed from one to the other, altho' the same thing be signified. The simple words originally are *γραμμαι*, Lines, and *πλαῖραι*, Sides, between which EUCLIDE makes this distinction. For speaking in general of strait-lin'd Figures, he says, *εὐθύγραμμα ἄρματα*. *Strait-lin'd Figures*. But next when he does divide, and specificate *The strait-lin'd Figures*, he says not *τρίγραμμοι* & *τετραγραμμοί*, Trigramma & Tetragramma, that is, *Trilineal* and *Quadrilineal*, or *Three lin'd* and *Four lin'd Figures*; but *τρίπλευρα* & *τετράπλευρα*, Tripleura & Tetrapleura, that is, *Trilateral* and *Quadrilateral*, or *Three sided*, and *Four sided Figures*; changing the general word *Lines* into the particular name *Sides*. And again in the *Specification of Trilateral Figures*, he gives to Them anew the name TRIANGLE in the manner following.

Of Trilateral Figures.

DEFINITION XXIII.

A *Equilateral Triangle is a Figure which hath three equal Sides.*

ANNOTATIONS.

That is, A strait-lin'd Trilateral Figure of *three equal sides*, I call an *Equilateral Triangle*.



DEFINITION XXIV.

A *Equicrural Triangle is That which hath only two Sides equal.*

ANNOTATIONS.

That is, A strait-lin'd Trilateral Figure of *only two equal sides*, I call an *Equicrural Triangle*.



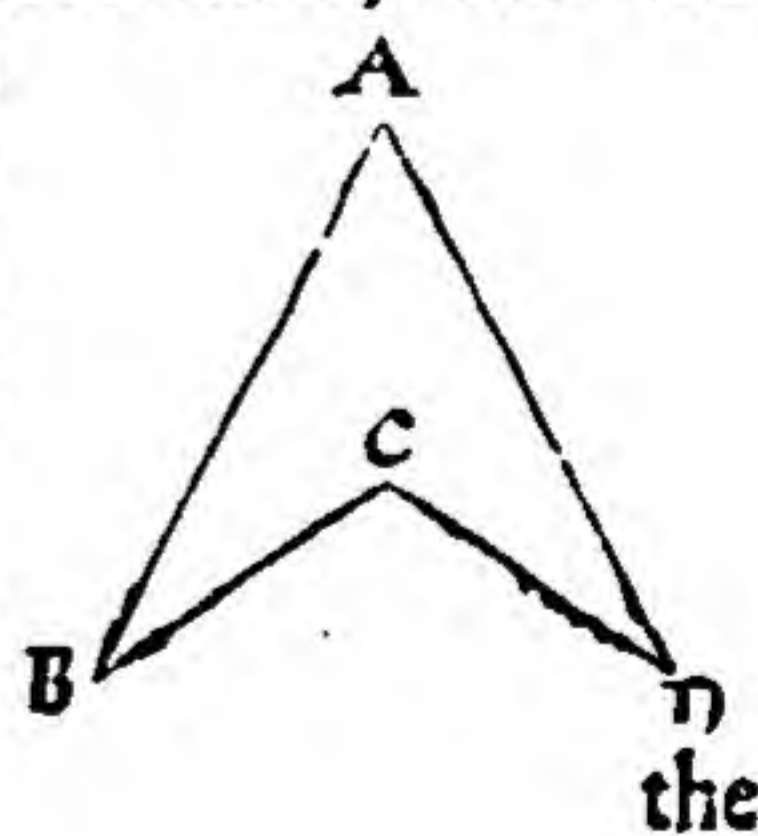
DEFINITION XXV.

A *Scalene Triangle is That which hath the three sides unequal.*

ANNOTATIONS.

That is, A strait-lin'd Trilateral Figure of *three unequal sides*, I call a *Scalene Triangle*.

After the division of strait-lin'd Figures according to the number of their sides into Trilateral, Quadrilateral, and Multilateral, *Euclide* begins with the Trilateral, being the first of all strait-lin'd Figures. And the Trilateral Figures he divides into several Species from all the possible changes that can be made, of their three sides; Which is into three kind of Triangles. For now these *particular Trilateral Figures* he calls *Triangles*. And by a Triangle is to be conceived a Figure of three sides, tho' the word implies three Angles; And only that Figure is named a Triangle, which is Trilateral. For there may be a Figure, which has only three angles yet is not Trilateral, but comprehended by four lines, or more. As the Figure A B C D comprehended by the four lines A B, B C, C D, D A, has notwithstanding only three angles at A, B, and D. For the angle B C D is not commonly taken to belong to the Figure; but to have its respect



G 2

the contrary way wholly without the figure: And this is call'd ἀκιδειδὲς, *The Arrow-headed Figure*. There may be after this manner Triangles of five sides, and six sides. In general such kind of figures are called κειλογώνια, *Hollow-angled Figures*. But concerning this double face of an angle we have already taken notice, that it is useless in Geometry.

Now in every Triangle we are to observe, and distinguish the names of *Sides*, *Leggs*, and *Base*, or *Fulciment* upon which the Other two sides are supposed to stand. In a Triangle of *three equal sides*, if *any one* of the sides be put for a *Base*, the *Other two* shall make *equal Leggs*. This therefore has three several changes of two equal Leggs: And so takes not a Name from *the equality of two Leggs*, which may be three ways variable, but from *the equality of all the three sides*, and is called an *Equilateral Triangle*.

In a Triangle of *two only Equal Sides*, the Third Side is called the *Base*; and the *Other Two* the *Equal Leggs*. And in this case the *Base*, and *Equal Leggs* are determined, being only to be made one single way: Therefore a Triangle only of two equal Sides is specially call'd ἰσσκελὲς, *Isosceles*, or *Equicrural*.

But in a Triangle of *three unequal Sides*, let Any side be put for a *Base*, the Other two shall make unequal Leggs every way: And therefore It is called σκαληνὸν τρίγωνον, *Scalenum Triangulum*, A *Lame*, or *Haulting triangle*, πάλαιθεν χωλεύει, *Undequaque claudicat*, says *Proclus*. So a rugged, and uneven way (as it is cited in *Erasmus* his *Adagies* out of *Plutarch*) is called σκαληνὴ ὁδός, *A Scambling way*.

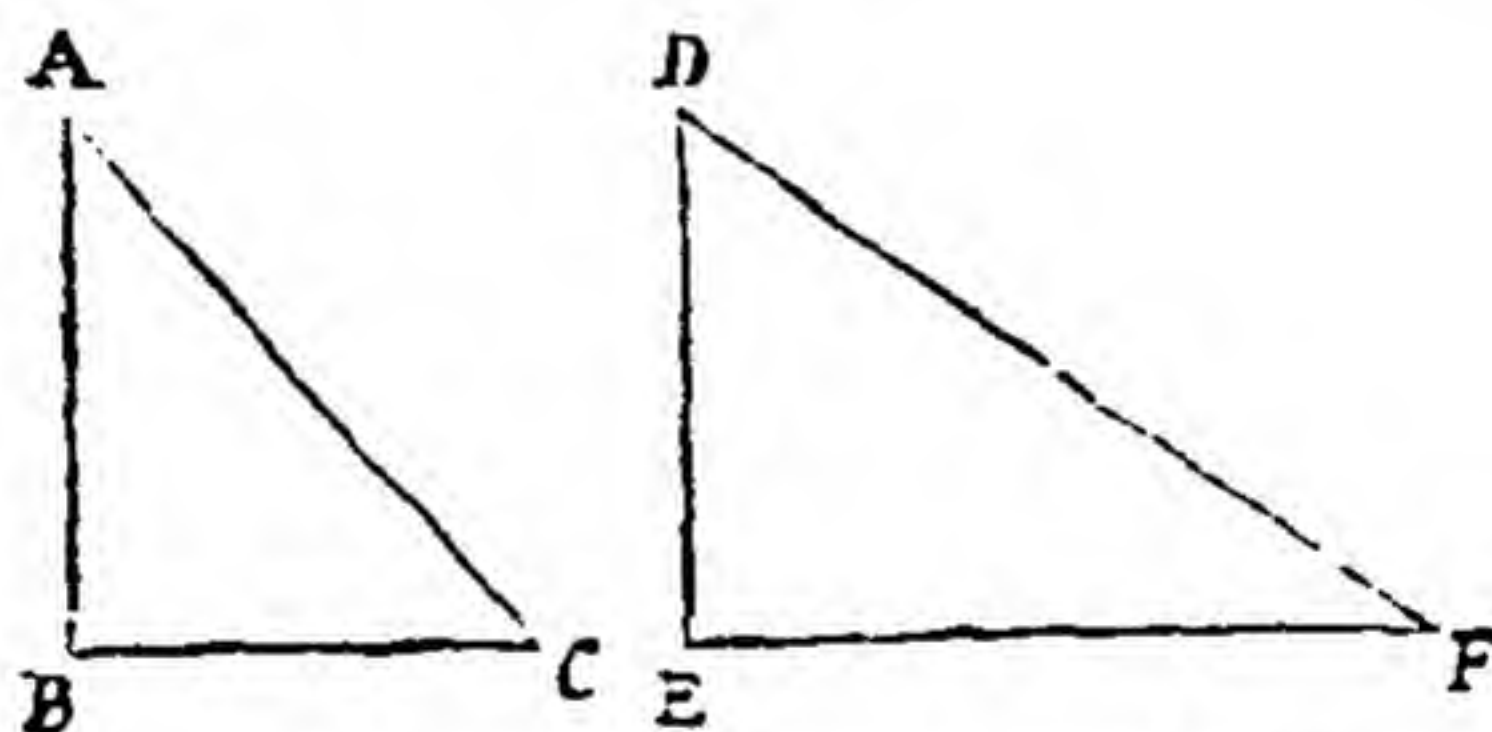
Moreover of Trilateral Figures.

Magnitude, as before hath been noted, is in It self indeterminate, and only by Termination becomes Figurate, therefore *Euclide* has first distinguished Trilateral figures from the condition of their Terms in respect of their Equality, or Inequality to one another.

And next, for the Quality of their Angles he lays down here another distinction, and denomination. As follows.

DEFINITION XXVI.

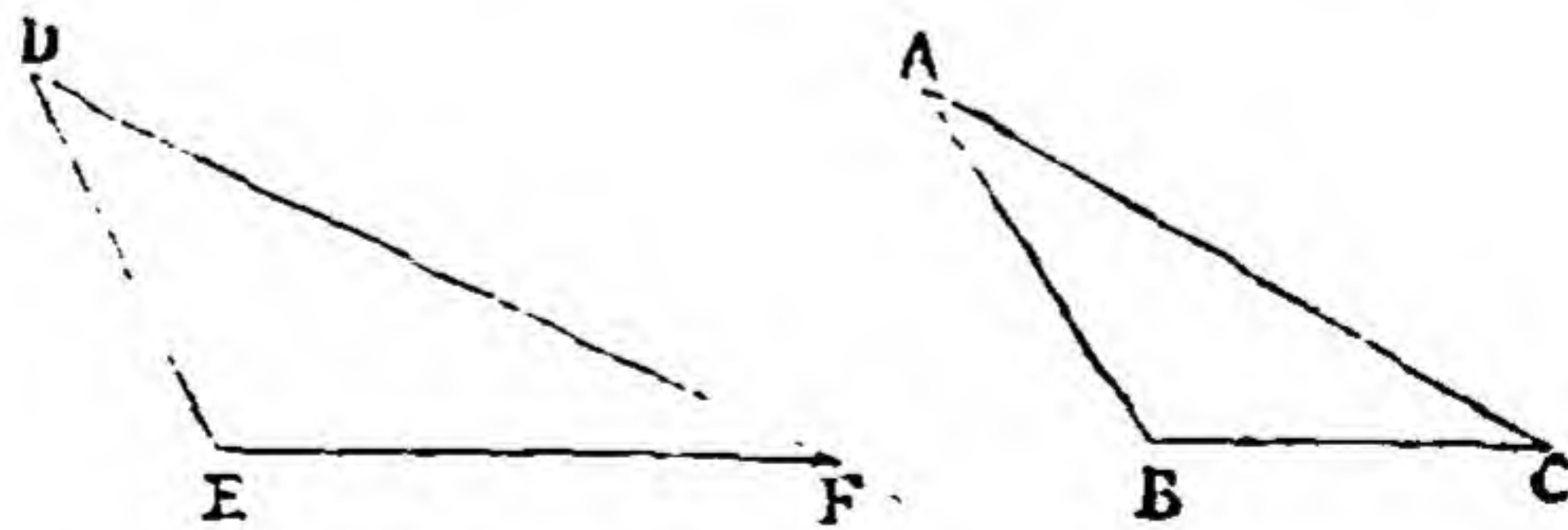
A Right angled Triangle hath a Right angle.



And such a Triangle may be Equicrural, as ABC, or Scalene, as DEF.

DEFINITION XXVII.

AN Obtuse angled Triangle hath an Obtuse angle.



And such a Triangle may be Equicrural as ABC, or Scalene as DEF.

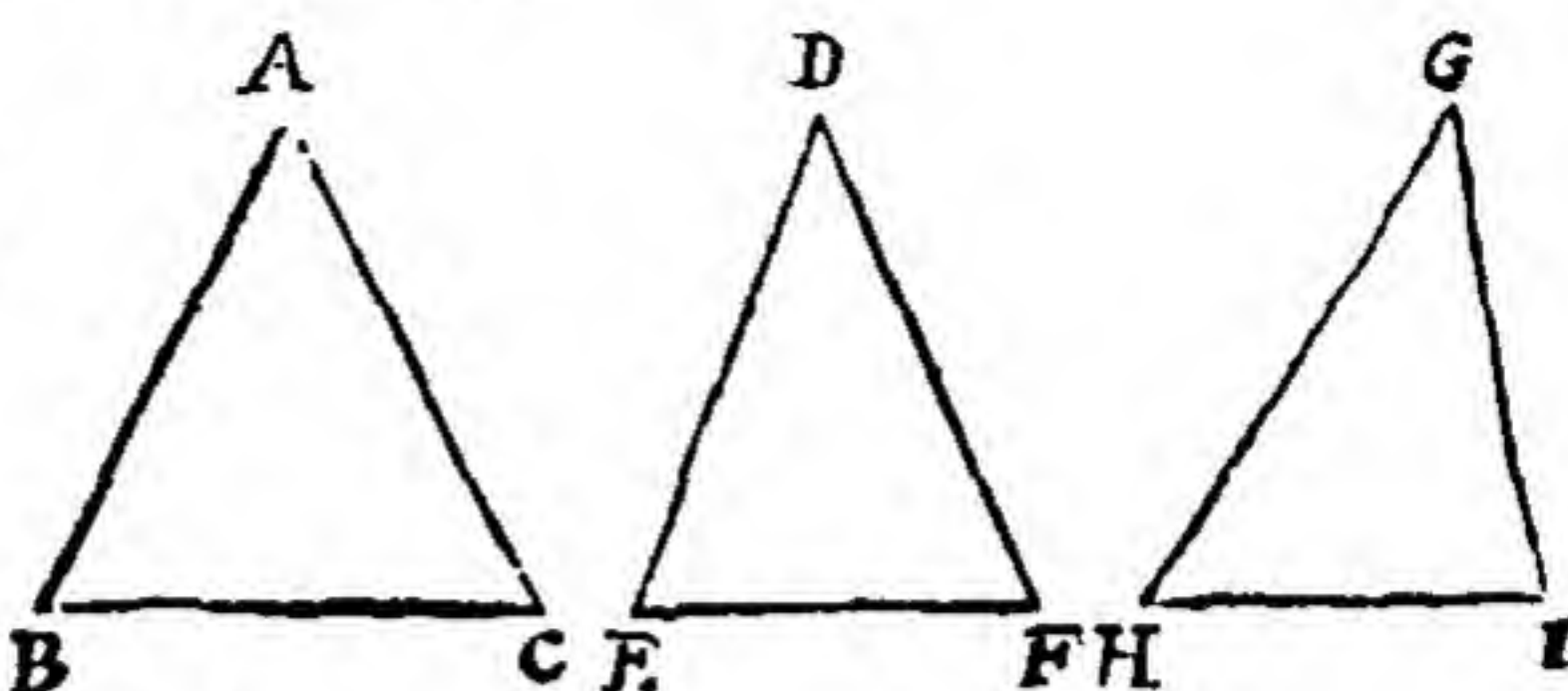
DEFI-

DEFINITION XXVIII.

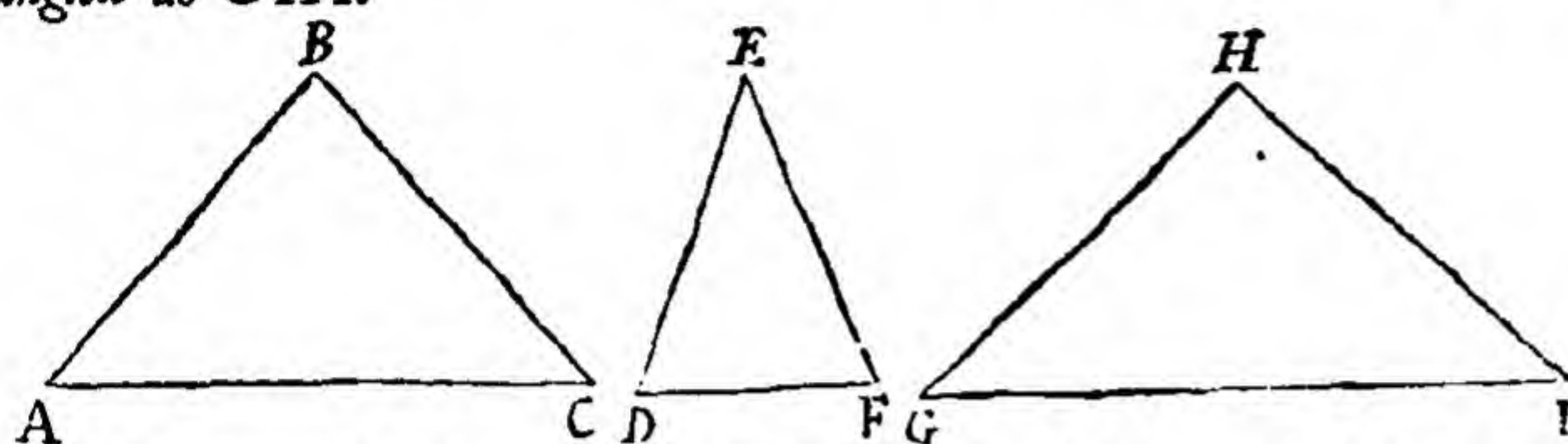
A *N Acute angled Triangle hath three Acute angles.*

ANNOTATIONS.

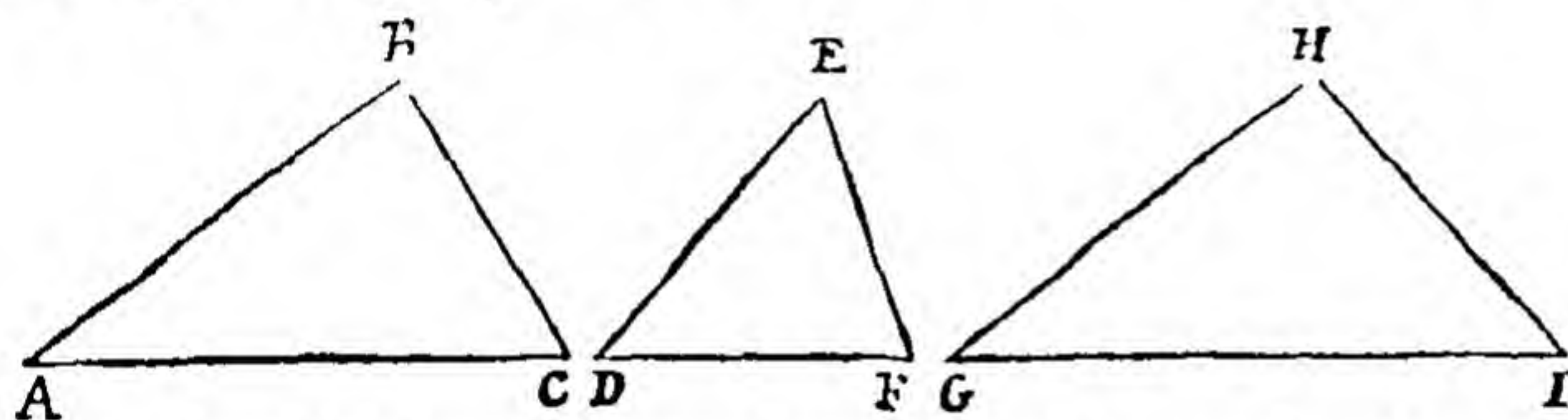
And such a Triangle may be Equilateral as ABC, or Equicrural as DEF, or Scalene as GHI.



From these divisions It appears, that there are seven sorts of Strait-lined Triangles. For the EQUILATERAL Triangle is *Singular*, and only *Acute angled*. The EQUICRURAL may be *Right angled* as ABC, or *Acute angled* as DEF, or *Obtuse angled* as GHI.



And so also may the SCALENE Triangle be *Right angled* as ABC, or *Acute angled* as DEF, or *Obtuse angled* as GHI.



To a Triangle which has *Three equal angles*, or *Two equal angles*, or *All Three unequal angles*, Euclide has given no name, as he hath to a Triangle, which has *Three equal sides*, or *Two equal sides*, or *All Three sides unequal*, because a Triangle of three equal angles is ever Equilateral, and a Triangle of two equal angles is ever Equicrural, and a Triangle of three unequal angles is ever Scalene, as will be shewn hereafter. And therefore if Triangles had received a distinction and name from the number of their Equal, or Unequal angles, as they have from the number of their Equal and Unequal sides, there had been given two Names to the same thing.

Of Quadrilateral Figures.

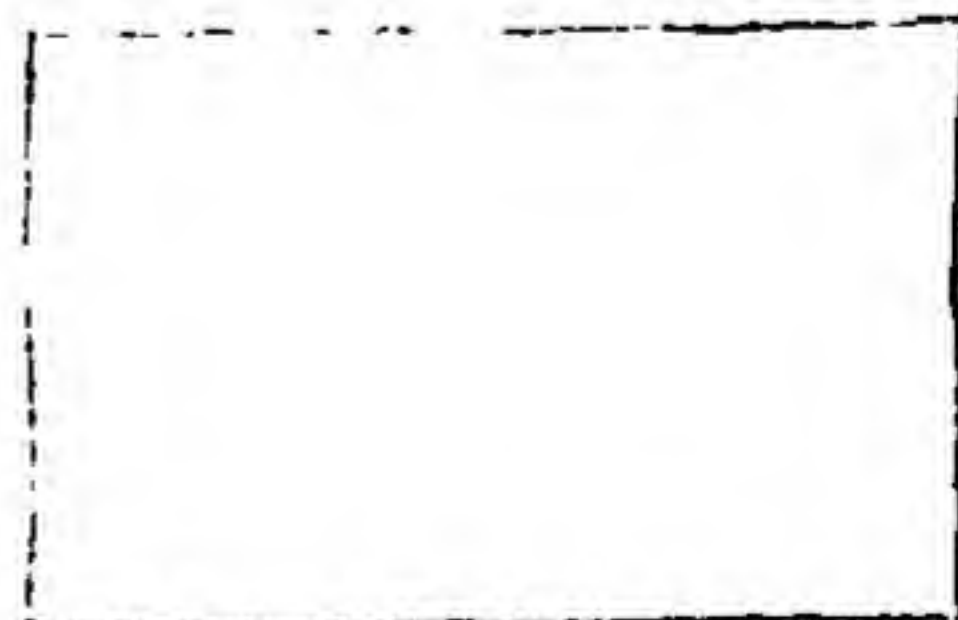
DEFINITION XXIX.

A *Square is That, which is both Equilateral, and Rectangular.*



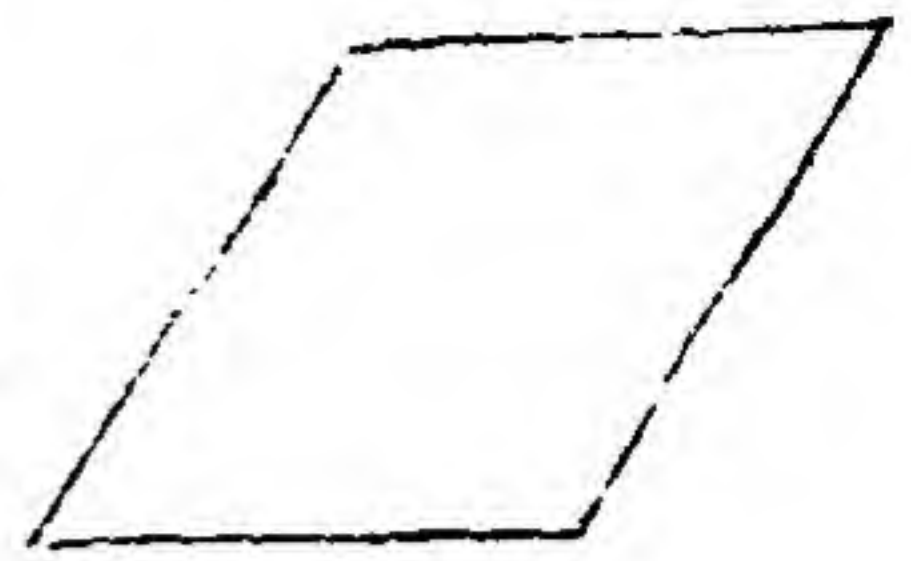
DEFINITION XXX.

A *N Oblong is That, which is Rectangular, but not Equilateral.*



DEFINITION XXXI.

A Rhombus is That, which is Equilateral, but not Rectangular.



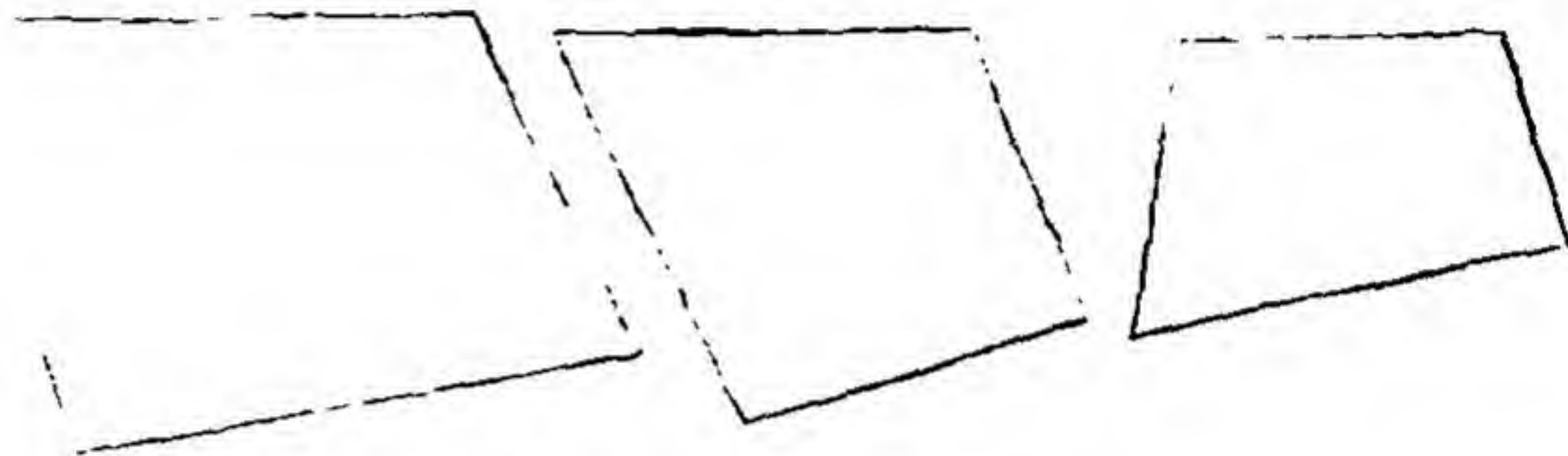
DEFINITION XXXII.

A Rhomboid is That, which having the opposite sides and angles equal to one another, is neither Equilateral, nor Rectangular.



DEFINITION XXXIII.

L Et all other Quadrilateral figures [besides These Four] be called Trapeziums, or Tablets.



ANNOTATIONS.

Of Trilateral Figures, every Species has the common name of *τρίγωνον*, a *Triangle*. And as the Species has besides a proper Epithete for distinction sake, as *Triangle Equilateral*, *Triangle Equicrural*, *Triangle Scalene*, &c.

But of these Quadrilateral Figures One only Species is called *τετραγώνον*, a *Quadrangle*, and That, from its singular equality of all its Sides; and the Rectitude of all its Angles. The other Quadrilateral Figures, tho' they be all Quadrangular; yet have they not the general name of Quadrangle with an Epithete annexed as the Triangles have. But each Species has an other distinct Appellation, as a *Quadrilateral Oblong*, *Rhombus* and *Rhomboid*. Which three with that Figure called *κτὶ ἐξῆς*, *τετραγώνον*, a *Quadrangle*, and commonly translated a *Quadrate*, or *Square*, make the four Regular Quadrilateral Figures. All the other Quadrilateral Figures have one Name in general *Trapezium*, *A Tablet*.

Parallels.

DEFINITION XXXIV.

P Arallels are strait lines, which being in the same Plane, and produced infinitely either way, do neither way meet One with the Other.



ANNOTATIONS.

The word *Parallels* is a meer Geometrical term of Art, which according to a literal exposition of *παράλληλοι*, is, *Lineæ ad se invicem posite*. Lines placed against one another. But the Geometrical Notion and Thing to be conceived by this word requires these four Conditions.

1. *Parallels* are to be strait lines.
2. They must lye in the same Plane.

For if they be in different Planes, as One in a Plane above, The Other in a Plane

Plane beneath, then the strait lines may be infinitely drawn forth both ways, and never meet, yet are They not such as *Euclide* calls Parallels.

3. They are producible infinitely both ways.
4. After this imaginary production both ways infinitely, (that is, indefinitely, or indeterminately further and further at pleasure) They are never to meet together.

Let therefore the Plane be one and the same, The production free, and both ways infinitely, *εἰς ἄπειρον*, *In infinitum*, without any restriction or qualification in the manner of the Production, and neither way let there be any Concurrence; Then the Strait Lines having these Conditions are called PARALLELS.

In Parallelism therefore the Subject is Strait Lines in the same Plane, and the Attribute of these Strait Lines is Nonconcurrence. So that upon any mention of Parallels the Geometrical Notion to be conceived under that name of Parallels is Nonconcurring Strait Lines in one and the same Plane.

This Definition, at present, only supposes such Lines to be in nature, but before these kind of Nonconcurring strait lines are brought into any use, *Euclide* demonstrates in the 27th. Proposition of this First Element, that there are such kinds of strait lines (called by him Parallels) which both ways infinitely produced shall never meet, and in the 31st. Prop. he shews how to draw them.

But now in this matter of Parallelism there are two mistakes made by most Interpreters. One in respect of the name and translation of the word *παράλληλοι*. The Other is in the Notion of the Thing, which is to be conceived under that Name and Term of Art.

For first concerning the Name, many Translators, *Latines*, and Others, take *Parallels* to be of the same signification with *Equidistant lines*, saying, *Parallels*, or *Equidistant lines*, are &c. going then forward with *Euclid's* Definition; As if the Subject of the Definition, or the Name Parallels signified Equidistant lines, and that *γραμμὰι παράλληλοι*, and *γραμμὰι ἴσων ἀφιστάμεναι*, or *ἰσοδιαστηθεῖται* were indifferently to be taken for the same; and that here *Euclide* had defined Equidistant lines by Nonconcurrence. Whereas *Euclide* neither uses the word Equidistant, nor by the word Parallels understands Equidistant lines: But only lays down the Conditions of some certain strait lines: Which Lines in a *Signal Term of Art* he calls *Parallels*.

And because the *Latines* have not any word, that answers to it, therefore the Greek name ought to be retained, saying, *Lineæ Parallele*, for *γραμμὰι παράλληλοι*; and not *Lineæ Equidistantes*. For, as we have said, by the words *γραμμὰι παράλληλοι*, is only signified *γραμμὴ πρὸς γραμμὴν*, *linea adversus lineam*, *Line against Line*, or *Line to Line*. Therefore commonly *Archimedes*, *Apollonius*, *Pappus*, sometimes *Euclide* also, say, *ἐστὶν ἡ αὖς πρὸς τὴν γ δ'*, when they mean, That *AB* is parallel to *CD*; which phrase of *ἡ αὖς πρὸς τὴν γ δ'*, cannot in any propriety of Speech among the Greeks, bear the interpretation of Equidistant lines: Neither was it so understood by Those ancient Geometricians.

Again, the second mistake is of Those, who rightly take the word Parallels for *γραμμὴ πρὸς γραμμὴν*, *LINE TO LINE*. and meerly for a Term of Art, not as a common word signifying Equidistant lines, but, changing *Euclid's* Notion, do define the Term *Parallels* by the conception of *Equidistancy*, in the place of *Euclid's* *Nonconcurrence*.

So *Posidonius*, as we find in the Commentaries of *Proclus* on this matter, thus defines Parallels.

Another Definition of Parallels.

Parallels are strait lines in one Plane, neither Inclining nor Reclining, But having all the Perpendiculars equal, which are drawn from the points of either of the lines unto the other line.

But all strait lines, which make the Perpendiculars less, shall at length meet together. For the Perpendiculars only can determine the Altitude of spaces, and the distances of Lines; wherefore the Perpendiculars being equal, the Distances of

the Lines are equal. But the Perpendiculars being made greater or less, the Distance is made greater or less, and they shall meet together that way, where the Perpendiculars are less.



Thus much *Posidonius*, who defines Parallels to be Equidistant strait lines from the equality of all their Perpendiculars.

But now we are next to consider, and demand of *Posidonius* and his Followers, to what use the Notion of Equidistancy serves in these Elements. We find nothing advanced by *Posidonius* or Others in the Doctrine of Parallelism upon this new Definition: And certainly if from Equidistancy any Thing had been made better, and more firm than from Nonconcurrency, such a kind of improvement could not have altogether perished.

But yet thus much we do acknowledge, that *Euclid's Nonconcurring strait lines are Equidistant*; and moreover, that Equidistancy is the Physical Cause of their Nonconcurrency. But yet the Equidistance of Parallels is no where apply'd to any Proposition throughout all these Elements. The only Notion of Parallels used or useful in Geometry, is *An unlimited production of strait lines both ways without concurrence*. And therefore this affection of Nonconcurrency, *Euclid* the great Master of his Art, lays down to be only represented to our Imagination upon the naming of Parallels. But whether these strait lines called Parallels be equidistant, or not, or what is the distance of Parallels, he thought not fit to consider, because those Considerations served to no further use in any of his Geometrical demonstrations. If it be said that strait lines every where equidistant shall never meet, 'tis true and obvious. But this is to define a Geometrical Term in one sense, and to use it in another: To define Parallels by Their *Equidistance*, but ever after to apply them in their Nonconcurrency. A gross and intolerable absurdity in the Mathematics, or in the Definitions of any Science.

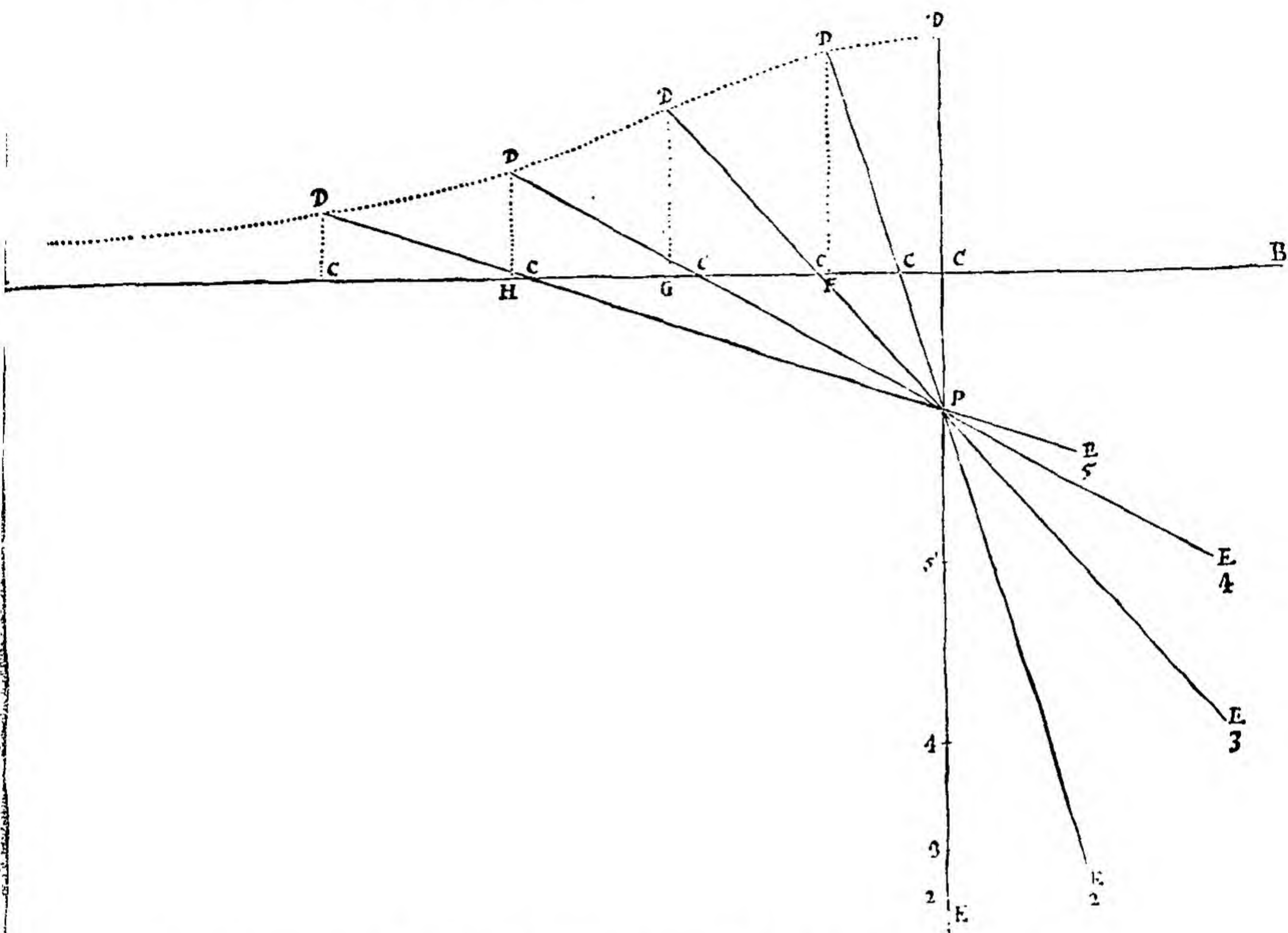
We are moreover to observe, that altho' *Equidistancing strait lines* be the proper cause of their Nonconcurrency, yet *Equidistancy* is not in general the adequate, or only cause of Nonconcurrency; so that whatsoever lines infinitely produced either way shall never meet, the same are to be always equidistant. For on the contrary, it is certain, that in one and the same Plane there may be two lines produced infinitely, which Lines shall never meet together, tho' they be not Equidistant, but do continually approach nearer and nearer to one another. As the *Conchoidal* line of *Nicomedes* describ'd by *Pappus* in *Prop. 22. lib. iv. Math. Collect.* And also by *Eutocius* in his Commentary upon *Prop. 1. lib. 11. Archimed. de Sphæra & Cylindro*, which curved Line draws nearer continually to a certain strait line, with which notwithstanding it shall never meet.

And because this Proposition seems very strange, and yet may be easily made evident to any common understanding, without a strict Mathematical demonstration; therefore to satisfy the Curious (Others may pass it over) we shall here only explain it, referring the legitimate proof therefore to *Pappus* and Others.

In a plain Superficies let a fixt point be put as *P*: And let there be a strait line without the point *P*, both ways infinite, as *AB*. Then from the point *P* suppose a strait line *PCD* to make right angles with the line *AB* at *C*; and that the same line *DCP* be directly continued infinitely from *P* towards *E*.

Now imagine the line *DCPE* to move along upon the line *AB* either way, towards *A*, or towards *B*, in such manner that the point *C* may lye always in the line *AB*, and thereby keep the line *DC*, in every place, of the same length.
And

And moreover let the same line DCPE be conceived to pass along through the point P. Now upon this supposition we are to note, first, That in the line DCPE The point C is determined to the line AB: and secondly, that The Whole line DCPE is determined to the point P.



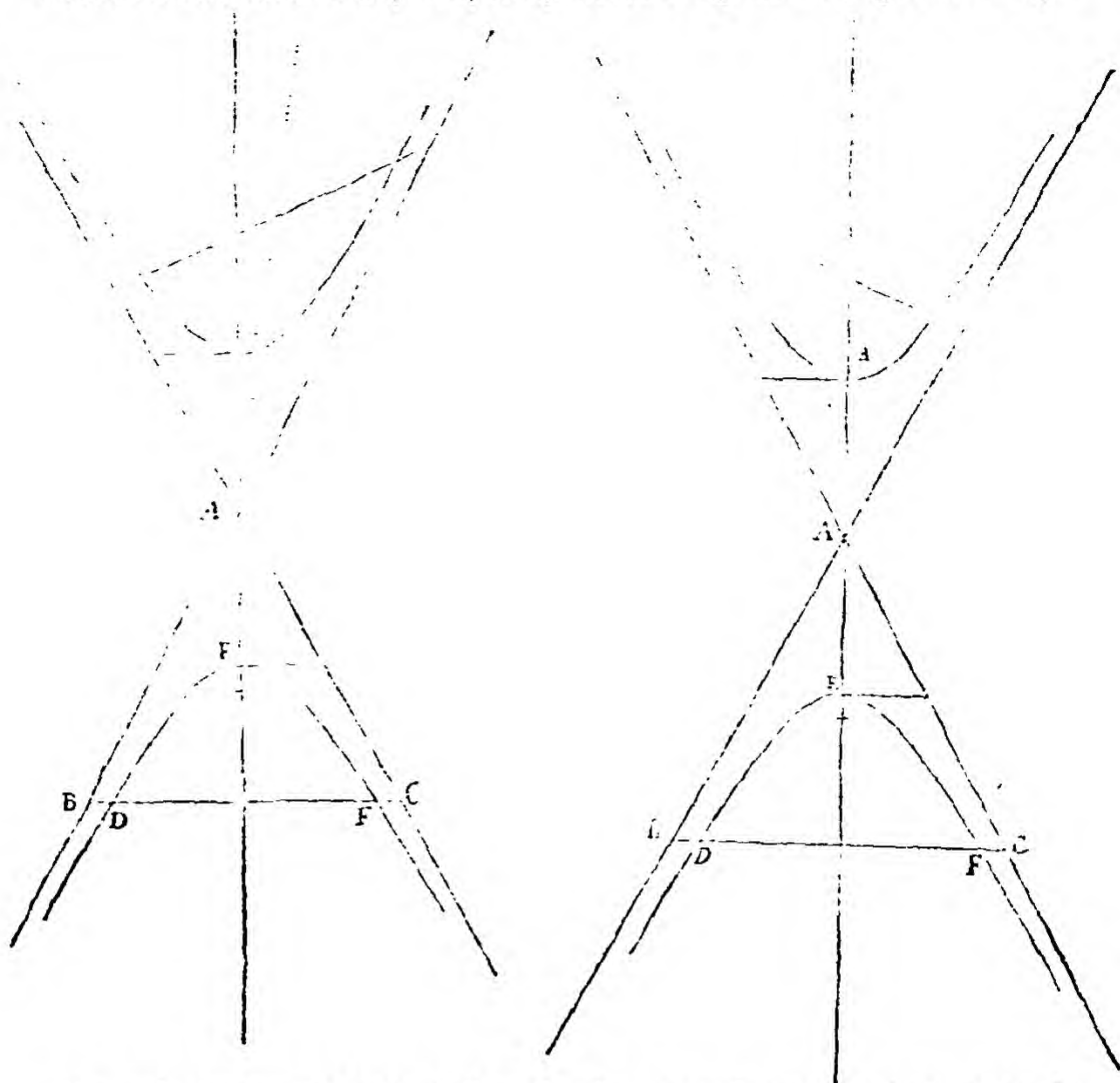
For in the motion of the line DCPE, the part DC shall always lye beyond AB. First, for that the point C is moved still forward in the line AB, so as to keep CD every where at the same length: And secondly, because the strait line DCPE can never come to be coincident with the strait line AB; for that the point P, through which the strait line DCPE always passes, is fixed at a certain distance from AB.

But now It is manifest, that in this motion of DCPE, the point D inclines continually nearer and nearer to AB, making the Perpendiculars DF, DG, DH, &c. shorter and shorter.

Wherefore the point D describing the *Conchoidal* curve line DDD &c. shall never bring that line to meet with AB, tho' it draws continually nearer and nearer to AB.

The Point P may be called the Pole. AB the Normal Line. DCPE the Arrow. The Point D the Arrow-head, which describes the Conchoidal line. The Point C the Button which holds the line CD at the same length. And under these names this matter may be fitly explained, and discoursed of: so as to be easily understood and acknowledged for a certain truth: Which indeed is founded upon the subtilty of Magnitude, being in its nature infinitely divisible. Whereof among many Others this Conchoidal line is a demonstrative Argument.

Such also are the *Asymptotes* of an Hyperbola; As for example, let the curve line DEF be an Hyperbola. There is a certain point, as A, from which may be drawn certain strait lines, AB, AC, which being infinitely produced toge-



ther with the Curve DEF, shall continually approach nearer and nearer to the same; yet shall They never meet one with the other: As *Apollonius* demonstrates in *Prop. 1. Lib. II. Of the Conic Elements*. And then in *Prop. 4.* he shows moreover how within any two strait lines making an angle an Hyperbola may be stated, to which the same lines shall be *Asymptotes*. These strait lines are called *Asymptotes*, that is, lines Noncoincident from this property, that altho' they come nearer and nearer infinitely to the Hyperbolical curve line, yet shall they never meet with it. A most true and wonderful Mystery in Geometry.

And thus much for the Explication of the Definition of Parallels, and of all the Other Definitions of the First Element of Geometry.

Of Mathematical Propositions

Demonstrable and Indemonstrable.

In the Mathematical Sciences are used two kind of Propositions called *Problems*, and *Theorems*.

A Problem is a Practical Proposition, in which Something is proposed to be done.

As, To find the Center of a Given Circle.

This Proposition is called a Problem: And it is the first Proposition of the Third Element.

Element. Where, according to the Question proposed, First the Center of any circle is by a Geometrical Practice certainly found, and after that, the Problem is demonstratively proved to be Done, or Effected, namely, that the Center of the Given Circle is found.

A Theorem is a speculative Proposition, in which Something is pronounced to be True.

As, the Diameter of a Circle is the greatest of all strait lines in the same Circle. This Proposition is called a Theorem : And It is the 15th. Proposition of the Third Element ; and there demonstrated to be True.

For in the first place we are to be informed, that whatsoever in these Geometrical Elements is proposed, whether in form of a Problem, or a Theorem, the same is either undeniably demonstrated ; Or it is at the First assented to from its own self-evidence without any further proof. For in all humane Reasonings every argumentation must be grounded upon some Thing, which is in it self Indemonstrable, that is to say, is incapable to be made more manifest to us from any other thing, than it is evident of it self to every common understanding. For if there were not a power of self-evidence in some things, which force upon Us an immediate Assent, no rational discourse, nor any demonstration could ever be framed, or have an unquestionable Beginning.

And therefore in this place there are premised such general Principles, upon which, and the like, the Mathematician builds his Demonstrations.

Of Mathematical Principles.

Of Principles in the Mathematics, some are Problems, some Theorems : like as the Demonstrable Propositions are.

The Problematical, or Practical Principles are called *Αιτήματα*, Petitions, or Postulates. Because in these Propositions some Things are required, or postulated to be Effected, or done, without any proof of their Being, or Construction ; for that their Being, and Construction is most simple, and manifest. *As*

To draw a strait Line from point to point.

The Theoretical, or Speculative Principles are called *Κοινὰ ἔννοιαι*, Common Notions as being obvious Conceptions, and generally received. For when Men from the particular Experience of their Senses, have naturally an agreement in their perceptions, and use of Things, they are then by one, and the same common, and innate Reason, alike enabled to deduce from those sensations the same Universal Propositions ; which therefore, whensoever proposed to Others, and the words understood, are presently without hesitation Assented to. *As that*

The Whole is greater than its Part.

Such kind of Propositions the Latine Philosophers call Maxims, the Greek *Ἀξιώματα*, *Axioms*, or *Dignities*, for that They are sentences of Worth, so Signal, and so Dignified, as to carry their own Authority, and Credit along with themselves, whereby to force an Universal Assent.

Now *Euclide* thus begins the Principles of Geometry.

The

The Practical Principles Postulates, or Petitions.

POSTULATE I.

L *Et it be granted, From any point unto any point to draw a Strait Line.*

ANNOTATIONS.

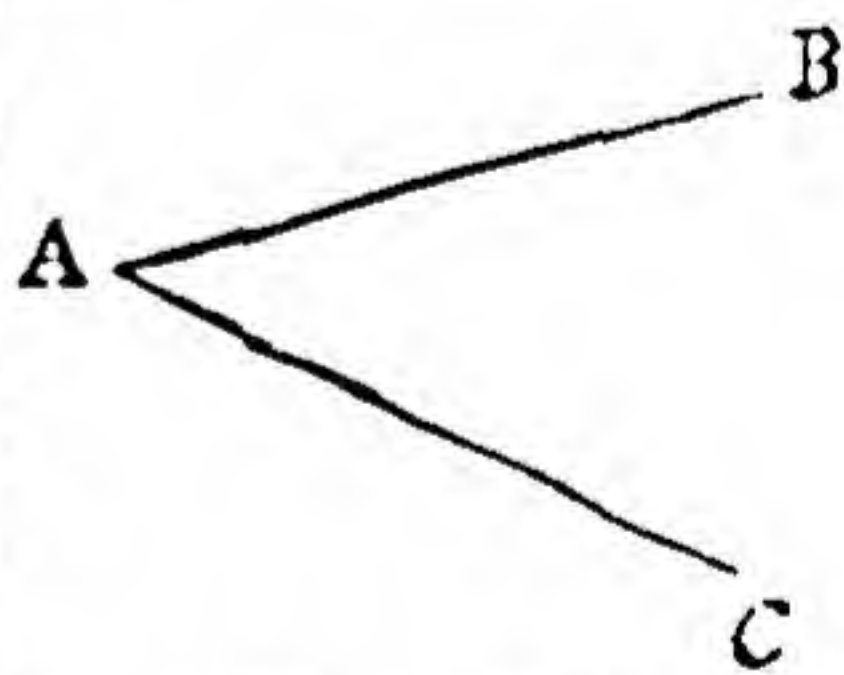
Euclide here first tacitely presumes the putting of a point any where: or jointly with the drawing of a strait line he does also postulate the putting of a point, or points at pleasure. And moreover because a point put to a point is still but a point; therefore if two points be distinctly put, as in this Postulate, then it is a point Here, and a point There; so that some kind of Length, or Space must be conceived to lye between Here and There, when *Euclide* says, From a point to a point.

Now in the 4th Definition *Euclide* tells us, what kind of Length he means by the name of a strait line: And in this Postulate he requires that such a Length, as he calls a strait line, may be put, and join any two points together. This indeed is *Aequum Postulatum*, a very reasonable Request, and as justly to be granted, as a point Here, and a point There. For between Here, and There, tho' we may make infinite deviations, and By paths; yet we naturally conceive but One only Singular, and direct Way, and that can be nothing else, but what *Euclide* calls a strait line.

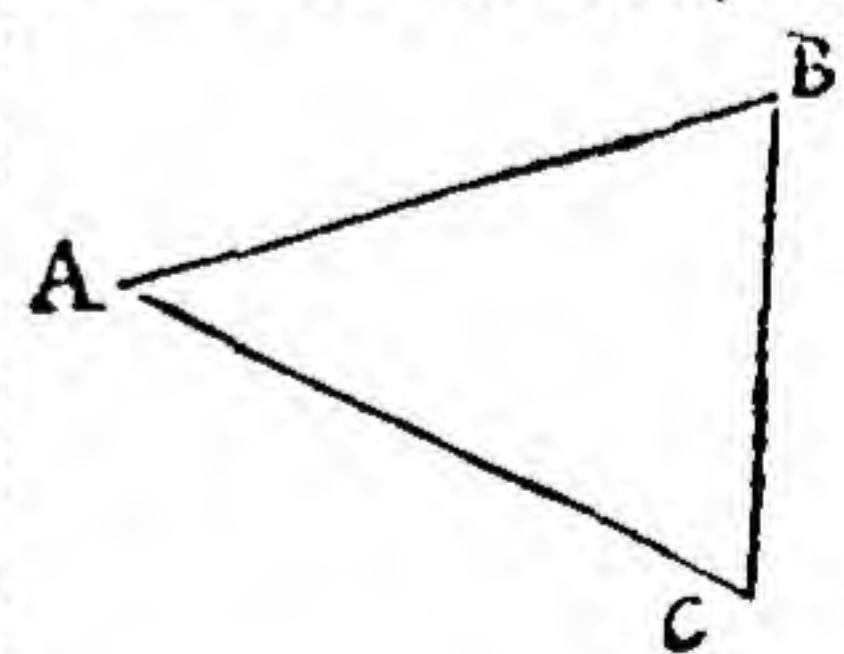
Moreover we are to know, that *Euclide* only means a Mental Duction, or Position of that strait line between any two points; not a draught of the hand from point to point by the help of a Ruler: Which does but imperfectly imitate that Geometrical exactness, which we conceive in a Mathematical strait line; For a strait line actually never was, nor ever can be drawn. But in Practical Geometry sufficient it is *Pro Accurato ponere quam proxime Accuratum*. And not only this postulated Problem; but all Geometrical Problems, whether postulated as Principles, or are from Principles to be demonstrated, are likewise all supposed to be effected in our Imagination only, without the help of our outward Senses, or of a Manual operation, or any material Instrument. Yet the Ruler, and Compasses have always been allowed to a Geometrician; not because Geometry needs them, but only to assist our Understanding by the Mechanical construction of a sensible Figure, whereby we may go the easier through an Intellectual demonstration.

And further from this Postulate, it is especially to be observed, that the Existence of strait-lined Angles, and also of strait-lined Figures, as well as of strait lines themselves, do naturally follow, and is here tacitely presumed.

For if from any point a strait line may be drawn to any point, as from the point A to the point B: It is likewise as evident, that again from the same point A an Other strait line AC may be drawn to an Other point, as C: and so make an Angle, as BAC. And therefore it had been frivolous to have postulated the making of an Angle in general.

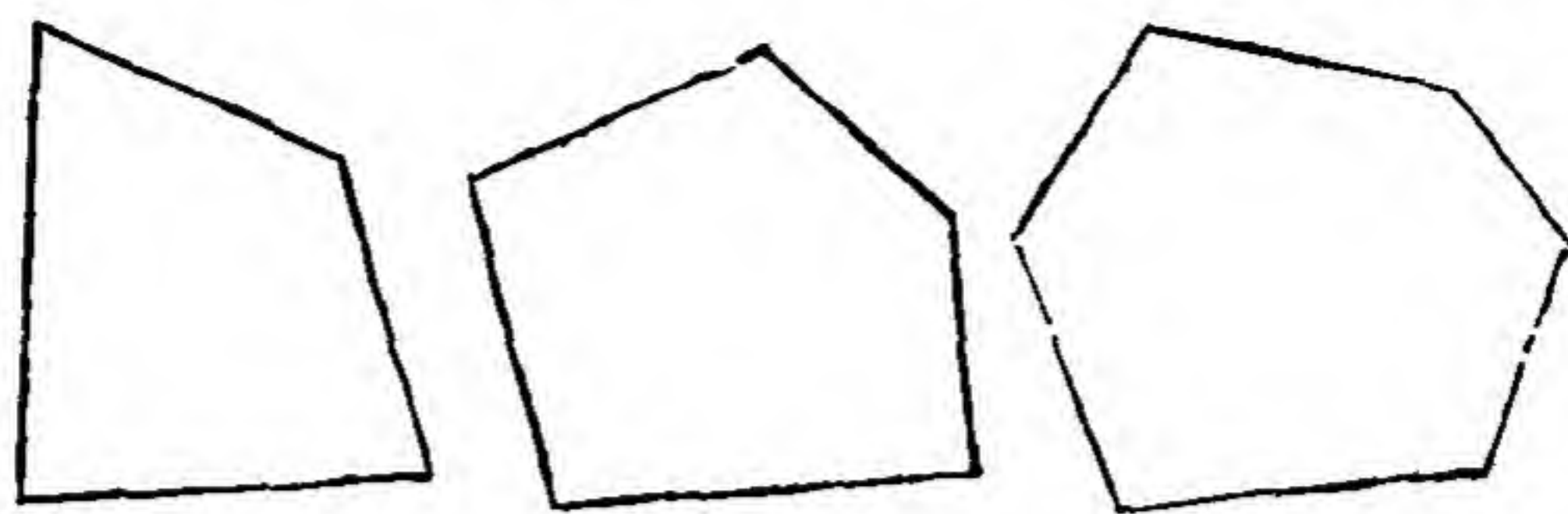


Again, if there be put three points not directly situated to one another, as A, B, C, then by this Postulate, They may be joined by three strait lines, and so there is constituted a Figure of three sides, called a Triangle. Wherefore *Euclide* neither Postulates, nor Demonstrates the construction of a Triangle in general. In like manner if there be put four points, or more at pleasure; and they be all joined by strait lines drawn from point to point, there will in common sense arise Quadrilateral, and Multilateral strait-lined Figures. So that their existence is evident



dent from this Postulate without any further demonstration of their structure.

But for the several species of strait-lined Figures defined by *Euclide*, as an Equilateral Triangle, or Square, &c. Also a Right Angle, Parallel line, &c. *Euclide* never makes use of any of them, till he hath first manifested their Being, and in particular their Construction. So accurate is the *Elementator* in his Method, as neither to be superfluous nor deficient in any matter.



POSTULATE II.

TO Continue a finite Strait Line directly onward.

ANNOTATIONS.

After that a strait line is allowed to be drawn from any point to any point; It is as much, or more evident, that a Finite strait line, that is, a strait line determined by two extream points, may be from those points conceived to be either way farther produced and continued at pleasure in the same direct course: And therefore the continuation of a strait line ought in common reason to be granted as well as a Strait line, which is the only Thing here required.

Consentaneous to the Second Postulate, This likewise might be added.

To put two strait Lines directly One to the Other.

That is, to conceive two strait Lines so situated to each other, that they may both together make one strait Line. This is frequently made use of in the Elements. But because in this case there are only two Given strait Lines imagined to be placed in a certain position towards one another; and not any other Magnitude *de novo* created, as in the foregoing Postulate there is to a Given strait Line a New One in a direct continuation to be joined; Therefore *Euclide* does not postulate This as an Other, and distinct practical Principle: but upon occasion assumes the liberty of Position in Lines Given; Sometimes of one strait Line to another directly; Sometimes of One strait Line upon another, as it may best suit to the demonstration of those Propositions, which require such an *Apparatus* of Situation toward their demonstrations.

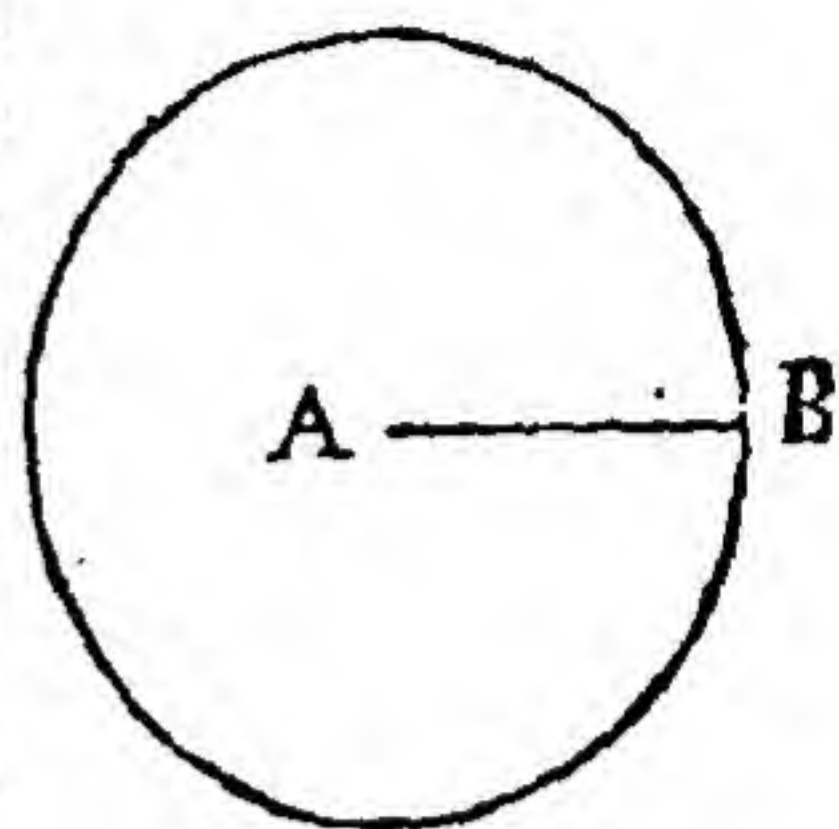
POSTULATE III.

From any Center, and to any Distance to describe a Circle.

ANNOTATIONS.

In the 5th. Definition *Euclide* means, that by a Circle should be conceived a Figure bounded on one single Term, having within it a Point, from whence all strait Lines drawn unto that Term are equal to one another, that is, having within a middle Point, which he names a Center, every way equally distant from that Term, which he names a Circumference. Now this Definition imports no more than that whensoever he mentions a Circle, we are to conceive such a kind of Figure. But here now *Euclide* farther postulates the construction of a Circle (according to this Definition) to be granted him, as a Figure of an easie construction, and very manifest of it self to be in nature. For in order to make plain the formation of a Circle, he first puts a point, which in relation to a Circle he had before, and so does here call a Center. Secondly from that point he supposes any Distance to be taken (Distance in this place is understood in the common acceptation of the word: and therefore it is not by *Euclide* defined among the

Geometrical Terms of Art) Now the natural conception of Distance is the shortest Tract between things distant from one another: So that the distance from any point to any point, is in common sense taken by a strait, and not by any crooked Line. And it is already granted in the first Postulate, that from any point to any (how near, or how far soever off) a strait line may be drawn. Put then the Center A, and from the Point A, let be drawn the strait line A B to set forth any distance. Now the Center and distance being thus laid down, the description of a Circle which is the thing required to be granted, does apparently arise. For if we conceive the line A B to move on the point A, as fixed and immoveable, till the same line return to the place, from whence it began to move; then 'tis evident, that this imaginary motion of the line A B hath described a Circle, whose Center is A: and that the point B hath delineated the bound, or circumference of the same Circle. For because the line A B in its revolution must be coincident, and the very same with all the strait lines that can be imagined to be drawn from the point A to the circumference; therefore they must all be equal to A B, and to one another, according to the definition of a Circle.



Thus therefore the structure of a Circle needs not any artifice, or ratiocination to prove its Being: but it is in it self so simple, and obvious, That it may be as justly postulated, as the Duction of a strait line from point to point. For indeed they are alike evident, there being in nature only two simple Motions, the Strait, and the Circular, and thereby are created the most simple of all Lines, and Figures, a strait Line, and a Circle.

Moreover, in relation to this intellectual construction of a Circle, we may observe that the Radius of a Circle, is by *Aristotle* in his *Mechanics* always named *ἡ γράφουσα τὴν κύκλον*, *The Line describing the Circle*. Whereas *Euclid's* phrase is *ἡ ἐκ τοῦ κέντρου*, *The Line from the Center*. *Aristotle's* Appellation respects the *Genesis*, and *Euclid's* the Definition of a Circle. And again, from this *περιφορά*, or Circumlation of the Radius, in the creation of a Circle, the curve Line described, and bounding the Circle is called *περιφέρεια*, a *Periphery*, a name very proper to the nature of the Thing. For any curve Line, which by a regulated Motion returns into it self, is significantly called a Periphery, or Circumference. But it is named by *Archimedes* in his *Cyclometries*, *ἡ ἀκμαιομένη*, as measuring the Circle in its Ambit round about: like as *ἡ διαμέτρος*, the *Diameter* is so called, for that it measures the Circle throughout at its utmost wideness. *Archimedes* therefore calls the Bound of a Circle the Perimeter, in order to the mensuration of the Area of a Circle, which was his present business; And *Euclide* the Periphery from the manner of its Generation.

Now answerable to this speculative Formation of a Circle, is the Mechanical description thereof made by help of the Compasses, and first invented by *Perdix* the Nephew of *Dadalus*, as *Ovid* reports. Which Instrument with its use he has most accurately expressed, *Metamorph. Lib. viii.*

*Perdix ex uno duo ferrea brachia nodo
Junxit, ut aequali spatio distantibus Ipsis,
Altera pars stare, pars altera duceret Orbem.*

These three easy Problems, first to draw a strait Line, Then to continue the same at pleasure, And lastly to describe a Circle, are the only practical Principles laid down by *Euclide*, to effect all his Geometrical Constructions, and all those excellent and subtil Problems, which are demonstrated in these Elements.

Lastly 'tis specially to be remarked, that *Euclide* most judiciously chose rather to make the Genesis of a Circle to be a Postulate, than with some of our modern Geometricians, an useless, unapplicable Definition: the absurdity whereof we have shewn before.

A strait Line therefore, and a Circle are the only Instruments of plain Geometry, and the only two Things, which *Euclide* Postulates to be granted him. But *Solid* Geometry requires moreover to make use of *Conic Sections*, the *Parabola*,
Hyperbola,

Hyperbola, and *Ellipsis*, for the Effect of Problems of an higher, or more compound nature, than what can be perform'd by a strait Line, and a Circle: which difference in this matter ought to be well considered and observed.

For a foul error it is in a Geometrician (whereof some of our Moderns have been too guilty) to undertake the Solution of such kind of Problems (as the Duplication of a Cube, and the like) by strait Lines, and Circles, whereas they may be readily effected by the *Conic Sections*, which are as truly Mathematical, as a Circle, and have a Genesis, as purely Geometrical: Both arising from simple Motions: the Circle being created by a mental revolution of a strait Line upon a fixed point; and the *Conic Sections* from a mental Motion of a Plane cutting a Conical Superficies. As is shewn in the Conic Elements of *Apollonius*.

The Speculative Principles, Common Notions, or Axioms.

A X I O M I.

T *Hings equal to the same, are equal to one another.*

ANNOTATIONS.

As if A be equal to B, and C be equal to B; Then shall A and C be equal to one another.

Now to express this briefly in Characters, commonly named *Symbols*, or *Species* after the manner of Analysts, with which to be timely acquainted is very useful to a Geometrician;

Let the Sign of Equality be this =

Then shall the Proposition be thus signified.

If $A = B$ and $C = B$. Then shall $A = C$.

In words thus. If A be equal to B and C to B, then shall A be equal to C.

| | | |
|---|---|---|
| | | |
| A | B | C |

A X I O M II.

I *F Equals be added to Equals the Wholes are equal.*

ANNOTATIONS.

Let A be equal to C, and B to D, then shall A and B added together be equal to C and D added together.

In Symbols thus it is.

Let the Sign of Addition be this +

Then shall the Proposition be thus signified.

If $A = C$ and $B = D$, then $A + B = C + D$.

In words thus. If A be equal to C, and B to D, then A more B (that is more by B) shall be equal to C more D (that is more by D.)

| | |
|--------|--------|
| | |
| A
B | C
D |
| + | + |
| A
B | C
D |

A X I O M III.

I *F Equals be taken from Equals, the Remainders are equal.*

ANNOTATIONS.

Let A, B, be equal to C, D, and A be equal to C: then A taken from A, B, and C taken from C, D, shall leave the remainders B, D, equal to one another.

To express this in Symbols,

Let the Sign of Subtraction be this —

Then shall the Proposition be thus signified.

If $A + B = C + D$, and $A = C$, then $A + B - A = C + D - C$;

That is, $B = D$.

| | |
|--------|--------|
| | |
| A
B | C
D |
| — | — |
| A
B | C
D |

THE FIRST ELEMENT

In words thus. If A more B be equal to C more D. And A be equal to C, then A more B less A (or less by A) shall be equal to C more D, less C (or less by C) That is, B shall be equal to D.

A X I O M IV.

If Equals be added to Unequals, the Wholes are unequal.

A N N O T A T I O N S.

Let B, D be unequal to one another, B the greater D the lesser; and let A be equal to C. Then shall A added to B be unequal to C added to D. So that A, B, together is greater then C, D, together, the inequality being still the same as before it was between B and D.

To express this in Symbols,

Let the Sign of Greater be this $>$.

Let the Sign of Lesser be this $<$.

Then shall the Proposition be thus signified.

If $B > D$, and $A = C$, then $B + A > D + C$.

In words thus. If B be greater than D, and A be equal to C, then shall B more A be greater than D more C.

| | |
|---|---|
| 7 | 5 |
| B | D |
| 3 | 3 |
| A | C |

A X I O M V.

If Equals be taken from Unequals, the Remainders are unequal.

A N N O T A T I O N S.

Let A, B; C, D, be unequal to one another, A, B, the Greater, C, D, the Lesser, and let A be equal to C. Then A taken from A, B, and C taken from C, D, shall leave the Remainders B, D, unequal to one another; B the Greater, D the Lesser, the inequality remaining the same that it was at first.

In Symbols thus it is.

If $A, B > C, D$, and $A = C$, then $A, B - A > C, D - C$. That is, $B > D$.

In words thus. If A, B, be Greater than C, D, and A be equal to C. Then A, B, Less A shall be greater than C, D, less C. That is, B shall be greater than D.

| | |
|---|---|
| 7 | 5 |
| B | D |
| 3 | 3 |
| A | C |

In these four last Axioms 'tis naturally evident, that *Equality* and *Inequality* are not changeable by Addition, or Subtraction of Equals. So that after such Additions, or Subtractions, the things are as at first Equal, or Unequal.

These are the general Maxims of this nature, which were thought fit by *Euclide* to be laid down in Form. Altho' there be used hereafter some other Propositions of the very same kind: Principles indeed as evident and as necessary as the foregoing. But because they are not Primary Notions, but only manifest Confectaries from these here now mentioned, or else that they are not of so general an use; *Euclide* at the present passes those over, and only assumes them upon occasion, as the matter in hand requires. Which order is less troublesome to Beginners, and therefore ought rather to be followed, than that of *Clavius*, and some Others after him, who have gathered these kind of Principles out of several places in *Euclide*, and do usually pack them all together; without a just consideration had of Principles *Primitive*, or *Derivative*, more, or less *General*.

A X I O M VI.

Things which are Double of the same are equal to one another.

A X I O M

A X I O M VII.

T*Hings which are Halves of the Same, are equal to one another.*

A N N O T A T I O N S.

In these two Axioms *Euclide* instances only in the *Duple*, and in the *Half*. For in laying down these common notions, he judged it sufficient to put the Principal notion in General, leaving the Confectaries, which do naturally follow to every ones common understanding. Also superfluous it is, and besides too trivial, to intermix with the general Principles every obvious Consequence. For in the naming Double and Half, who does not presently conceive the same evident truth in Triples, Quadruples, Quintuples, &c. And so in a Third, a Fourth, a Fifth, or any like part of the same thing, or of *Equal Things*: Which last also of *Equals*, *Euclide* could have as easily added, as his Commentators. But he would here intimate that what is said of one and the same Thing, is alike to be understood of Equal Things; for that *Identity and Equality, are to be indifferently taken, and used in Geometrical demonstrations.*

The Axioms hitherto laid down are more general, and common to several Sciences: But these which follow are purely Geometrical.

A X I O M VIII.

M*agnitudes Congruous one with the other are equal.*

A N N O T A T I O N S.

That is, If two Magnitudes be imagined to be applied One upon the Other, and after this mental application it be demonstrated, that neither does any ways exceed the other; but that the Intermedial parts of the One do agree with the Intermedial parts of the Other, and the Extrems with the Extrems, then these Magnitudes are said to be Congruous. And Geometricians do justly assume for a rational Principle, that Magnitudes being so far proved Congruous, are then to be concluded equal to one another. Thus *Euclide* is to be understood in the Speculative and true Geometrical use of this Axiom concerning Congruous Magnitudes.

But moreover, there goes along with this Speculation a very natural, and common Mechanical use of the same Axiom. For in the practice of Artisans They Mechanically fitting one Magnitude to an other do judge by their Eye, or Hand how one agrees with the other, and accordingly do determine their equality. As in the mensuration of Magnitudes by a Foot, a Cubit, a Perch, &c.

The like Mechanical Congruency is made use of in the measure of Liquids, of Grain, and the like, by Pints, Gallons, Pecks, Bushels, &c. The equality of these kind of things being judged by congruous Vessels, or Places, which do contain them. So that this Mechanical Application of Magnitude to Magnitude is a natural, and universal practice: And in common sense Congruency is a standing Rule of Equality.

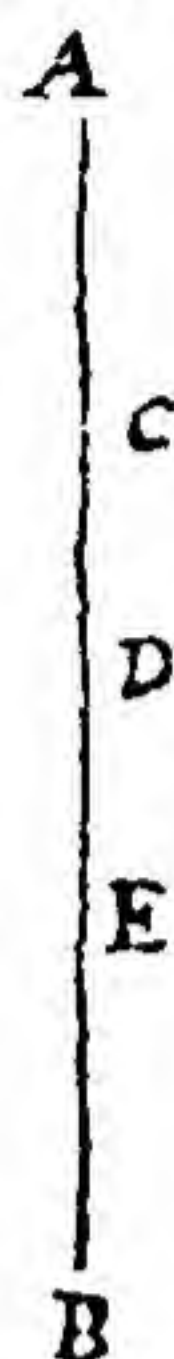
But the Geometrical Congruency in this place understood, arises only from an Intellectual application of one Magnitude to an other: And then after such an application there is made by argumentation a demonstrative proof of their Congruency, both in their Extrems, and Intermedial parts, without any judgement taken from our outward Senses. So that their Congruency being thus rationally demonstrated, we do then from this natural and common notion of Congruency, conclude their Equality. As we shall find in Prop. 4th. and 8th. of the first Element: In Prop. 24th. of the third Element &c.

A X I O M IX.

The Whole is greater than its Part.

A N N O T A T I O N S.

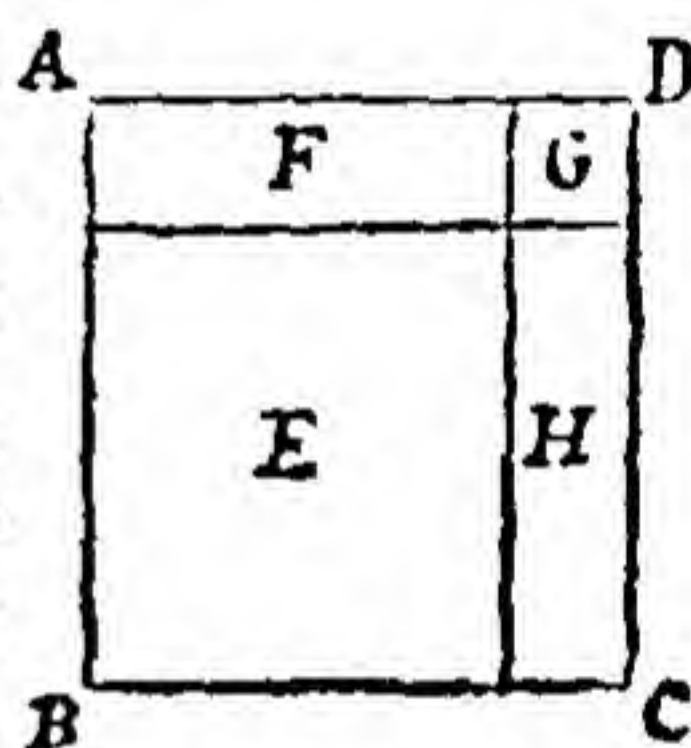
Every magnitude is in it self *Quid Unum & Continuum*, only one entire Thing: Neither to be called Great, or Little, a Whole, or a Part. But because magnitude is infinitely divisible into smaller magnitudes, therefore every magnitude, tho' in it self but one, may yet by imagination be supposed to consist of certain less magnitudes contained within the same. These are called the Parts, and That the Whole, These parts are not really separate from the whole; but are all united by common terms, which are conceived the End of one part, and the Beginning of the other. As let AB be put for any finite magnitude, whose Extrems are A, B. Now this magnitude AB is in It self but one: yet It may be distinguished into parts, as, AC, CD, DE, EB, whose intermedial limits are C, D, E; here C being the End of AC, and the Beginning of CD, so D of CD, DE, and E of DE, EB. Thus the parts are to be taken, and understood in continued magnitudes. And the same truth is alike manifest in any *Discrete, Collective, or Aggregate Totum*. As in numbers in a Peck of Corn, and the like, 'tis naturally evident that the Whole is greater than its Part.



Moreover, it is commonly added as an Axiom by the Commentators, That

The Whole is equal to all its Parts, or all the Parts are equal to the Whole.

Yet *Euclide* having no occasion to use these Propositions in these direct Terms; but only upon some particular argumentation to infer *ex Diagrammate*, from the Diagram it self, that such and such Parts all together are the Whole; As E, F, G, H, are the whole Square ABCD, It seemed not proper to place This signally for a distinct Maxim among the rest of his General, and Common Notions.



A X I O M X.

ALL Right angles are equal to one another.

A N N O T A T I O N S.

Truth, and *Rectitude* have the same property. For as one Truth cannot be more true than an other; so one strait Line cannot be more strait than an other: nor one Right angle more Right than an other. So *Martianus Capella lib. vi. de Nuptiis Philologiae & Mercurii*, says, *Angulorum natura triplex est, Nam aut Justus est, aut Angustus, aut Latus*. The Acute, and Obtuse are here called *Angustus*, & *Latus*; *Quorum uterque semper est mobilis*, says he, always changeable in their increase, or decrease; there being no Obtuse angle, but that there may be a more, or less Obtuse, nor any Acute angle, but that there may be a more, or less Acute: Only the Right angle *Justus est & semper Idem*.

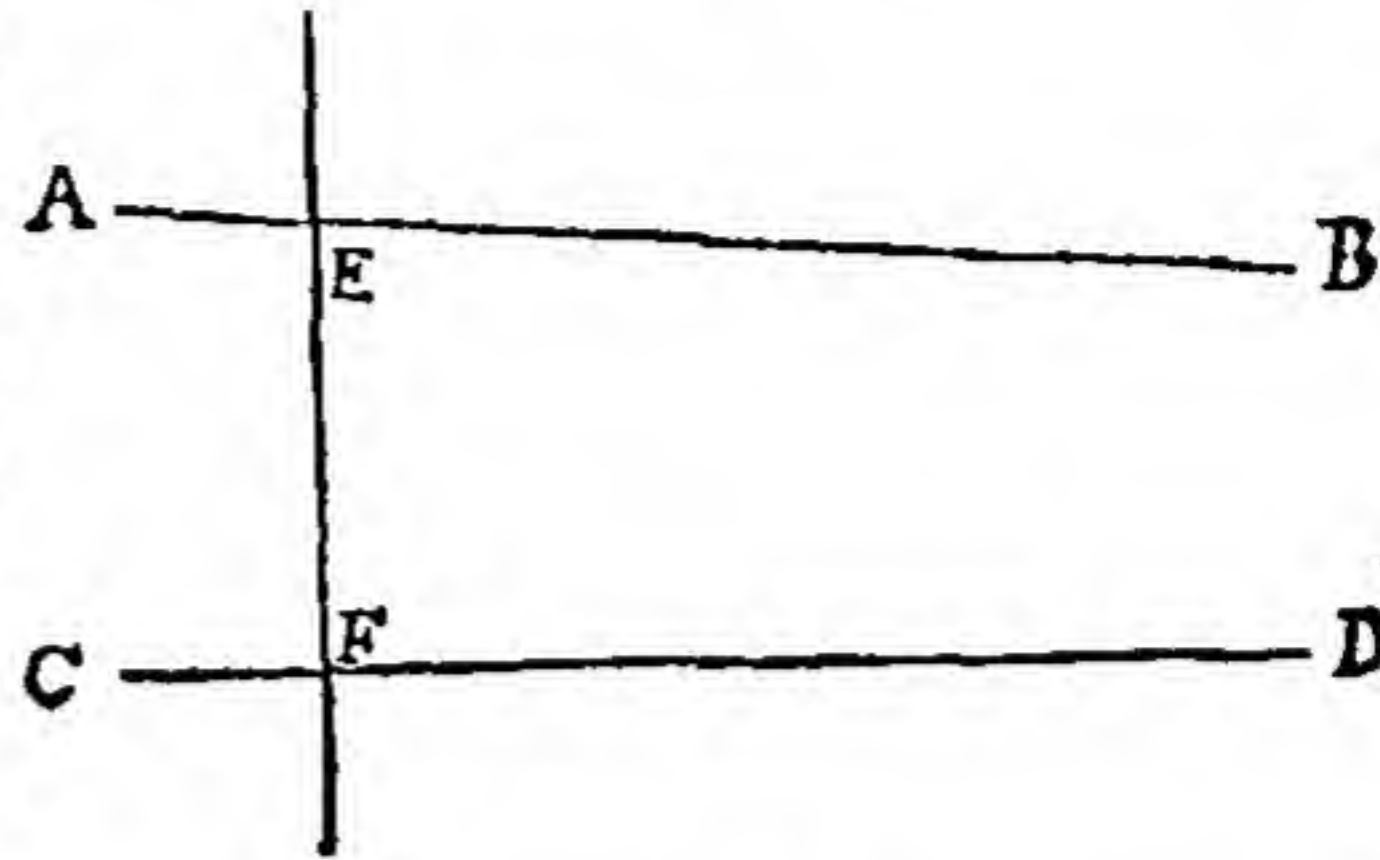
A X I O M XI.

IF upon two strait lines a strait line falling, does make the internal angles on the same side less than two right angles; Those strait lines being infinitely produced, shall meet on that side where the angles are less than two Right angles.

A N-

ANNOTATIONS.

Upon the lines AB, CD , let the strait line EF fall, making the internal angles BEF, DFE , less than two Right angles: If then AB, CD , be the same way produced indeterminately, that is, onward, and onward in an undetermined free course, it is here put as a manifest notion, that the strait lines AB, CD , shall at length meet together towards the parts B, D .



Tho' this Proposition be a most certain truth, yet it hath been generally excepted against for want of the just *Evidence* of a Principle, tho' not of the *Certainty* of the Thing. But yet let us consider that here are put two strait lines AB, CD , under such conditions, which do clearly shew that They have a *Tendency*, and *Inclination* towards one another: And therefore it may be justly assumed for a common notion, That two strait lines inclining each to the other, and being that way infinitely produced, shall at length meet together. This is the substance of *Euclid's* 11th. Axiom, tho' expressed in other words more suitable to the Form of his demonstrations.

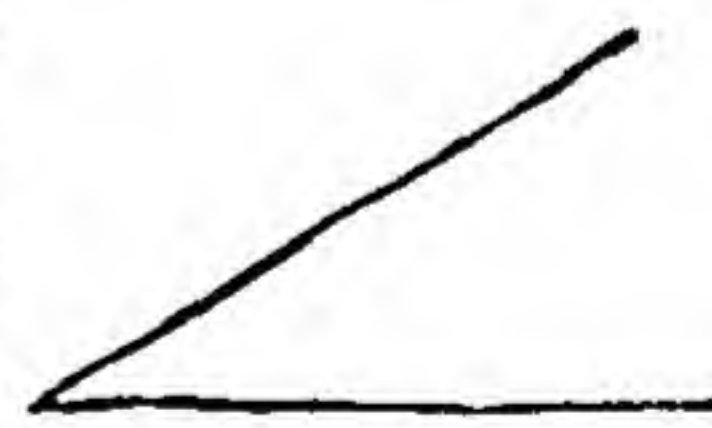
But here are made many Objections. In the first place, it has been already shewn, that there may be two Inclining lines infinitely produced, and continually approaching nearer and nearer to one another, which notwithstanding shall never meet together. A thing very strange, and at the first view hardly credible: yet afterwards certainly found to be true. Wherefore seeing that Inclination, and perpetual Approximation force not a Concurrence, it may be doubted, whether the same may not also happen in *strait lines inclining, and continually approaching towards one another*: Inſomuch at least that the Concurrence of two Inclining strait lines cannot well be admitted for an evident Principle. Besides this, there are several other Objections, of which we shall have occasion to speak hereafter. Only at present, I say that they who have endeavoured to mend this matter, have with much trouble, and disturbance of *Euclid's* excellent method taken great pains to little better purpose.

A X I O M XII.

Two strait lines do not comprehend a Space.

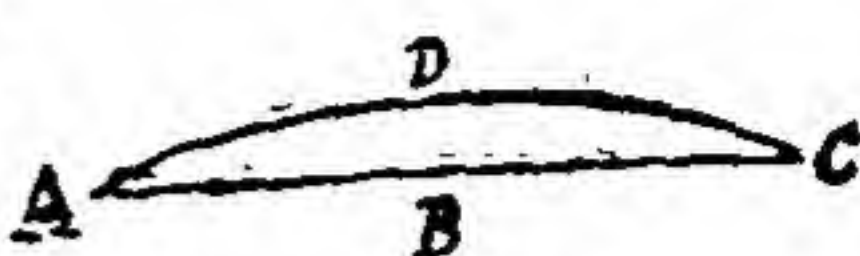
ANNOTATIONS.

If the two lines be Parallels, there is an open space both ways between them, which is neither way boundable by those strait lines, for that parallels are Nonconcurring strait lines. Again, if the two strait lines any where meet, there is made a strait-lined angle: But then the angular space is not thereby comprehended, being one way infinitely open, nor is it imaginable, that the same space can again be inclosed by a progress of Those two strait lines, which contain the angle; unless there be conceived some Flexure, or Vergency of the same strait lines towards one another, whereby they being produced at pleasure may again meet together. But this is to destroy the natural conception of Rectitude in the strait lines themselves. Therefore in common sense there must naturally intervene a third line for the inclosing of a space.



Again, for as much as every strait line does in all its parts lye Evenly to all its points [Def. 4.] therefore two distinct strait lines cannot have the same Extream points.

points. As let the strait line ABC have its extreame points A, C : then the line ADC , having the same extreams (and so both together bounding a space) shall not be likewise a strait line, for that the intermedial parts of those two lines ABC, ADC , cannot in common sense be conceived to lye alike in both the lines, Evenly to the same points A, C , but that the parts of the One of them must deviate from the Even, and direct Course, which lyes between the points A, C .



As also farther according to the *first Postulate*, a strait line drawn from point to point, from A to C , can be but One, and the same singular strait line; and is there understood so to be. Or if we suppose two strait lines applyed to one another, to have the same extreams A, C , these strait lines, *quatenus* strait, must be wholly coincident with one another, so that they cannot intercept any imaginable space.

Here *Euclide* ends the Principles.

In the Definitions (which are by Commentators commonly accounted among the Principles) was laid down the subject Matter, or the particular Things to be treated of in the first Part of these Geometrical Elements.

After the Definitions next follow the Postulates, and Axioms, that is, Principles *Practical*, and *Speculative*. These are rightly called *Principles*, as being the foundation upon which this Science builds its demonstrations, and are in the first place made use of in the doctrine of those Magnitudes, and Figures expounded in the foregoing Definitions. For to speak properly These Definitions are not Principles of common Reasoning premised for demonstration sake; but in truth they are a small part of the ample Subject of Geometry.

As in the Definitions of an Angle, of a Circle, of Trilateral, and Quadrilateral Figures, of Parallel Lines, is explained what kind of Things are to be understood by those several Names. And these Things are here laid open for an Entrance into Geometry, as being Matters most simple, and easily to be taught, and apprehended. But now in pursuit of a perfect understanding of Them, the Postulates, and Axioms serve as Natural, and general Principles of Reasoning, whereby we are enabled to demonstrate the manner of their Construction, their Properties, and Affections. The contemplation whereof is the business of this first Element of Geometry.

The Postulates, and Axioms, or common Notions, are clearly intelligible to every ones capacity, altho' some Annotations made with Examples, and Instances may be to Beginners useful for Ease, and Illustration sake. But for the Propositions themselves, there is nothing in them further, or otherwise to be expounded, than what the literal sense, and common meaning of the words import. As *Tertullian* says upon a like occasion, *Definitiones, ac sententiæ, quarum aperta est natura, non aliter Sapiunt quam Sonant.*

THE FIRST ELEMENT.

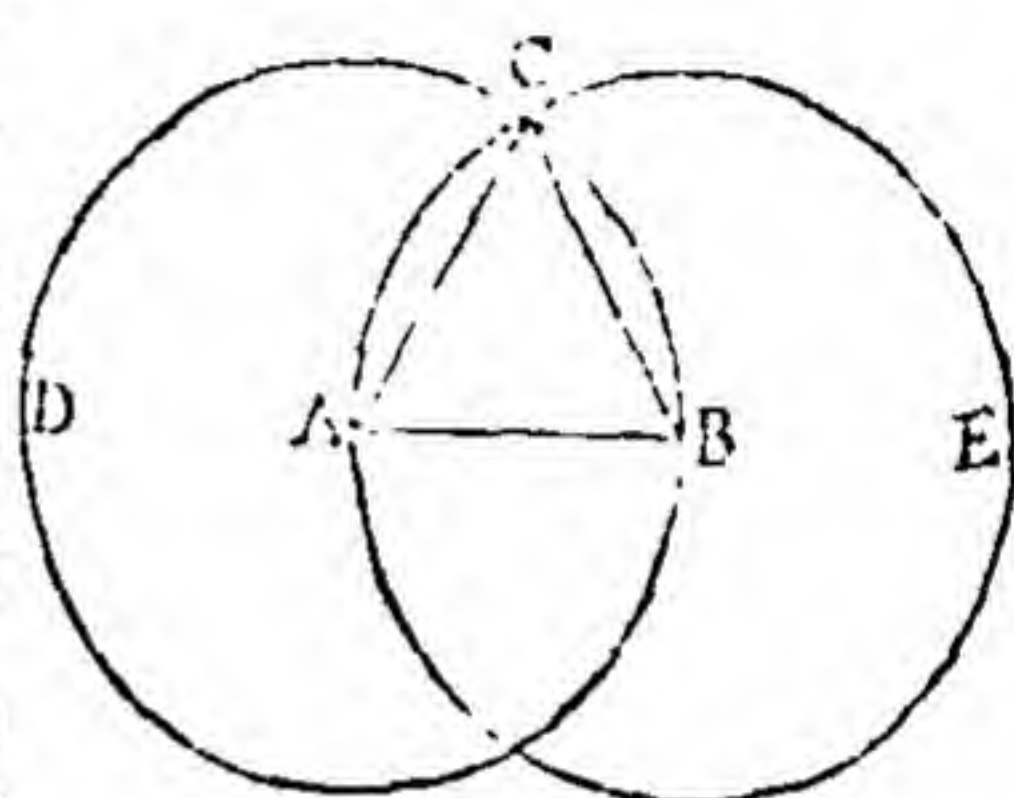
PROPOSITION I.

ON a given finite strait Line to constitute an Equilateral Triangle.

Exposition. Let the given strait Line be AB .

Recognition. It is required on the Line AB to constitute an Equilateral Triangle.

Construction. The Center A , and the distance AB , let be described the Circle BCD . [by Postul. 3.] And again the Center B , and the distance BA , let be described the Circle ACE . [by Postul. 3.] Then from the point c where the Circles cut one another, to the points A, B , let be drawn the strait Lines CA, CB , [by Post. 1.]



Determination. I say that the Triangle ABC is Equilateral.

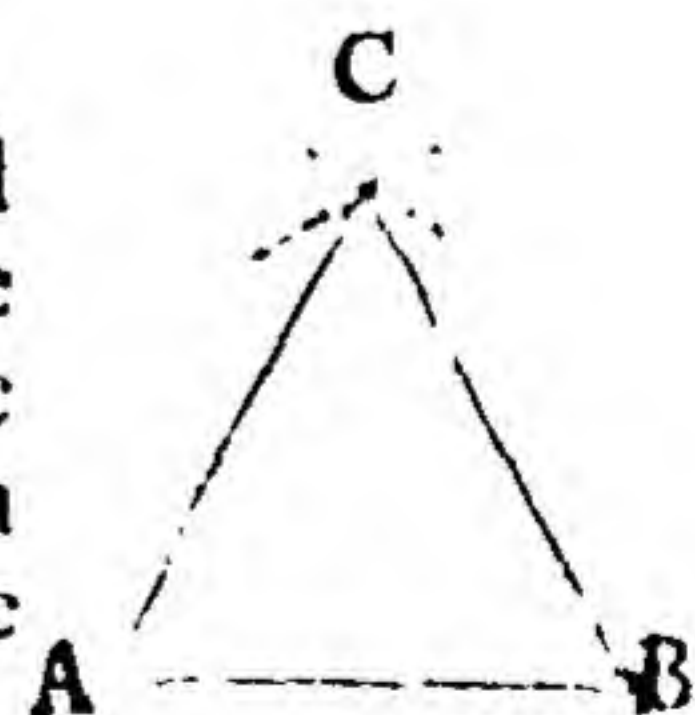
Demonstration. Forasmuch as the point A is the Center of the Circle BCD , therefore the Line AC is equal to the Line AB . [Def. 15.] Again, because the point B is the Center of the Circle ACE , therefore the Line BC is equal to the Line AB . [Def. 15.] But it has been proved that the line CA is equal to the line AB ; therefore each of the lines CA, CB , is equal to AB . But things equal to one and the same thing are also equal to one another, [Ax. 1.] and therefore CA is equal to CB : wherefore the three lines CA, AB, BC , are equal to one another.

Conclusion. Therefore the Triangle ABC is Equilateral, and is constituted on the given finite strait line AB . Which was to be done.

The Practice.

To make an Equilateral Triangle.

Open the Compasses to the length of the Given line AB , and fixing one foot on the point A , describe on either side of the line AB an Arch. Again, fixing a foot on the point B , describe on the same side an other Arch cutting the former; And from the point of Intersection C , draw CA, CB . Thus is made the Equilateral Triangle ABC .



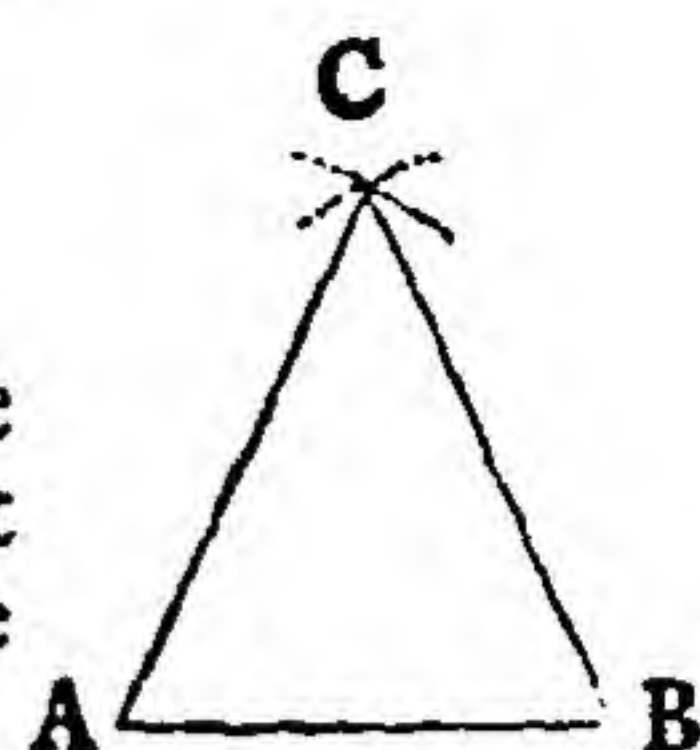
By the like practice may be formed an Equilateral Triangle.

K

To

To make an Equicrural Triangle.

Open the Compasses to any distance beyond half of the line AB , and describe Arches as before. Then from C the point of Intersection draw CA , CB , making an Equicrural Triangle ACB .



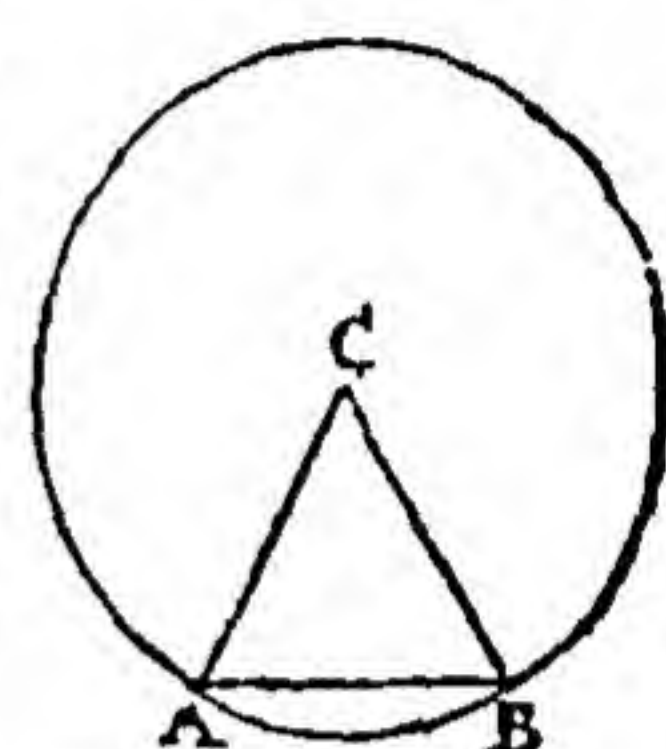
A practical Corollary.

From hence it is manifest, how to effect this following Problem.

A Problem.

By any two given Points to describe a Circle, whose circumference shall pass by the given Points.

Let the given points be A , B . And [by Postul. 1.] drawing the line AB , let thereon be constituted an Equilateral Triangle ABC , [by Prop. 1.] and [by the 3^d. Postulate] from the point C as a Center, and at the distance either of CA , or CB , a Circle being described, the circumference shall pass by the given points A , B , for that the lines CA , CB , are equal to one another.



How to describe a Circle, whose circumference shall pass by any three given points, not lying in a strait line, will be shewn hereafter.

ANNOTATIONS.

The being and the structure of a Circle, *Euclide* has before postulated to be granted him, from the natural simplicity, and evidence of its generation. The next simple, and uniform Figure, is an Equilateral strait-lin'd Triangle; with which therefore the *Elementator* begins. Whose Genesis, because there is used some Artifice, and Composition in the work, beyond that of a Circle, requires a demonstration to prove the Triangle constructed to be Equilateral. And this *Euclide* evidently deduces from a Circle, that primary Figure already postulated to be allowed without any demonstration. For Mathematicians proceed by Proposition after Proposition: First from certain Propositions as natural Principles, unto others as they may, with most facility and evidence, be deduced one from the other: The foregoing Propositions serving to demonstrate the following.

The method in all Propositions, Problems, as well as Theorems, is much the same, and consists of certain distinct parts. As for example, in this first Problem the parts are thus to be distinguished.

I. *The Proposition*. Which proposes in general Terms, a Thing given, and a Thing required. In Theorems the *Quasitum* is a thing required to be demonstrated as an undeniable truth. In Problems the *Quasitum* is first required to made, or constructed; and then the construction of the same is to be undeniably demonstrated. As in this Problem, *On a given strait line to constitute an Equilateral Triangle*. The subject given is any finite strait line in general: the thing required is an Equilateral Triangle to be constituted upon that given line.

II. *The Exposition* of the thing given. This *Expositio Dati* is an instance in special of what in the Proposition was given in general. As in the Proposition was given a *finite strait line* in general. Then next in the Exposition, is laid down in particular a finite strait line, as the finite strait line AB . The use of the Exposition is to facilitate the whole matter of the Proposition and Demonstration, by setting it forth in a sensible Diagram for the readier information of the Intellect; which may apply the same as universally as it was propounded; for that the line AB may denote any finite strait line whatever.

III. *The Recognition* of the thing required. After the Exposition laid down in a single instance, there follows in all, and *only* in Problems, that part which we call *Recognitio Quasiti*, a Recognition of the thing required: The phrase is, $\Delta\epsilon\delta\eta$, *Oportet*.

Oportet. In which *Oportet* we are reminded, what ought next to be done in particular upon the exposed instance. As upon this exposed *Datum* the line AB, it is now required in particular to constitute an Equilateral Triangle; so that in this Recognition the sides of the Equilateral Triangle are confined to the length of the exposed line AB, and the like in all Problems there is included in this *Oportet*, a Confinement of the *Quasitum* to the quantity of the *Exposed Datum*.

This, which, as a distinct part, we call Recognition, is by *Clavius* annexed to the Exposition, and by *Dasypodius* called the first Determination. Yet it cannot properly belong to either; for that the thing required is not as yet in being. But in reference to the foregoing *Datum* now exposed, here is next a special Designation of what is to be, as it was before in general required in the Proposition: And for a distinction from that part, called properly the Determination, we have named it *Recognitio Quasiti*.

IV. *The Construction.* This is a Geometrical Operation made out of the Postulates by an intellectual drawing of strait lines, and describing Circles. In Problems the construction does both effect the thing required: and also serves to demonstrate the same to have been rightly effected. As in the Construction of this Problem. The Equilateral Triangle ABC, is first tacitly constructed by describing the Circles BCD, ACE, and drawing the lines CA, CB. Then from the manner of its Construction the Triangle is next pronounced, and determined to be Equilateral.

V. *The Determination* of the thing required. This Part is only a Declaration that the thing required is now in a special Diagram exhibited. The phrase is *Ἀεγῶ*, I say, or Pronounce. As here in *Specie* the Triangle ABC (having been just before tacitly constructed upon the exposed *Datum* AB) is now determined, and pronounced to be Equilateral: Which the following Demonstration makes Apparent.

VI. *The Demonstration* of the Proposition. This is the glorious part of a Mathematical Proposition, wherein is made an undeniable and indubitable proof of the thing proposed, and exhibited in a particular Diagram. As here is demonstrated, that the Triangle ABC is Equilateral.

The form of Argumentation is for brevity sake made in *Enthymems*; which may be reduced into perfect *Syllogisms*: as *Cunradus Dasypodius* has set forth the first six Elements in a compleat Syllogistical form.

VII. *The Conclusion.* This is first particular in reference to the present Diagram, on which the Demonstration proceeded: as, *Therefore the Triangle ABC is Equilateral, and constituted on the given line AB.* So that this particular Conclusion consists of the Determination, and Exposition.

And because after the same manner there may on any other finite strait line be constituted an Equilateral Triangle, therefore from the Logical Rule of Induction, there is lastly a general Conclusion deduced, that *on any given finite strait line may be constituted an Equilateral Triangle*, which is only the Proposition repeated as being now demonstrated. Therefore in a Problem there is subjoined to the Conclusion, *ὅτι πρὸς ἰδέα ποιῆται.* Which was the thing propounded to be done. And in all Theorems, *ὅτι πρὸς ἰδέαν δέχεται.* Which was the thing propounded to be demonstrated.

This is the regular course used by Geometricians in their Forms of Doctrine. So that in Problems there may be seven parts according to the forenamed Order. The Proposition, Exposition, Recognition, Construction, Determination, Demonstration, Conclusion. But in Theorems there can only be six, and in this Order. The Proposition, Exposition, Determination, Construction, Demonstration, Conclusion. The different nature of Problems, and Theorems requiring such a difference in the number, and order of their parts, and specially in the disposal of the *Determination*; Which part in Problems ever follows the Construction: but in Theorems follows immediately the *Exposition*.

It is commonly, but improperly said, that every perfect Proposition has all these parts. Whereas all Propositions, which are rightly demonstrated, are alike perfect; For that one Demonstration cannot be more a Demonstration than an other: But thus it is, every Proposition requires not all these parts. For some Theorems

need no Construction, by adding strait lines, or Circles to the Original Scheme of the Proposition, because the Scheme it self is often sufficient for the Demonstration of the Proposition. But where it is not, there is then superadded a Construction, which is only as an *Apparatus*, or Preparation made on purpose to help out the Demonstration of the Theorem, and belongs not to the Proposition either as any part of the *Datum*, or *Quasitum*. Now in these Cases, that Theorem is rather to be accounted more perfect than any ways defective, which is not forced to seek out a Construction, for the setting forth of its Demonstration.

Again, in Problems the Determination is oftentimes omitted, as being in it self not absolutely necessary, tho' for the more perspicuity it be convenient. So in these three first Problems the Determination is not in the Text of *Euclide*, yet in 9, 10, 11, and 12th. Propositions, which are Problems, the Determination is expressly set forth. Therefore accordingly we have inserted in this, and the two following Problems the Determination, to clear the matter for the ease of Beginners, that after the Construction, and before the Demonstration, there might be set forth, and determined, what by the Construction has been effected, and is next to be demonstrated.

Moreover, some Problems have only a *Quasitum*, and not a *Datum*, upon which in particular to work. And so there is neither an Exposition, nor a Recognition to follow, but only a Construction, Demonstration and Conclusion. As the tenth Proposition of the fourth Element, which is a Problem requiring *To constitute an Equicrural Triangle, having each of the Angles at the Base double to the remaining Angle*. Here now is required to be constructed such a kind of Triangle without any thing given. Yet notwithstanding there is no imperfection, or any defect in this Proposition; but that it is a subtil and admirable Problem. Therefore we are to understand, that every Proposition of the Elements is according to its nature compleat in it self, whether it has, or needs not to have all the forementioned Parts.

Of the DATUM in Geometrical Propositions.

In Geometrical Propositions, a thing may be given four several ways: In Position, In Specie or Form, In Magnitude, In Proposition.

A point having no Magnitude is only given in Position, that is to say, *HERE*. But every kind of Magnitude may be given all the four ways, jointly, or severally, in all, or in some.

A thing is said to be given in Position, when it is restrained to a certain Situation. As the Data in all Problems are: where whatsoever is given ought not, so much as in our imagination, to be removed from its given Situation; but according to the restriction of the Position given the Problem is to be performed.

Now on the contrary, in Theorems the Diagram of the Proposition is not tyed to a certain Position, but left as indifferent. For tho' Position be a special condition in the Structure of a Problem, whereby it is to be regulated: yet in a Theorem it appertains not at all to the Truth, or Falsity of the Theorem: and therefore the Position of the Diagrams is alterable at discretion. Insomuch that in some Theorems, where two things are given, there a certain Position of the one to the other, is sometimes a means to help out the Demonstration: And therefore an arbitrary position of the Data in Theorems, is allowed to the Demonstrator for a kind of Construction. As *Euclide* sometimes applys Figure to Figure, sometimes conceives two strait lines so directly situated to one another, as to become one strait line; with such like choice and change of position, as may best serve to the demonstration of the present Theorem.

In this Problem the line *AB* is not only given in Position, as the *Datum* in every Problem is, but also 'tis given in Specie, being proposed a strait line, to specify it from a crooked.

And moreover, 'tis given in Magnitude as a finite strait line, which also is understood to be so given under a certain Termination, that it may be compared as Equal, Greater, or Lesser, than an other: and in like manner any other may be compared

pared to it. For 'tis not meant to be given in such, or such a singular Quantity; as of one, or two, or three Inches, or Feet, in relation to any stated measure; but here That is taken to be given in Magnitude, which is proposed under some certain limitations, so that it may be said to be equal, or unequal to an other Magnitude. Yet after such an Indefinite manner the limited Magnitude is put; that it may be conceived to represent any Magnitude of the same kind, in any quantity whatsoever. As in this Proposition when we say, Let the given line be AB, there is meant by AB any length, by what measure soever estimated. Whether by Inch, Foot, &c. or not at all estimated by any distinct Quantity. So that in this Case by an Example, or Instance exposed in particular, the Universality of the Proposition, and its Demonstration are not destroyed, but still remain in their full Latitude, and general Extent.

The Methods of Composition and Resolution, as they are used by Geometricians.

There are two ways of Reasoning, whereby Man comes to the knowledge of things; And both the ways are established upon the same Foundation, that is; upon the Dictates of Nature, or common Notions among Mankind. The difference here only is, that in one way we begin our Discourses from those Natural Dictates, and in the other we end with them; treading the same path forward and backward. Both these ways we thus explain.

Ratiocination proceeding from Natural, that is, Self-evident Principles of Truth unto other Truths, made known to us from them, and so going onward by the help of these discovered Truths, to infer and make manifest Truths more remote, and as yet unknown, is called the Method of *Composition*, or *Synthesis*. For that in this way of Reasoning, we do from Notions most plain and simple, gather up by degrees Notions more intricate, and compounded: framing out of simple Materials firm, and stately Edifices.

Ratiocination proceeding from the Supposition of things uncertain, whether True or False, Possible or Impossible; and which by demonstrative Consequences deduced from that Supposition, does necessarily come to a manifest Truth, or Falsity; Possibility, or Impossibility, is called the Method of *Resolution*, or *Analysis*. For that a doubtful Supposition is hereby resolved into a certainty of Truth, or Untruth, of Being, or not Being.

If the Issue of our Argumentation terminates in an acknowledged Truth, then are we assured of the Being, and Verity of the Supposition upon which we argued.

And we may again take a beginning from the same Truth, wherein the Resolution rested; and from thence as a Principle, proceed in the Method of Composition, making a return in the very same steps, which in the Resolution of the Supposition were traced out before: till at length we arrive at that thing, which was at first Supposed, demonstrating in the common Compositive Method of Geometricians the Truth of Theorems, and the Geometrical effect of Problems. Thus Composition and Resolution, or Synthesis and Analysis, answer one another; the Analysis ending where the Synthesis begins, and the Synthesis ending where the Analysis begins. Like to an Ascent and Descent made in the same path step by step.

Resolution, or Analysis, is properly the Method of Invention, and the ready way of discovering the Truth, or Falsity of a Proposition, in any Art or Science.

Composition, or Synthesis, is the Method of Doctrine, or the way of Teaching, and therefore in this Method from allowed Principles, all Arts and Sciences are usually delivered. Like as lasting Buildings are raised upon sure Foundations; Whereof the Doctrine of these Elements is a most perfect pattern.

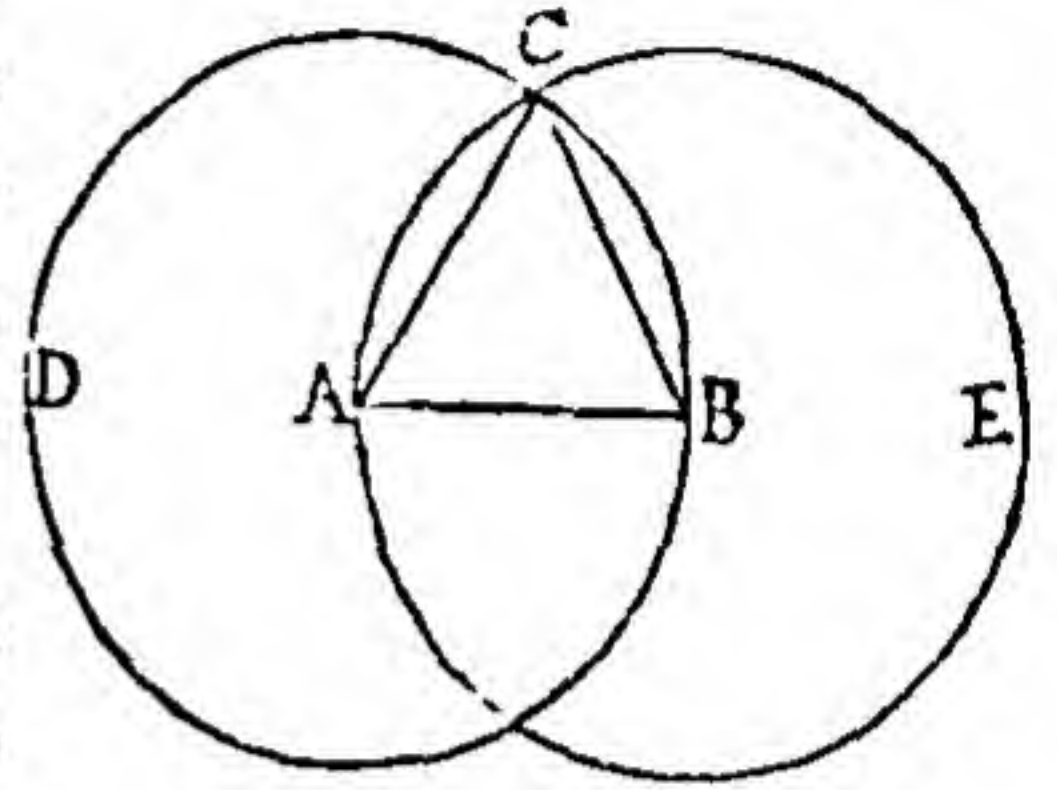
Therefore of this first Problem, we shall take a review, and nicely observe, by what Gradations of Composition it is to be effected and demonstrated.

And for an Entrance into this Inquiry, we are first to consider what from the

preceding Principles can be made use of, upon the given line AB , to effect this Problem, and construct an Equilateral Triangle.

First, the given line AB being finite, there are given the ends thereof. So that three Things are given, two Points, and an interjacent strait line. Next therefore we are to search out, what can arise to our purpose, from these three Data. And recollecting the former Definitions, Postulates, and Axioms, (for in the Mathematics memory and reasoning are to go together) there must among the rest occur to our remembrance the third Postulate; *From any point unto any distance to describe a Circle.* And here in these present Data aptly appear a given Point A , or B , and a given Distance, namely the strait line AB given; therefore of the line AB taking one of the Extrems, as the Point A for a Center to a Circle, and the given line AB for a given Distance, we postulate a Circle to be described.

Again, taking the point B , the other extrem of the given line AB for a Center, and the given line for the same distance we postulate another Circle to be described. 'Tis evident, that *This Circle must cut the former, for that the line AB lyes wholly within both the Figures, and is a common Radius to both Circles.*



Now by the mutual Intersections of these Circles, there occurs a point common to both Circumferences, and let it be signed the point C .

Here then are three known points, A, B, C , and the line AB . What can, from these four things known, be reasonably deduced for the making an Equilateral Triangle, is next to be thought on.

The first Postulate cannot but come readily into our mind, which allows the drawing of a strait line from point to point. So that we are prompted to draw from the Found point C to the given points A, B , the two strait lines CA, CB .

Here at last is made a strait-lined Figure of three sides called a Triangle, which we are next to consider of what kind, or condition it may be: and whether answerable to the solution of the Problem.

The two Circles just now before described by the same Radius AB , and BA , are obvious to our consideration, and the Idea or Notion of a Circle delivered in the 15th. Definition, that the lines from the Center to the Circumference are all equal to one another. This Idea does readily lead us to infer, that the line CA drawn from the point C in the circumference of the Circle BCD to the point A the Center, is equal to the given line AB the describing Radius, or the primary line from the Center of the same Circle.

In like manner, and by the same means we are instructed to argue, that the line CB drawn from the point C in the circumference of the Circle ACE to the point B the Center, is equal to the given line BA , the describing Radius of the Circle ACE .

Wherefore finding that both CA and CB , are each equal to AB , we do naturally suggest to our selves the first Axiom, that things equal to the same are equal to one another, so that CA and CB , are equal to one another. And therefore all the three lines CA, AB, BC , are equal to one another, making an Equilateral Triangle, according to the 23^d. Definition.

Thus in the Method of Composition from the third and first Postulates, from the fifteenth Definition, and first Axiom, we have fully set forth the Construction and Demonstration of an Equilateral Triangle on a given finite strait line. And have together shown, upon what easy rational Grounds and Natural Suggestions this Problem may in this way alone be invented. So the like may be done in many other following Problems; for that the Invention of their Constructions and Demonstrations, is not to be far fetched, depending only upon some few foregoing Propositions, which may at once be brought into memory, and fitly applied to present Use.

But Problems more abstruse and intricate, tho' they may by a well exercised Geometrician be performed wholly in this Compositive Method, yet it is not the
readiest

readiest way to discover how a perplexed Problem may be extricated and effected.

In these Cases, instead of making our Gradations by *Composition* from the Principles, and other Propositions arising from them; Geometricians contrarywise order the matter after the Method of *Resolution*, and in the first place *do suppose the very Thing to be already done, which is propounded to be done.* FACTUM PUTANT QUOD FACIENDUM EST. And then they examine what can by just consequence be deduced from the *same Supposition*; demonstratively inferring one thing after another until they fall upon something, which evidently shews how to effect the Problem, or that it is impossible to be effected.

If by legitimate Argumentations we are brought to an impossibility; It is thereupon concluded in common reason, that the Problem is impossible to be effected; and that the *Thing supposed* is inconsistent with Nature. But if we meet with no such Obstacle, then are we assured the Problem is feasible, and that from the Supposition of the thing already effected, We may by necessary Inferences clear the way, and come to a certainty how to effect the same.

And therefore to give a glimpse of Light into this *Admirable Method of Resolution*, take here an Example thereof in the Invention of this first Problem. Altho' by reason of its Simplicity it is readily found out in the former Method of *Composition*, as we have already explained.

P R O B L E M I.

On a given finite strait line to constitute an Equilateral Triangle.

Investigated by the Method of Resolution.

In all Problems we are first to consider wherein the STRESS of the Problem does consist. Therefore well pondering the nature of this Question, it will appear, that the main matter is to find a Point without the given line, from whence two strait lines being drawn to the ends of the given line, shall each be equal to the same.

Now to begin, *Put a factum.* Suppose this done: and the point found let it be C, and the Triangle A B C, be Equilateral, having the sides C A, C B, each equal to A B the given line.

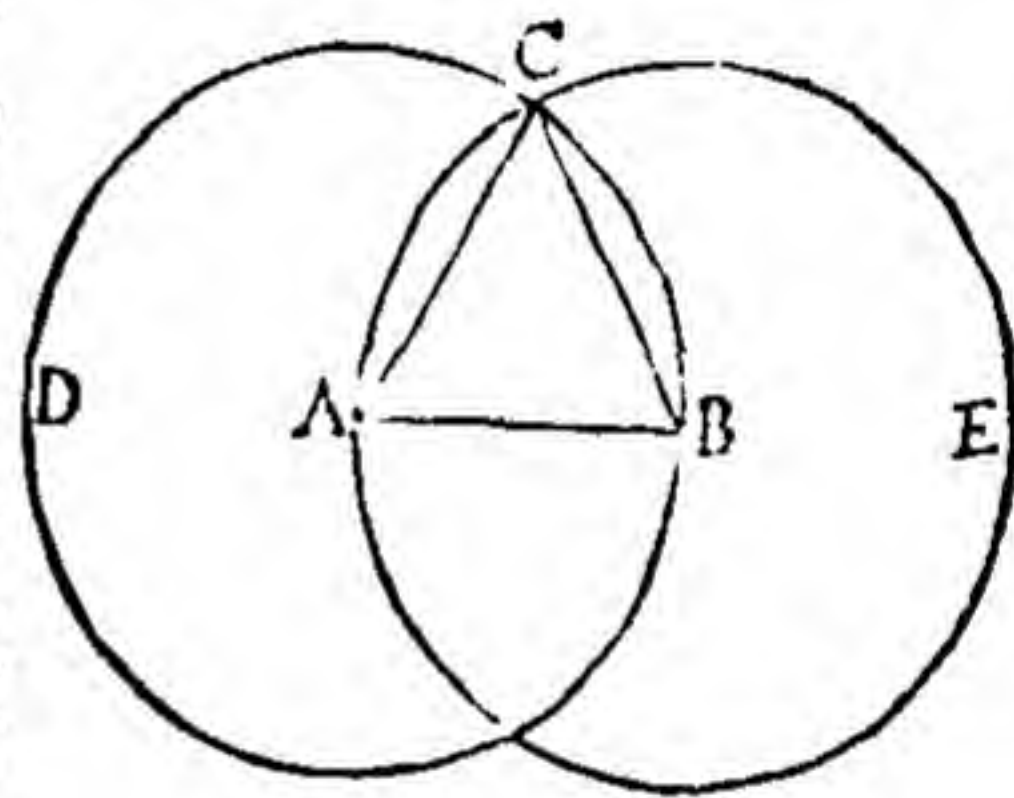
Having thus supposed the Thing in Question: Now for the Solution of it, as of every Problem, there is to be used a dexterous Sagacity of Thought in searching out something *latent* in the Question, which is in some sort known unto Us, and from whence we may by degrees arguing from one thing to another, make in the end a perfect discovery of That, which is wholly unknown. For in humane reasoning we can attain to the Cognition of Things unknown, and under inquiry, only so far, as they partake of, and secretly contain within themselves the nature of other things already known, from which we must argue; or else one thing could never be deduced from an other.

And therefore this Præexistence of the knowledge of something in the very things unknown, and sought for, is the foundation of all our Ratiocinations, and in this Case thus leads Us on.

Because the line A B is not barely supposed, as are the lines C A, C B, but is actually given in Position, and Magnitude: therefore upon this Datum, with a respect likewise had to the Tenor of the Supposition it self, the Force and Perspicacity of our Mind is to be exercised, in bringing forth something relating both to what is actually Given, and to what is only Supposed, that may open a way toward the Invention of the Thing Required.

Now of all the several Subject Matters of Geometry, and of the Figures before defined, there is not as yet any of them in being besides a strait Line and a Circle: both which are postulated to have a being. Therefore from one, or both of these two we are to begin the Work.

From a given strait line nothing else can arise, but either the *Continuation* thereof by the *Second Postulate*; or by its *Circumlation* the Generation of a Circle according to the



the *Third Postulate*. It is at first view manifest, that the Continuation of a strait line cannot serve to the framing of a Triangle. There is therefore nothing else existing but a Circle to help towards this matter.

Now we having had before an Idea of a Circle delivered to us, and superadding the conception of a Circle to the given strait line : Let us for an Essay try what may arise from them both, that is, From a strait line given and a Circle supposed.

Suppose then by Postulate the third from the Center A, at the distance of A B given, a Circle to be described. It must now occur to our thoughts from the Notion of a Circle in Def. 15. that the circumference thereof shall pass by the point C, the end of the supposed line A C, for that A C was supposed equal to the given Line A B. Let therefore be described the Circle B C D passing by the point C in the line A C.

Again, the Center B, and distance B A being likewise given, if we suppose a Circle to be described, the circumference thereof shall pass by the point C, the end of the supposed line B C, for that B C was supposed equal to B A. Let therefore be described the Circle A C E passing by the point C in the line B C.

And because the same point C is common to the supposed lines A C, B C, and is moreover in the circumference of the Circle B C D, and also in the circumference of the Circle A C E; and that the Circles B C D, A C E, have nothing common but their Intersection, therefore the point C is in the Intersection of the Circles B C D, A C E, now described.

Wherefore the point C is found : And *thereby* the Equilateral Triangle is found.

For the three Points A, B, C, being now known, and in Position given; the three sides of the Equilateral Triangle are also given by the *first Postulate*, From any point to any point to draw a strait line. Let therefore from the point C, thus found, be drawn the strait lines C A, C B (lines only before supposed) which lines by Ax. 1. are equal to one another, because each of them is equal to A B, by Def. 15. And so all the three strait lines are equal to one another, C A, A B, B C.

Thus the Construction of an Equilateral Triangle on a given finite strait line, is naturally found out in the Method of Resolution. From whence does arise a Theorem, as a RULE OF PRACTICE, how demonstratively to construct the same, as *Euclide* has done, in the Method of Composition, after this manner.

The THEOREM deduced from this Resolution, and teaching how Geometrically to effect the Problem.

If from the ends of a given line be described two Circles, at the distance of the given line, and from the point of their Intersection be drawn two strait lines to the ends of the given line, there shall be constituted an Equilateral Triangle on that given line.

In the progress of this Problem, tho' it be very easy, and its Invention obvious, so that it might have been carried on in short by continued Inferences without any Comment, or interwoven Observations: Yet here as we pass from one thing to an another by several Gradations, we have thought fit to intermix some Advertisements on purpose to direct, and set forth, after what manner, and with what circumspection Problems more abstruse and difficult, ought to be managed in the Method of Resolution: Encompassing in our thoughts all things possible to be comprehended within the nature of the Question, by which Sagacious search most wonderful and occult Truths may out of that Obscurity, wherein they lye involved, be brought to Light, and as it were hammered out.

Ut Silicis venis abstrusum excudimus ignem.

PROPOSITION II.

AT a Given Point to put a strait line equal to a strait Line Given.

Let the given Point be A, and the given strait line be B C.

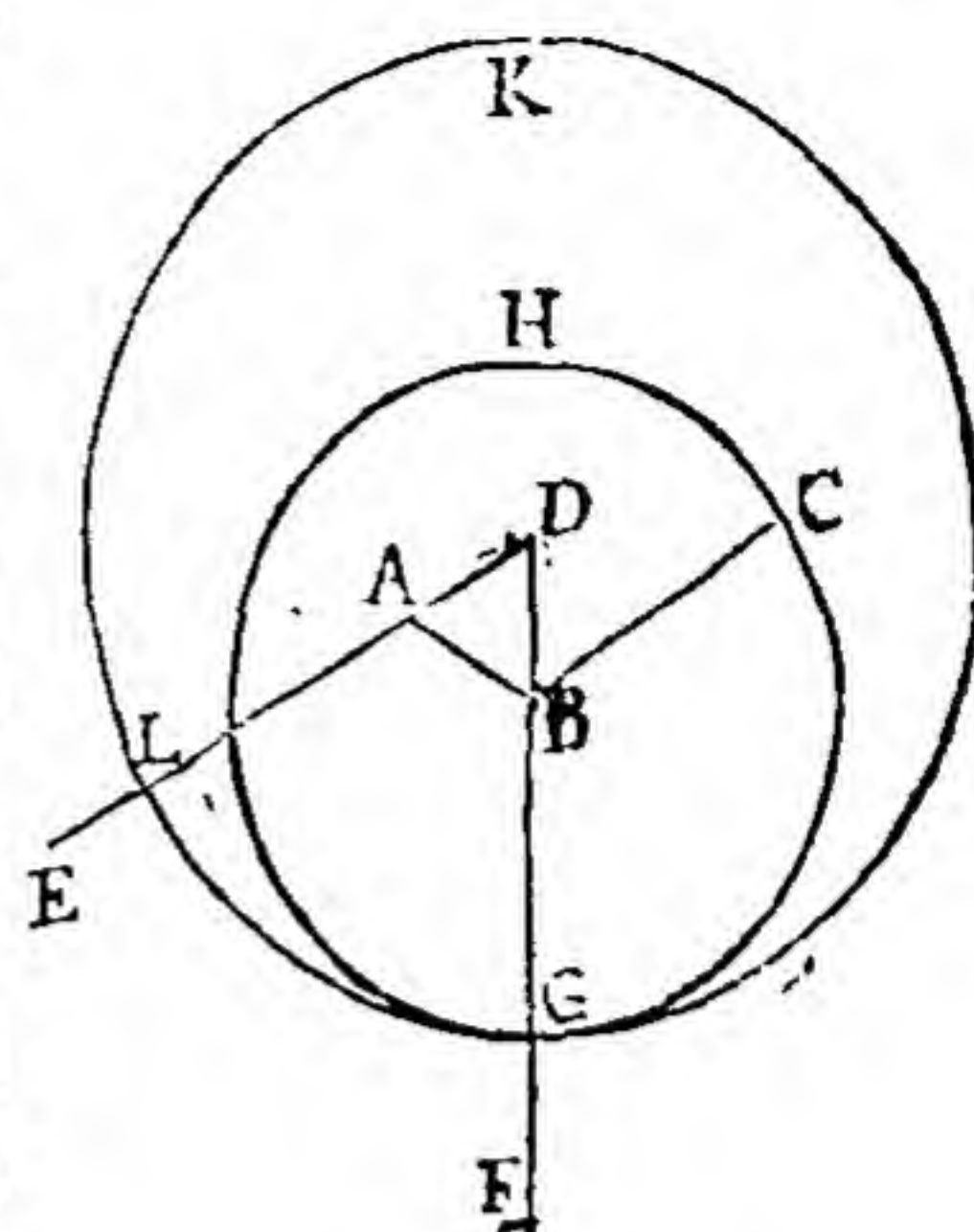
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It is required at the point A , to put a strait line equal to the strait line BC .

From the point A , to the point B , let be drawn the strait line AB . [Post. 1.]

And on the line AB let be constituted the Equilateral Triangle DAB . [by Prop. 1.]

And to DA, DB , let be continued directly the strait lines AE, BF , [by Post. 2.] Then the Center B , and distance BC let be described the Circle CGH , [Post. 3.] And again the Center D , and distance DB let be described the Circle GLK . [Post. 3.]^G



I say that at the given point A , is put a strait line AL equal to the given strait line BC .

Forasmuch as the point B is the Center of the Circle CGH , therefore BC is equal to BG , [Def. 15.] Again, because the point D is the Center of the Circle GLK , therefore DL is equal to DG . [Def. 15.] Of which DA is equal to DB , [by Constr.] wherefore the remainder AL is equal to the remainder BG . [Ax. 3.] But it has been proved that BC is equal to BG . Therefore each of the lines AL, BC , is equal to BG .

But things equal to one and the same thing, are equal to one another. [Ax. 1.] And therefore AL is equal to BC .

Wherefore at the given point A is put a strait line AL , equal to the given strait line BC . Which was to be done.

ANNOTATIONS.

In this Problem there are three Cases, according to the various situations of the given point A , in respect to the given line BC .

For the point A is either without the line BC , as in the Figure used in the demonstration.

Or it is within the given line, or else at one end of it. In which last Case a Circle only described from the same end as the Center, and to the distance of the given line, effects the Problem without the construction of an Equilateral Triangle.

The Practice.

The practice is obvious. For opening the Compasses to the length of the given line BC , and then placing one foot on the given point A , set forth the same length with the other foot to L , and draw AL . This is the sensible and Mechanical Operation.

But we are again to be reminded upon this occasion, that in the pure Geometrical Solution of Problems, no use is to be made of Ruler and Compasses, or of any outward Sense. And moreover, that whatsoever things are given in a Problem ought to remain in the position given. And according to that stated position of the *Data*, and the Tenor of the Problem, every thing is to be transacted in the Mind, as if we neither used our Hands, nor Eyes. And therefore by some of the Ancient Geometricians Problems are also called Theorems, for that their Operations are only Speculative and Intellectual, in a subject wholly abstracted from Matter. Yet we are to know, that they are the sure and demonstrative Grounds of Material and Manual Practices, in the Mensuration of all kinds of Magnitude

in Architecture, Fortification, Navigation; In all sorts of Mechanism, infinitely useful to Mankind, both for Necessity and Curiosity.

PROPOSITION III.

Two unequal strait lines being Given, to take from the Greater a strait line equal to the Lesser.

Let the two unequal strait lines given be AB , and c ; of which let AB be the greater.

It is required to take from AB the greater, a strait line equal to c the lesser.

By the preceding Problem, let at the point A be put a strait line AD , equal to the strait line c .

Then the Center A , and distance AD , let be described the Circle DEF . [by Post. 3.]

I say, that from the strait line AB the Greater, is taken a strait line AE equal to c the Lesser.

Forasmuch as the point A is the Center of the Circle DEF , therefore the line AE is equal to the line AD . But the line c is equal to the line AD : so that also AE is equal to c . [Ax. 1.]

Therefore two unequal strait lines being given, AB and c , there is taken from AB the greater, a line equal to c the lesser. Which was to be done.

The Practice.

The practice of this is as before: Opening the Compasses to the length of the given line C , and setting the same off from the point A to E .

These three Propositions are only ministerial Problems, and therefore here premised for their general use through all Geometry.

PROPOSITION IV.

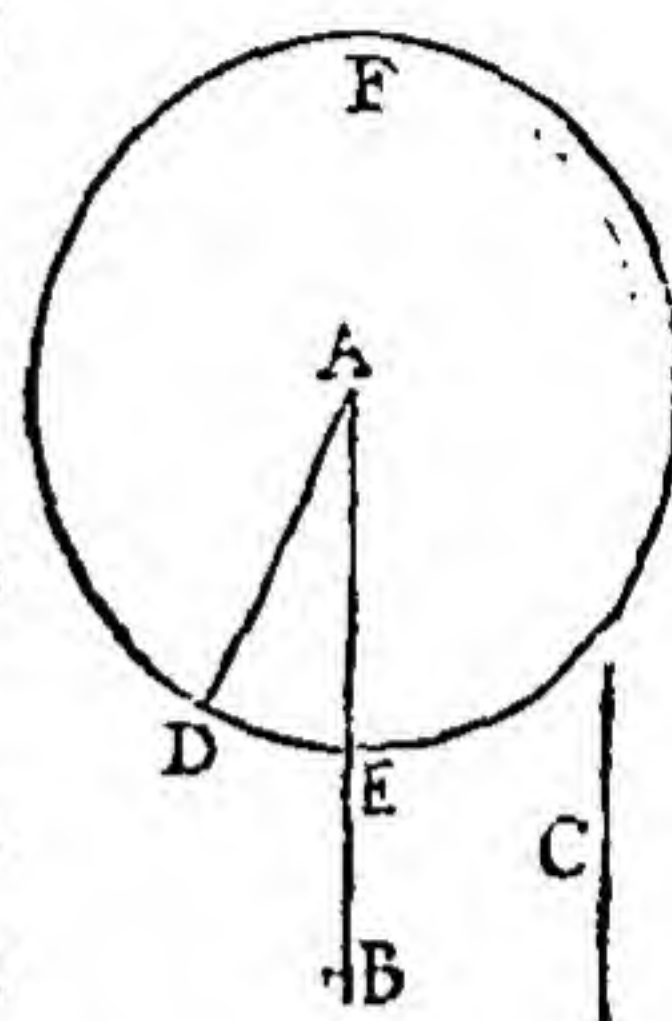
If two Triangles have two sides equal to two sides, each to each, and have an angle equal to an angle, namely that, which is contained by the equal lines.

Then shall they have the base equal to the base, and Triangle shall be equal to Triangle, and the remaining angles shall be equal to the remaining angles, each to each, under which are subtended equal sides.

Exposition. Let the two Triangles be ABC , DEF , having the two sides AB , AC , equal to the two sides DE , DF , each to each, namely AB to DE , and AC to DF , and the angle BAC equal to the angle EDF .

Determination. I say, that the base BC , is equal to the base EF . And the Triangle ABC shall be equal to the Triangle DEF , and the remaining angles shall be equal to the remaining angles, each to each, under which are subtended the equal sides, namely the angle

ABC



ABC to the angle DEF, and the angle ACB to the angle DFE:

Construction. For the Triangle ABC, being applied to the Triangle DEF: and the point A put on the point D; and the straight line AB on the straight line DE;

Demonstration. Then shall the point B agree with the point E; for that AB is equal to DE. [by *Supposition*.]

Now AB agreeing with DE, AC shall also agree with DF, for that [by *Supposition*] the angle BAC, is equal to the angle EDF.

So that also the point C shall agree with the point F; for that the line AC is likewise equal to DF, [by *Supposition*].

But now also the point B had agreed with the point E; so that the base BC shall agree with the base EF.

For the point B agreeing with the point E, and the point C with the point F, if the base BC shall not agree with the base EF; Then two straight lines shall comprehend a space. Which is impossible, [by Ax. 12.]

The particular Conclusion. Wherefore the base BC shall agree with the base EF; and therefore shall be equal to it, [by Ax. 8.]

So that also the whole Triangle ABC, shall agree with the whole Triangle DEF, and therefore shall be equal to it, [by Ax. 8.]

And the remaining angles shall agree with the remaining angles, and therefore shall be equal to them, namely the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

The general Conclusion. If therefore two Triangles have two sides equal to two sides, each to each, and have an angle equal to an angle; namely, That which is contained by the equal lines: Then shall they have the base equal to the base, and Triangle shall be equal to Triangle, and the remaining angles shall be equal to the remaining angles, each to each, under which are subtended equal sides. Which was to be Demonstrated.

Annotation on the Proposition.

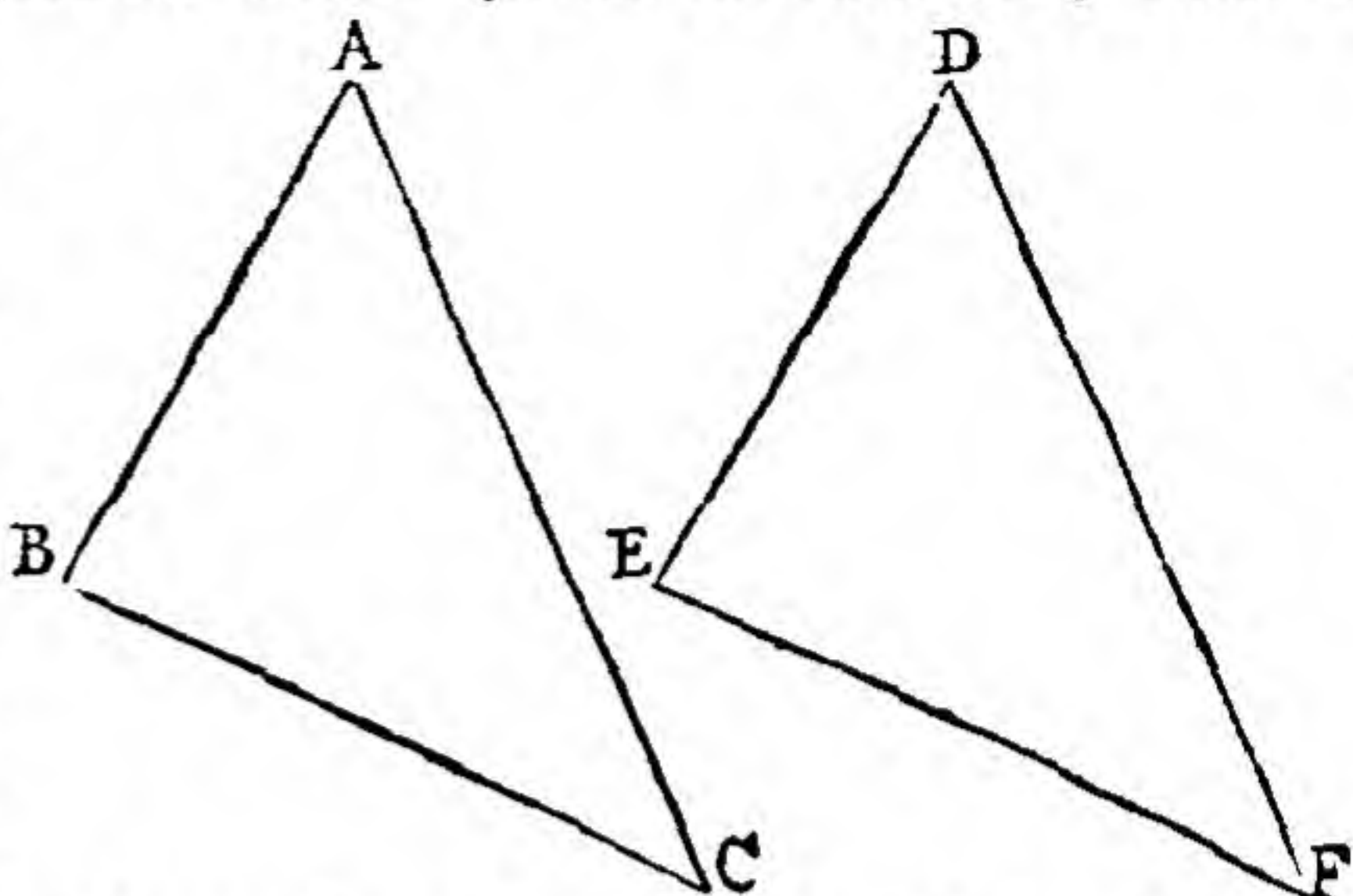
This Proposition is the first Theorem, and Foundation of all Geometry, which therefore with the whole manner of its demonstration, we shall specially endeavour to explain.

In a Triangle there are *Seven Things* to be considered.

The three Sides, the three Angles, and the Area, or Space comprehended by the sides. Of these Seven here are three Things in one Triangle, namely, *two Sides, and the angle contained*, which are supposed equal to three the like Things in the other Triangle. And from the supposed equality of these Three, is demonstrated the equality of all the other Four, that is, of the third side to the third side, of Area to Area, and of the two other angles to the two other angles, each to its correspondent angle.

Annotation on the Exposition.

What Things were in the first part of the Proposition *Given*, or *Supposed* in general (namely, in two Triangles two sides equal to two sides, and the contained angles equal) Those are now in the Exposition exemplified in the Triangles ABC, DEF, and in particular set forth: Side to side, AB equal to DE, AC equal to DF, and angle to angle, BAC equal to EDF (Each to each &c.) That is, the two sides in one Triangle taken singly, and compared respectively to the two sides singly, in the other Triangle. For otherwise the two sides in one Triangle taken both together, may be equal to the two sides in the other Triangle taken both together; that is, the sum of the two sides in one Triangle may be equal to the sum of the two sides in the other Triangle, and the angles contained be equal to one another: Yet from thence the other parts in those Triangles cannot be proved equal to one another. As in one of the Triangles, if one side be 2, and the other be 5, which together are 7: And in the other Triangle if one side be 3, and the other be 4, making also together 7: And let them contain equal angles, yet the Triangles shall not in the other parts be equal to one another. Therefore it is here specially said *Each to Each, singula latera singulis lateribus, ἑκάστην ἑκάστην*.



Annotation on the Determination.

As the *Exposition* did set forth particularly, in the Triangles ABC, DEF, what was in the first part of the Proposition supposed, and given in general Terms: so what was in the second part of the Proposition laid down in general, as to be demonstrated from what was supposed, and given in general, That is now in the *Determination* specified in the particular parts of the same Triangles ABC, DEF: and pronounced to be true. Saying, The base BC is equal to the base EF, and the Triangle ABC &c. Now the truth thereof is to be made good by the following Demonstration.

Annotation on the Construction.

After the Exposition and Determination, there often follows in Theorems a *Construction*, wherein are added to the simple Figure of the Proposition some strait lines, or Circles, or both, to make way for the Demonstration. This Proposition requires no such kind of Construction, having all lines requisite to the demonstration, laid down at first in the Exposition and Determination. But here *Euclide* uses another sort of Construction, or *Apparatus* towards his Demonstration, by a mental Application of one Figure to an Other: which is by the Greeks called *ἑφαρμογή*, *Epharmosis*; an Adaptation, Apposition, or Application of one magnitude to another: And in the manner following is thus made use of.

For the Triangle ABC being applied to the Triangle DEF; and the point A put on the point D.] It is not meant, that this Application be made by the use of Hands and Eyes: but by an imaginary position, first of the point A on the point D; and then of the line AB on the line DE, as followeth.

And the strait line AB on the strait line DE.] In the 4th. Def. *Euclide* tells us, that by a strait line we are to conceive such a line as lyes evenly to all its points. So that upon this conception of a strait line, if we imagine one strait line to be placed upon another strait line, we must also conceive, that no part of the one does any where swerve, or any ways deviate from the other; And as agreeable to this conception of a strait line, we may observe the exactness of *Euclid's* expression,

sion, how that the *lines* AB, DE, which before in the Exposition of the Proposition were called *Sides*, in relation to a Triangle, are now in respect of their Application to one another, called *strait Lines*, not *Sides*, because from the notion of a *strait line* it follows, that if strait lines be apply'd to one another, They must be imagined to be entirely coincident with one another. Thus far proceeds the Construction by way of Application, or *Epharmosis*. That is, *first the imaginary position of the point A on the point D. And secondly of the strait line AB, on the strait line DE.* From whence by a just Ratiocination *Euclide* demonstrates the equality of the two Triangles, in the Whole, and in every Part. As followeth.

Annotation on the Demonstration.

Then shall the point B agree with the point E, for that AB is equal to DE.]

Here is first to be observed *the difference between the agreeing of two strait lines, and the agreeing of their extreame points.* The Former is in respect of being strait lines, the Other in respect of their equality to one another. For all strait lines are conceived to have their parts Congruous: And all equal strait lines to have also their extreams Congruous. And therefore *Euclide* upon the Application of the strait lines AB, DE, to one another, having presumed from the notion of Rectitude the mutual agreement of their intermedial parts; He next urges the agreement of their extreame points from their supposed equality. For the extreame point A of the line AB, being conceived to lye on the extreame point D of the line DE, and the line AB on the line DE; if of the line AB the other extreame point B, does not agree with the other extreame point E of the line DE: then one of these points falls short of the other: so that one of the lines shall be a part of the other line. And because the lines AB, DE, are supposed equal, therefore the part shall be equal to the whole. Which is impossible by the 9th. Axiom. Therefore the point B shall agree with the point E.

Now AB agreeing with DE, also AC shall agree with DF; for that the angle BAC is equal to the angle EDF.]

For otherwise, if AC agrees not with DF; then shall AC fall either within, or without DF: so that one of the angles BAC, EDF, shall be a part of the other. And because the angles BAC, EDF, are supposed equal; therefore the part shall be equal to the whole. Which is impossible. Therefore the line AC shall agree with the line DF.

So that also the point C shall agree with the point F.]

For the same reason as before, that the point B did agree with the point E, AC being supposed equal to DF, as AB was to DE. Here again is to be observed, how *Euclide* distinguishes between the agreement of the lines AC, DF, and the agreement of the extreame points C and F. For from the equality of the angles BAC, EDF, he proves the coincidency of the lines AC, DF: And from the equality of the lines AC, DF, he proves the coincidency of the extreame points C, F.

For the point B agreeing with the point E, and the point C with the point F, if the base BC shall not agree with the base EF.]

Then must BC fall either within, or without EF, so that the strait lines BC, EF, shall comprehend a space. Which by Ax. 10. is impossible. Therefore BC shall agree with EF.

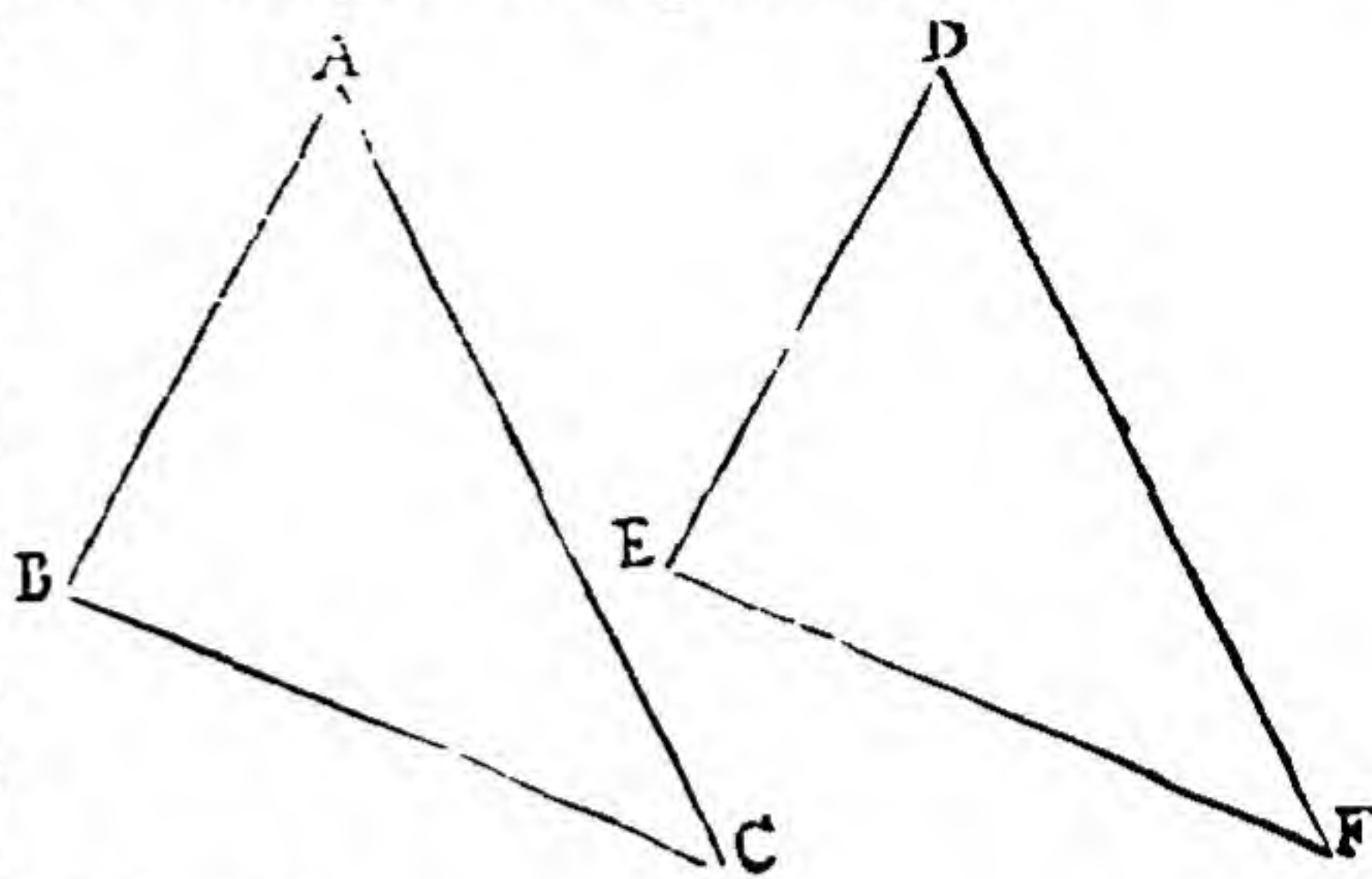
The subtil changes made in the course of this demonstration are remarkable. First, from the Definition of a strait is implied the coincidency of the intermedial parts of two strait lines, which are applyed to one another, namely of AB and DE. Secondly, from the supposed equality of two strait lines, is proved the agreement of their extreame points. Otherwise the part would be equal to the whole. Thirdly, from the supposed equality of two angles, is proved the coincidency of their containing lines. Otherwise the part again would be equal to the whole. And lastly, from the agreement of the two extreame points B with E, C with F, of two strait lines BC, EF, is concluded the agreement of the lines Themselves, namely of BC and EF. Otherwise two strait lines would comprehend a Space.

Annotation on the Conclusion.

We are here to observe a double Conclusion. The first particular, The other general. The *particular Conclusion* is the *foregoing Determination* repeated word for word, but so as being now demonstrated; and therefore concluded true in those particular Triangles ABC, DEF. The *general Conclusion* is the *Proposition it self* repeated, and concluded with, as justly following upon that particular Conclusion already demonstrated.

For according to the Mathematical method, as before in Problems, so here in Theorems, is in the first place laid down the Proposition in general terms.

Then next, of this Proposition there is made an *Exposition*, and a *Determination* in a particular Instance, or Example. An Exposition of what is given: and a Determination or Specification of what is to be proved. As here in the two Triangles ABC, DEF, is set forth particularly by name, *every Thing given*, and *every Thing proposed to be demonstrated*. As we have distinguished them in the several Paragraphs, of *Exposition*, and *Determination*.



Now upon this Instance in the Triangles ABC, DEF, the demonstration proceeded: and therefore *Euclide* first concludes particularly, as to that Instance in the said Triangles ABC, DEF. And thereupon implying the like reason in the like matter, He concludes by the Logical Rule of Induction, with the Proposition it self in general, as at first it was laid down. For what is now particularly demonstrated in the Triangles ABC, DEF, the same may be proved after the same manner in any other two Triangles, which have the same conditions. And therefore the Proposition is in general concluded to be true.

This is the regular form of a Geometrical demonstration in all Theorems; which we have here explained at large, once for all.

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Thus with more Artifice than is commonly taken notice of, *Euclide* manages this Demonstration. And indeed the nearer any Proposition comes to a Principle, or an evident Truth, as this Proposition doth, the more difficult it is to be demonstrated; because the *Mediums* for the proof of such Propositions, which are so near to Principles, can be but few, and those *Mediums* not much more manifest than the Proposition it self. Therefore in these kind of Propositions the manner and management of the Demonstration is more exquisite, and requires an extraordinary meety on the Masters part clearly to demonstrate, and a greater attention in the Scholar, for the right understanding of the subtilty of such a demonstration.

Wherefore, some for want of due consideration have unjustly cavill'd at the demonstration of this 4th. Proposition, as being in a manner Mechanical. Whereas they have not rightly considered, what in the Application of magnitude to magnitude is Geometrical, or purely Mathematical, and what is Mechanical. To apply any Measure, as a Foot, Cubit, &c. to any other magnitude, as Carpenters, and such like Artificers do, or in general to adapt one magnitude to another, and then from their *visible and sensible Congruency* or *Incongruency*, to conclude their equality or inequality, This indeed is plainly Mechanical. But in the demonstration of this Proposition there is nothing of this kind, no use made either of Hands or Eyes. Only there is of two Triangles compared together an imaginary position of a point on a point, and a strait line on a strait line: and from this mental Application the demonstration takes its beginning, and by clear Ratiocination from one necessary Inference to another, proves the entire Congruency of the two Triangles, equally convincing

Cecos,

Cæcos, and *Oculatos*: for that here the Congruency is not enforced by any evidence of Sense; but only by an Intellectual demonstration. And the *Epharmosis*, or Application of one Triangle to the other, is also intellectual, and necessarily to be presupposed, in order to the proof of their *Exact* Congruency. Now if such an *Epharmosis* could justly be excepted against, then the 8th. Axiom of the equality of Congruous magnitudes, which even *Ramus* himself allows to be *maxime Geometricum*, were altogether useless in Geometry. For that there must in some manner an Application be conceived of some part of one magnitude, to some part of another magnitude, in the way of a Construction or *Apparatus* to the demonstration, before we can proceed to demonstrate the Congruency of all its parts, and from *Thence* to conclude the equality of the whole to the whole, by the 8th. Axiom.

Briefly then to determine in this Case.

Congruency is a natural Rule of equality; and the certainty of Congruency, if made from Sense, is Mechanical; if by a just Reasoning proved, it is truly Mathematical. Which distinction, if some of our Commentators had well observ'd, they would not in this matter have made such frivolous Objections against *Euclide*, nor committed such Paralogisms in their vain Attempts to amend his demonstrations of this kind.

PROPOSITION V.

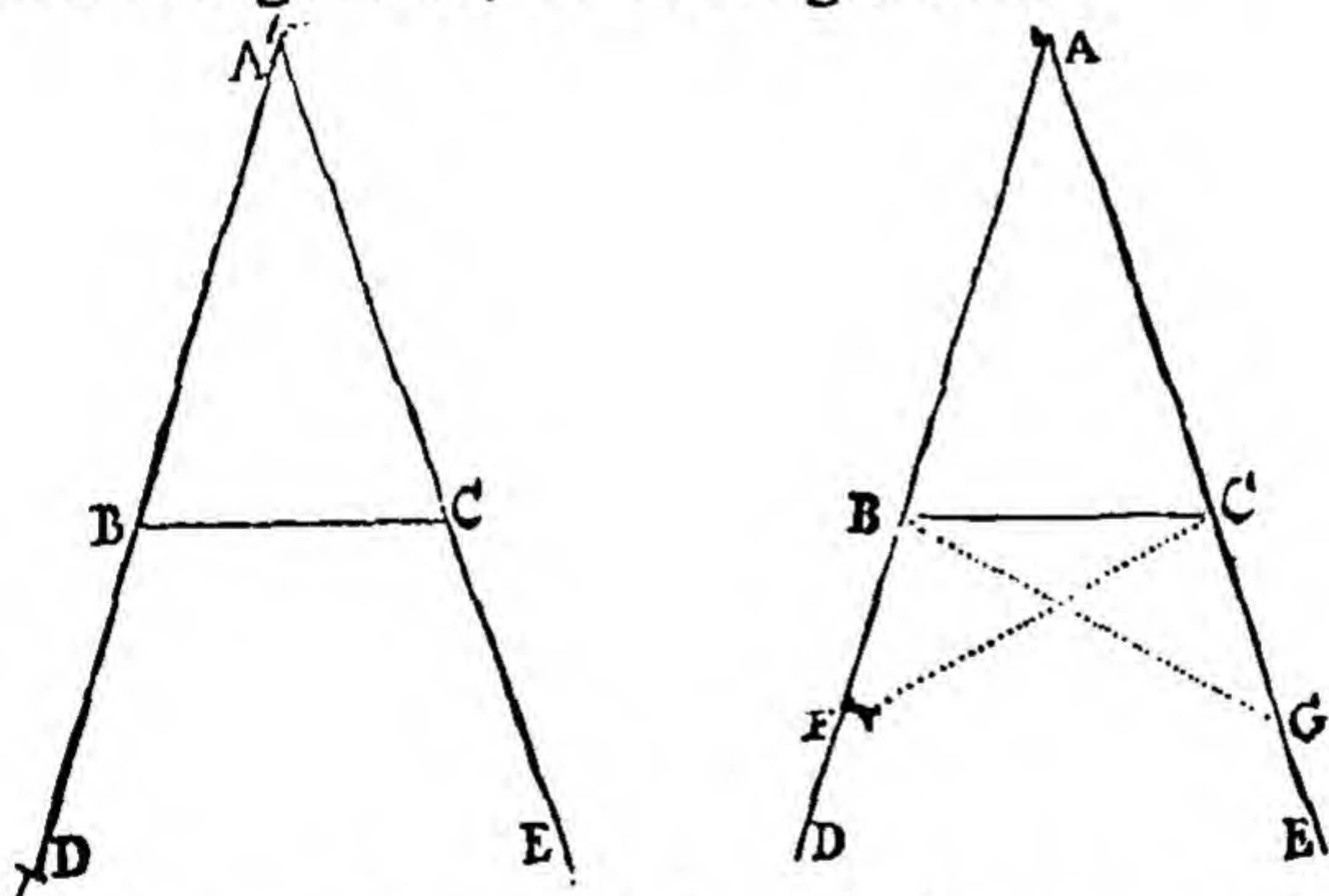
OF Equicrural Triangles, the angles at the base are equal to one another.

And the equal strait lines being produced, the angles under the base shall be equal to one another.

Let there be an Equicrural Triangle ABC , having the side AB equal to the side AC ; and to the strait lines AB , AC , let be continued directly the strait lines BD , CE , [by Post. 2.]

I say, that the angle ABC is equal to the angle ACB .

And also the angle CBD , to the angle BCE .



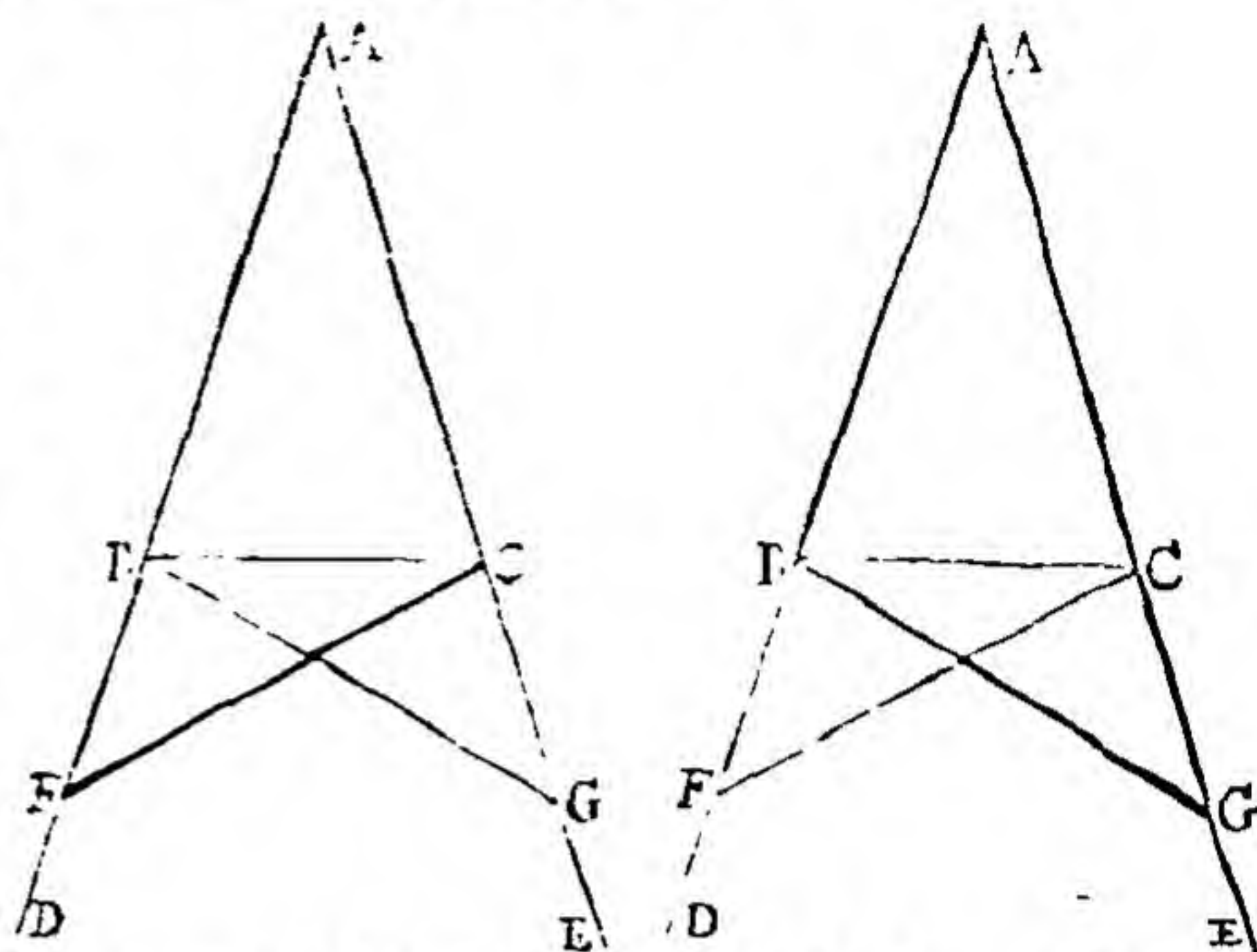
For in the line BD , let any point be taken, as F . Then from the greater AE , let be taken AG equal to AF the less, [by Prop. 3.] and let be joyned the strait lines FC , GB , [by Post. 1.]

Now forasmuch as AF is equal to AG [by Construction] and AB to AC ; [by Supposition]. Therefore there are two lines FA , AC , equal to the two lines GA , AB , each to each, and they contain a common angle FAG . Therefore [by Prop. 4.] the base FC is equal to the base GB ; and the Triangle AFC shall be equal to the Triangle AGB ,

$\angle AGB$, and the remaining angles shall be equal to the remaining angles, each to each, under which are subtended equal sides, namely the angle $\angle ACF$ to the angle $\angle ABG$, and the angle $\angle AFC$ to the angle $\angle AGB$.

And now because the whole line AF is [by *Construction*] equal to the whole line AG , of which AB is equal to AC , [by *Supposition*] Therefore the remainder BF is equal to the remainder CG , [Ax. 3.] But also it has been proved, that FC is equal to GB .

There are therefore the two lines BF , FC , equal to the two lines CG , GB , each to each, and the angle $\angle BFC$ equal to the angle $\angle CGB$, and BC is their common base; therefore the triangle BFC , shall be equal to the triangle CGB ; and the remaining angles shall be equal to the remaining angles, each to each, under which are subtended equal sides.



Wherefore the angle $\angle FBC$ is equal to the angle $\angle GCB$. And the angle $\angle BCF$ to the angle $\angle CBG$.

Now whereas the whole angle $\angle ABG$ has been proved equal to the whole angle $\angle ACF$, of which the angle $\angle CBG$ is equal to the angle $\angle BCF$.

Therefore the remaining angle $\angle ABC$ is equal to the remaining angle $\angle ACB$. And they are at the base of the Triangle ABC . *But also it has been proved, that the angle $\angle FBC$, is equal to the angle $\angle GCB$. And they are under the base.*

Therefore of Equicrural Triangles, the angles at the base are equal to one another.

And the equal strait lines being produced, the angles under the base shall be equal to one another. Which was to be demonstrated.

Corollary.

From hence 'tis manifest, that of Equilateral Triangles, the three angles are equal to one another.

For every side may in order be put for a base, and accordingly, as in an Equicrural Triangle, the three angles may be proved equal to one another.

This Proposition seems very difficult to young Geometricians, by reason of the cross interfering of the Triangles, which are compared to one another. We have therefore endeavoured to clear the matter by making separate Figures, in which the several Triangles compared together, may be more easily distinguished.

PROPOSITION VI.

If two Angles of a Triangle be equal to one another, then shall the sides subtended under the equal Angles be equal to one another.

Let

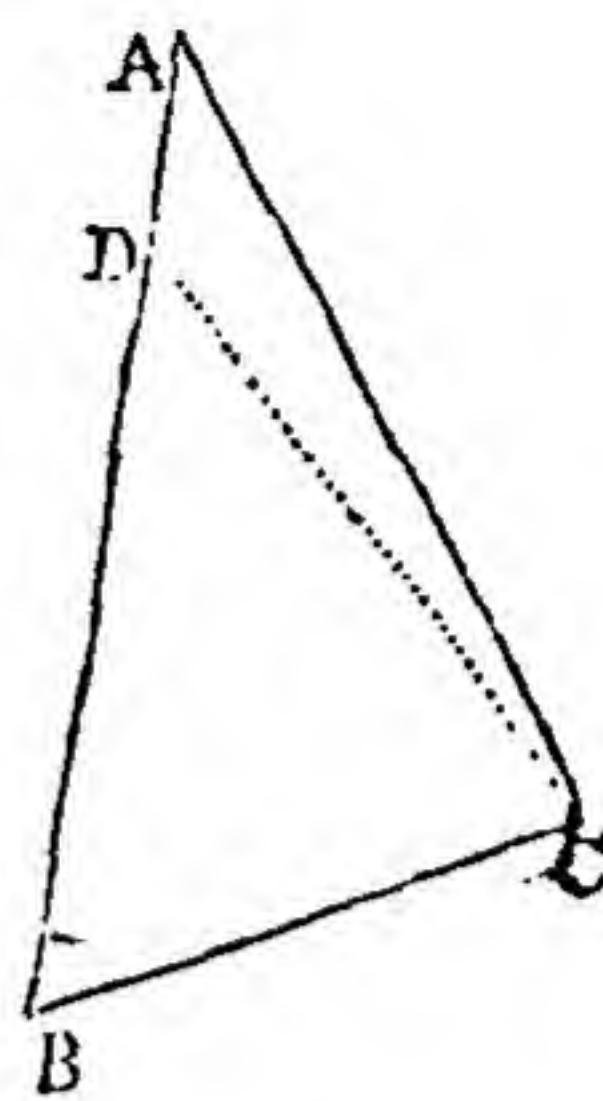
Let the Triangle be ABC , having the angle ABC equal to the angle ACB .

I say, that the side AC is equal to the side AB .

For if AC be unequal to AB , one of them is the greater. Let the greater be AB .

And from AB the greater, let be taken DB equal to AC the less, [by Prop. 3.] and let be joined DC .

Now forasmuch as DB is equal AC , and BC is common; therefore there are the two lines DB, BC , equal to the two lines AC, CB , each to each, and the angle DBC is equal to the angle ACB , [by Supposition] Therefore [by Prop. 4.] the base DC is equal to the base AB ; and the Triangle ABC shall be equal to the Triangle DCB , the greater to the less. Which is absurd.



Wherefore AB is not unequal to AC . Equal therefore it is.

If therefore two Angles of a Triangle be equal to one another, then shall the sides subtended under the equal angles be equal to one another. Which was to be Demonstrated.

Corollary.

From hence 'tis manifest, that if the three Angles of a Triangle be equal to one another, the Triangle shall be Equilateral.

ANNOTATIONS.

Of the Conversion of Geometrical Propositions.

This Sixth Proposition is the Converse of the Fifth.

One Proposition is said to be the Converse of another, when the Position of one is the Conclusion of the other; and the Conclusion of this is the Position of that. As the fifth Proposition puts two sides of a Triangle equal to one another, and thence concludes, that their opposite Angles are equal. So this sixth Proposition puts two Angles of a Triangle equal to one another, and thence concludes that their opposite sides are equal. Thus in the Conversion of Geometrical Propositions, the Positions and Conclusions are reciprocal.

The fifth Proposition might have been here entirely converted, I mean in that latter part also of the Angles under the Base: from whose equality the Triangle may likewise be demonstrated to be Equicrural, as *Proclus* has shewn. But this Conversion being useless, *Euclide* does omit. Or rather, if the Form of this demonstration be strictly examined, it will appear that the second part of the fifth Proposition, was not *Euclid's*. But that because *Euclide*, to prove the equality of the Angles at the Base, doth first prove the equality of the Angles under the Base; This might give an occasion to some one afterward, for the subjoining of this property of an Equicrural Triangle to *Euclid's* Original Proposition. And this conjecture is very probable, for that the equality of the Angles under the Base, is not made use of in any of the following Propositions. Now certainly *Euclide* never laid down an Elementary Proposition useless in any part thereof: nor ever put that for one part of the Proposition, which is only used as a Medium to prove the other.

Of the two kinds of *Analysis* used in Geometrical Propositions.

The demonstration of this sixth Proposition is much different from those before. All which were made from true and known Principles, to prove other Truths unknown in the common Synthetical Method, but contrarywise this Theorem is wholly demonstrated Analytically.

We have before discoursed of the two Methods of Composition and Resolution; and shewn the use of the Analytical Method in the Invention and Solution of *Euclid's* first Problem: Now here again, we have occasion to take more particular notice of a twofold consideration of Analysis, not in respect of its Nature and Method in the course of Reasoning, which remains the same in both; but of the different issue or event, that may arise from any Analysis by a just Ratiocination. For if the Analysis of a Supposition ends in an acknowledg'd Truth, then the thing supposed was true. But if the Analysis of a Supposition ends in a certain Untruth, then the thing supposed was false. Therefore from these different endings, True or False, there are given two denominations to *Analysis*. One is said to be *Constructive*, the other *Destructive*.

The Constructive Analysis is so called, for that ending in a manifest Truth, we may again upon this true foundation in the Synthetical Method, construct a demonstration of *that Thing*, from whose bare *Supposition* the Analysis took its beginning. And in these Cases alone the Methods of Resolution and Composition answer to one another, the Constructive Analysis having always a correspondent Synthesis: So that what Truths we find out by the help of the Analytical Method, the same when so found we teach to others in the Synthetical Method. This *Marinus Gethaldus* has shewn in his admirable Books *De Resolutione & Compositione Mathematica*. And Examples thereof we may find among the *Scholia* in the 13th. Element of *Euclid*.

The Destructive Analysis is so called, for that ending in an evident Falsity, it destroys the Supposition, from whence the Analysis began to argue.

Now unto the Destructive Analysis there cannot be made any return in the Synthetical Method, because upon Falsity, wherein this Analysis ends, that is, upon Nothing, or no Foundation, cannot be raised any Structure.

Synthesis therefore is but of one simple consideration and denomination, for that it can only argue from Principles of Truth laid down. As we must build upon some foundation, as well Speculatively, as Mechanically.

But Analysis, which may argue demonstratively from Suppositions either True or False, and so accordingly must rest at Principles of Truth or Falsity, is therefore from its different ending distinguished, and (as before said) denominated Constructive and Destructive. *Quia* SUPPOSITIONEM ponit Analysis, aut destruit. Not as *CLAVIUS* says, *Quia* PRINCIPIA ponit, aut destruit. For from what Supposition can the Principles be either confirmed or destroyed: But contrarily, these Principles must either confirm, or destroy the Supposition.

The Constructive Analysis is simply called Analysis. The Destructive Analysis is commonly called ἀπαγωγή εἰς ἀδύνατον, an *Abduction* or *Reduction to Impossibility*. Which way of Argumentation Geometricians thus make use of.

If a Proposition, altho' it be true, yet cannot be readily proved so to be: then They usually put, or *Suppose* its *Contradictory*, and disproving *This*, there necessarily follows the acknowledgement of the Proposition first laid down. For that of two Contradictory Propositions, if one be proved false, the other must be true, their being no Medium between Being and not Being.

So in this sixth Proposition are put two *Angles of a Triangle, equal to one another*: and upon this Position it is pronounced, that *Their subtending sides are also equal*. But now the equality of their subtending sides is not here directly proved. Only 'tis urged, that if the sides be not equal, then they must be unequal.

And now *supposing* the sides unequal, then *Euclid* demonstratively proceeds thereupon, and by gradual Consequences resolves *this Supposition* into a manifest impossibility, proving, That in a Triangle, if the sides subtending the equal Angles be unequal, then

then the whole shall be equal to its part: which is a manifest untruth; for that *The whole is greater than its part*, is a Principle of manifest Truth.

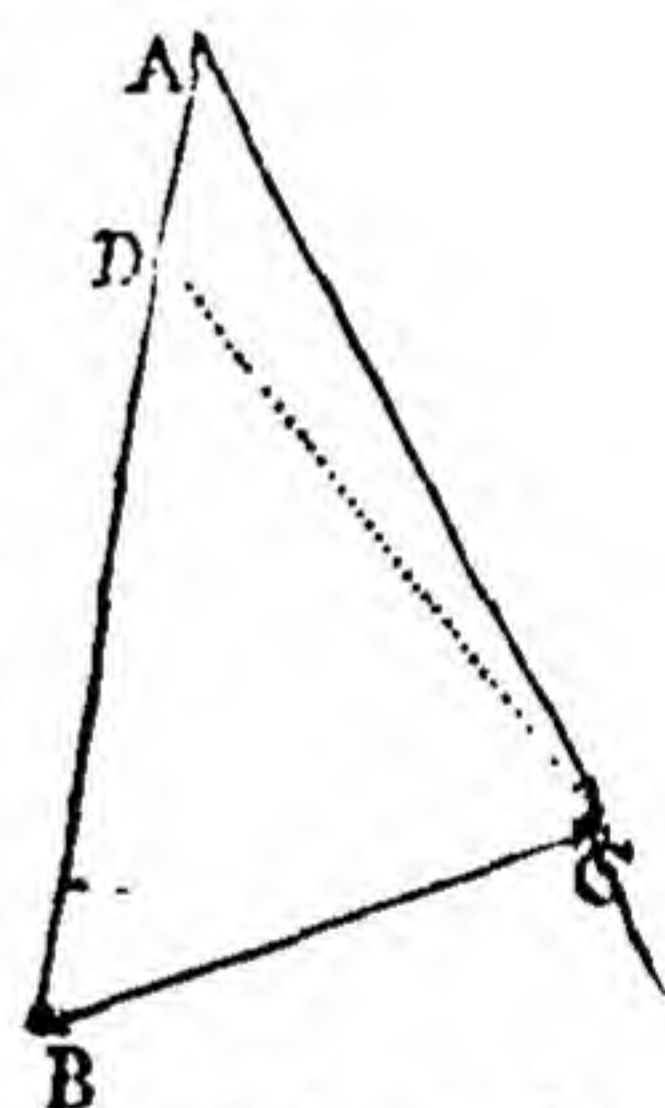
Therefore the *Supposition of the inequality of the two sides which subtend the two equal Angles of a Triangle*, being necessarily resolved into such an absurdity, was not true. For *Ex vero nil nisi verum*, From Truth no Untruth can follow.

Wherefore according to the sixth Proposition, Two Angles of a Triangle being equal, the sides subtended under the equal Angles, are therefore Logically concluded equal, for that it is demonstrated, that they cannot possibly be unequal. Equal therefore they are.

But in this Destructive Analysis, very often the Deduction to an impossibility is *Immediate*, and the absurdity forthwith seen, and urged against the false position without any further Argumentation. As before we may observe in the Close of the 4th. Prop. When we say, *If the base BC does not agree with the base DE, then two strait lines shall comprehend a space*, which Inference is immediate and contradicts the 10th. Axiom. Whereupon the absurdity is presently discovered; and therein that Supposition is immediately overthrown as untrue. And this is strictly to be called Deduction to Impossibility.

But now again, if the Proposition it self be not in the beginning of the demonstration, undertaken to be proved ostensibly, in some parts thereof, as the fourth Proposition is: but that in its stead, an Hypothesis contradictory to the Proposition, is put at the very first (as it is in this and the next Proposition, with many others) then the consequent absurdity does not immediately appear: but after intermediate Inferences which are made upon that Hypothesis *Ostensively*, step by step, we do fall at length into an Impossibility, and thereupon destroy the same Hypothesis, from whence we began to argue. And this is an *exact* Analysis: which to explain more signally, let us review the Analysis of this sixth Proposition, and particularly observe the several gradations of Argumentation, made upon the Supposition.

For now of the Triangle ABC if the sides AB, AC, be not allowed to be equal, notwithstanding that their opposite Angles at C and B were supposed to be equal: Then supposing them to be unequal, *Euclide* (by the 3^d. Proposition) takes from the greater AB a line BD equal to the less AC. Next (by the first Postulate) he draws a strait line DC; and so there is made the Triangle DCB. After this Construction he proves (by the 4th. Proposition) that the Triangles ABC, DCB, are equal to one another. Hitherto the demonstration goes from the Supposition fairly on *dénoués*. *Ostensively*, proving the equality of the Triangles ABC, DCB.



But here now after this *Ostensive* Proof, the next and immediate Result is, that one of these Triangles is but a part of the other; the Triangle DCB a part of the Triangle ABC, and yet upon the Supposition is justly demonstrated equal to the same. But this Equality and Inequality of the same things being inconsistent with Nature, must overthrow and null that Hypothesis, which by a positive demonstration, is resolved into this Impossibility; And therefore upon such an Evidence certain it is, that of the Triangle ABC, the sides AB, AC, cannot be unequal, and therefore equal.

But because there are some who allow not this Apagogical method of Argumentation to be so satisfactory, as the usual Synthetical manner of demonstration; therefore to clear this matter, we shall for the vindication of Geometricians, examine further the nature and force of all Demonstrations Mathematical and Philosophical: As we have done before concerning Definitions Mathematical and Philosophical.

Of Demonstrations Mathematical and Philosophical, as to Their several ends.

Every Man can reason no otherwise, than from the Ideas of outward things, according

ing as he receives them, and makes reflections on them. When the Ideas are the same in several persons, then they cannot but reasonably agree in their judgments: when they are different, as being differently received, then Men must disagree; tho' both may argue well, according to their own Ideas and Sentiments of those things. Besides, for want of due Observations, both may happen to be in the wrong, and that the very Truth is much otherwise. For we can only affirm, or deny, as things are apprehended by us: whether the things do in themselves agree, or not agree, one with the other.

Now many things there are which so clearly and distinctly present themselves to our common understanding, that they beget in us a ready and certain judgment of their Truth, or Falsity: and do so much urge our Natural reason upon their Self-evidence, that without any doubting we affirm, or deny. When with our Affirmation, or Negation, there is accordingly a real agreement, or disagreement between the things themselves, then is this Perception of ours true Knowledge, and an immediate conformity of the Mind of Man to the apparent Truth of things. These are then taken by us for Principles of knowledge, and so made use of in our common discoursings.

But again, numberless other things there be, whose Truths are less apparent, and with search and labour of Mind to be discovered; for which, we must, as in a Labyrinth, use a clew of Silk to find them out. This is our innate Power of reasoning, or Natural Logic, whereby every Man from the simple Apprehension of things, and those clear and independent Sentiments of Humane Understanding, is enabled to discourse, to hunt out and judge so exactly, as to acquire in many matters an assurance of Truth, and to rest therein. For tho' we are not born with Irrefragable Propositions in our Brains, as with Eyes in our Head; yet we come into the World with the advantage of such a Mind, that from those Eyes, and our other Senses, by the Natural Sagacity and Power of inferring one thing from an other, we can and do advance our Thoughts in the knowledge of things, far above all whatsoever our Sensations can reach unto.

But for a farther improvement of this our common Reason, *Aristotle* has, with great subtilty of Thought fram'd the Instrument of Instruments; and reduced the loose and familiar Discoursings of Men into an infallible Art, both for the Form of Argumentation by Syllogisms, as a Touch-stone to discover how truly or fallaciously they are made; and also for the matter thereof: having gathered all sorts of Arguments under certain Heads, and digested them into their proper Classes, called *Topics*. In which he shews what kind of Arguments are only *Probable*, and beget in us that which is called *OPINION*, the mischievous Mother of Dispute and Brangling. What again are *Demonstrative*, and infallibly Convincing, and do give us a Certainty of Knowledge. Of which there are two Degrees.

For this Knowledge, or Cognition is either absolutely perfect, and fully satisfying our Intellect; or else in some degree is less satisfactory: I mean not in the certainty of the Truth; but only in the manner of our Knowledge of this Truth. For besides the *certain Knowledge of a thing to be*; we naturally desire to know moreover, *Why it is so*, from the immediate constituent cause of such a Being.

To demonstrate any thing *ὑποκαταστατικῶς*, Scientifically, from its immediate Essential Cause, is the utmost perfection of humane Knowledge, if we could arrive unto it in a thorough and Methodical Contemplation of any matter.

This Scientific demonstration called by *Aristotle* *ἀπὸ τοῦ αἰτιώδους*, that is *From the very immediate Cause by which a thing is made to be what it is*, the Philosophers pretend unto: but with how little success and satisfaction to our Understanding, their performances are advanced, does evidently appear from the so many different Sects of Jarring Philosophers.

Whereas the Geometrical leaves nothing disputable, or uncertain; tho' his demonstration is very seldom *ἀπὸ τοῦ αἰτιώδους*, at which he aims not with Philosophers, but thinks it sufficient so far to satisfy our understanding, as undeniably to demonstrate the *Quod sit*, *The thing to be*; tho' it appears not in the demonstration of that Being, *Why it is*, or *How it comes to be*. If he had undertook this business, the

the Geometrician would have become as doubtful and miserable a Disputer, as the Philosopher.

But yet we may observe, that the Geometrician does sometimes give a demonstration $\Delta\iota\sigma\tau\iota$: as in the first Proposition. For there *Euclide* proves the Triangle to be Equilateral, because the sides thereof consist of the Rays of the same Circles: which is the immediate cause of their equality. Neither indeed could this first demonstration be otherwise than Scientifical, or $\Delta\iota\sigma\tau\iota$; for that an Equilateral Triangle being the Figure next in nature to a Circle, in respect of its simplicity and uniformity, so that there is no other medium possible to intervene between them; therefore it could have no other production, nor any other proof thereof, than from a Circle. The Demonstrations likewise of the two following Propositions are of the same kind from the same cause.

But generally the Geometrician takes $\tau\acute{o}\ \epsilon\sigma\tau\iota$, evidence and certainty of being to be abundantly sufficient, both for our pleasure and contentment in these Speculations, wherein we have an indubitable knowledge of some Truths in the secret and admirable properties of Magnitudes and Numbers. Altho' the famous *French* Logician, blames much the Geometricians, for being thus defective in their demonstrations: which he would have to be all Scientifical and $\Delta\iota\sigma\tau\iota$, and that the Elements ought to have been so methodized. And this he despairs not of; but that at some time it may be accomplished. This is indeed the fullness of Science, which only can make perfect the state of humane Nature: But such a knowledge it is, that seems beyond the reach of any finite Being. A chain of things (to speak with *Homer*) fastned to the foot of *Jove*; the frame of whose Links is only known to Him the Maker. I wish therefore that this *French* Writer had shewed himself in this point, as great a Geometrician as he is a Logician in others; and given some Specimen of that Perfection, which he requires in Geometry. Of which I shall have but little hopes, till he can shew me how an Acorn comes to be an Oak, and not an Elm.

But for this matter the ancient Geometricians, well exercised in demonstrable Speculations, fully knew their own strength, and contented Themselves with the certainty of Truth, rather than to venture at the Causes of that Truth, which would be ever lyable to dispute.

Euclide therefore never attempted to order these Elementary Propositions, in a natural dependence on one another, as the causes of each others Truth. Yet are they so disposed, as to be sure Guides to lead us along infallibly from one Truth unto an other, and in that order serve to prove indubitably a Thing to be, from the Evidence and Force of some thing before acknowledged, tho' not as a Cause of its Being; yet as a Cause of our necessary Assent to such a Being.

Demonstration $\tau\acute{o}\ \Delta\iota\sigma\tau\iota$, can be but one in any matter, for that of the same thing there is but one only Essential and Immediate Cause.

Demonstration $\tau\acute{o}\ \epsilon\sigma\tau\iota$, may be very various, some short and clear, others more puzzel'd and wandring about; according to the Sagacity of those, who endeavour to find out the readiest and aptest Mediums to prove the matter proposed. So that many times of the same Geometrical Proposition, there may be several demonstrations: And even a total change (as many have made to no purpose) of *Euclid's* Order in these Elements.

To conclude therefore concerning the Forms of Geometrical demonstrations. Forasmuch as $\tau\acute{o}\ \epsilon\sigma\tau\iota$ is what generally the Geometrician uses, both in the Synthetical and Analytical Method, and that in either way the proof of a Thing to be, is made no more Scientifically or $\Delta\iota\sigma\tau\iota$, than the proof of a Thing not to be, we are not much to prefer the one before the other; but may as the matter requires, for facility and brevity, make use of either; The course of Argumentation in both being equally Ostensive, and the Mediums of the same Nature and Force, that is, Propositions already demonstrated, or allowed Principles. 'Tis only walking through them forward or backward, sometimes from known Truths to prove an undiscovered Truth in the Synthetical Method: And sometimes by the same known Truths to disprove an undiscovered Untruth in an Analytical Deduction to Impossibility. A form of Argumentation most proper to overthrow a false Position: and thereby to establish the contradictory Proposition for a certain Truth. This

This then is the glory of the Geometrician, to demonstrate upon clear and unquestionable grounds, either Synthetically or Analytically. And besides, it is his wisdom not to adventure with the Philosopher, at a natural and necessary Series of things, from the immediate Causes to their immediate Effects, in which attempt all Philosophers have hitherto failed; but by irresistible reasoning exactly to perform what he undertakes, that is, to have such a Mastery over our Intellect as to convince. Insomuch that every Man shall in reason submit, and as readily yield his Assent, as that he knows he thinks: And before all other Sciences, this is the power and preference of Geometry.

PROPOSITION VII.

ON the same strait line, to the same two strait lines cannot be constituted two other strait lines, equal each to each, at another and another point, both points seated the same way, and the other two lines having the same ends with the two first lines.

For if it be possible, on the strait line AB , to the two strait lines AC, CB , let two other strait lines AD, DB , equal each to each, be constituted at another and another point, as at c and D , the points c, D , seated † the same way: And the lines AD, DB , having the same ends A, B , with the two first lines AC, CB . So that CA be equal DA : both having the same end A ; also CB be equal to DB : both having the same end B .

And now let be joyned CD .

First then in the Triangle ACD .

Forasmuch as AC is equal to AD [by Supposition] therefore the angle ACD is equal to the angle ADC [Prop. 5.]

*But the angle ACD is greater then the angle BCD , a part of the same angle ACD . Wherefore the angle ADC (equal to ACD) is also greater than the angle BCD : * And therefore the angle BDC , being likewise greater than its part the angle ADC , is MUCH GREATER than the same angle BCD .*

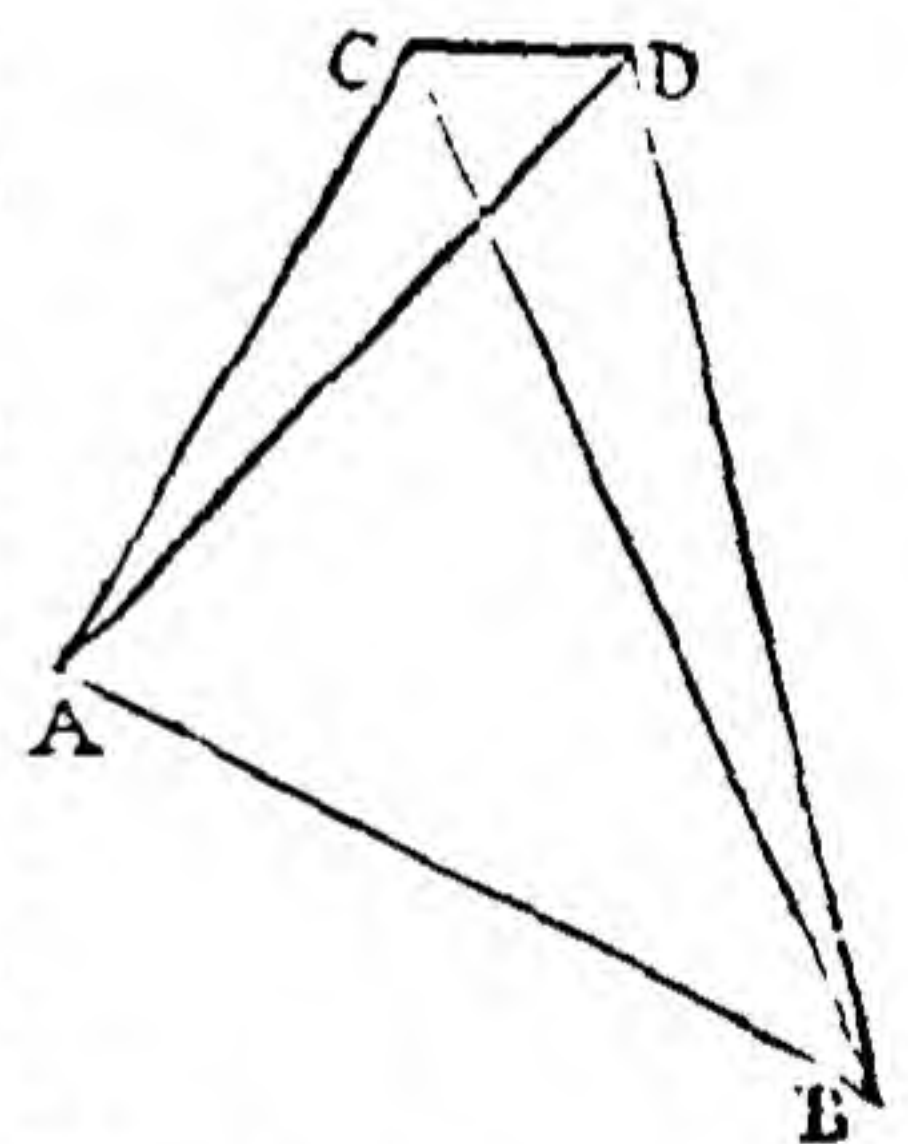
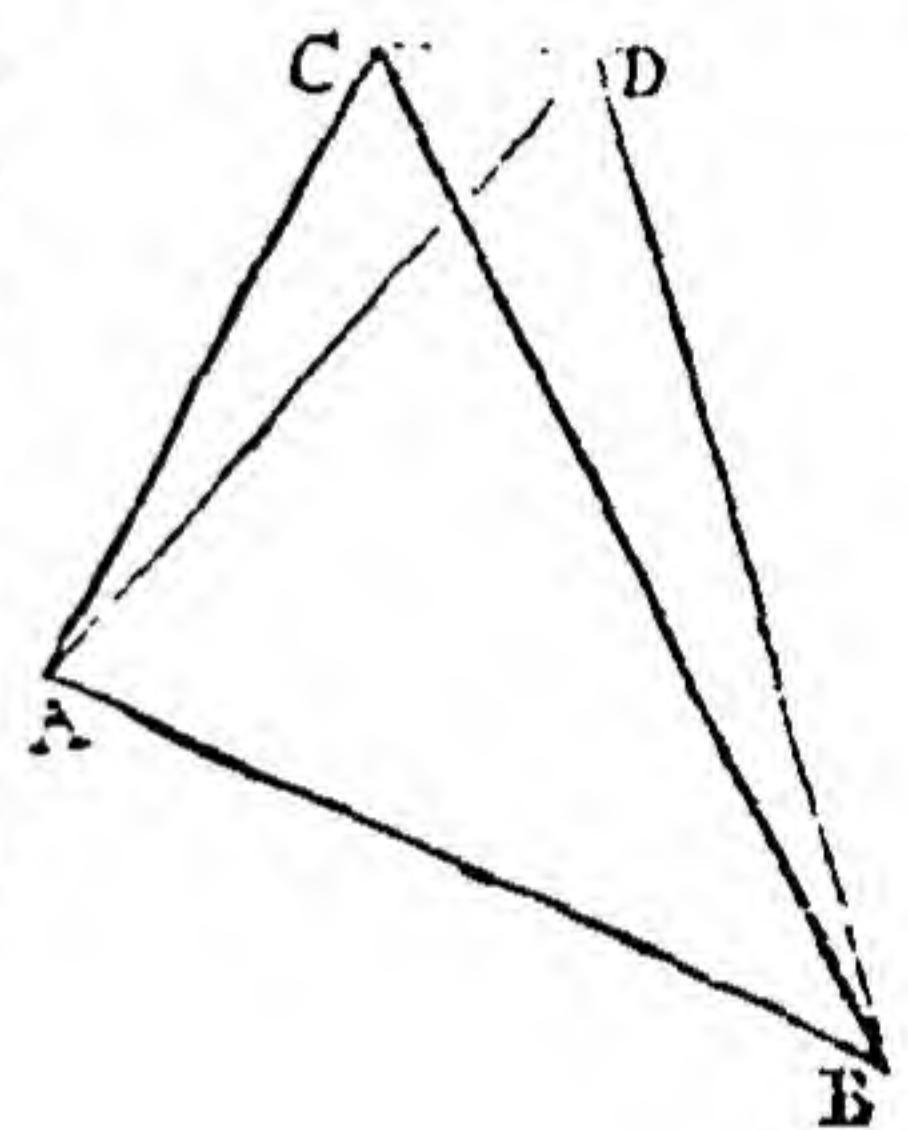
But again in the Triangle BCD .

Because BC is equal to BD [by Supposition] therefore the angle BDC , is equal to the angle BCD . [Prop. 5.]

* But the angle BDC has been now proved MUCH GREATER than the angle BCD . And it is impossible *to be equal and greater than the same*.

Therefore on the same strait line to the same two strait lines, cannot be constituted two other strait lines, equal each to each, at another and another point, both points seated the same way, and the other two lines having the same ends with the two first lines. Which was to be demonstrated.

† That is, on the same side of the line AB . For if the point c lyes
on



on one side of AB , and the point D on the contrary, then two equal lines may be constituted at those points, and have the same ends with the two first lines. As may be easily conceiv'd, if we imagine the Triangle ADB , to be turned over on the other side of the line AB .

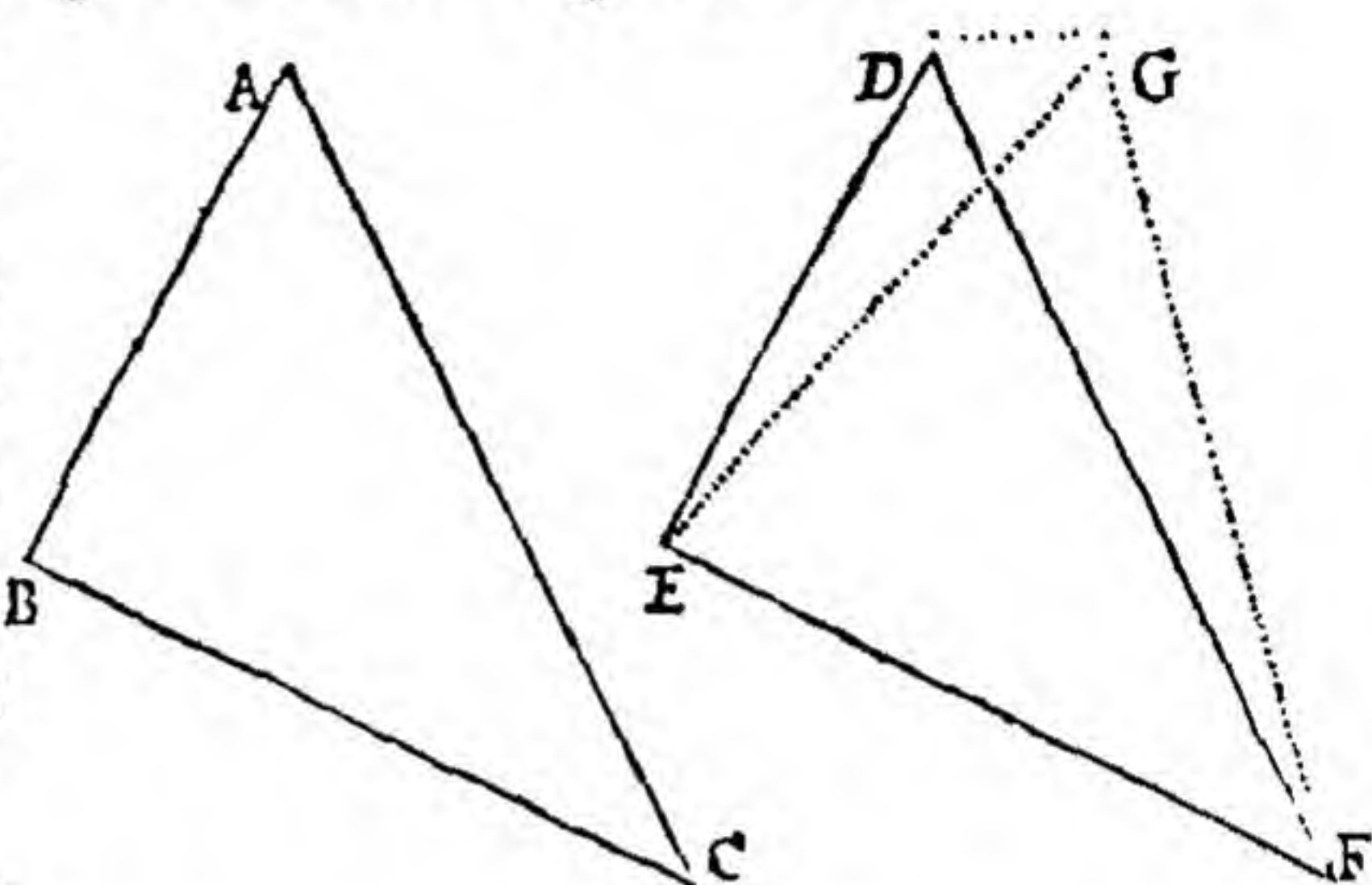
PROPOSITION VIII.

I*F two Triangles have two sides equal to two sides, each to each. And have also the base equal to the base; then shall they have the angle equal to the angle, contained by the equal lines.*

Let the two Triangles be ABC , DEF , having the two sides AB , AC , equal to the two sides DE , DF , each to each, that is, AB to DE , and AC to DF . And let them also have the base BC equal to the base EF .

I say, that the angle BAC is equal to the angle EDF .

For the Triangle ABC being apply'd to the Triangle DEF , and the point B put on the point E , and the strait line BC , on the strait line EF ; then the point C shall agree with the point F ; for that BC is equal to EF .



Now BC agreeing with EF ; BA , AC , shall also agree with ED , DF . For if the base BC shall agree with the base EF , and the sides BA , AC , do not agree with the sides ED , DF , but change their situation, as EG , GF .

Then on the same strait line to the same two strait lines, shall be constituted the same way, two other strait lines, equal each to each, at another and another point, having the same ends. But they cannot be so constituted. [by Prop. 7.]

Wherefore the base BC agreeing with the base EF , the sides BA , AC , shall not disagree with the sides ED , DF ; therefore they shall agree.

So that also the angle BAC shall agree with the angle EDF , and therefore shall be equal to it. [Ax. 8.]

If therefore two Triangles have two sides equal to two sides, each to each, and have also the base equal to the base, then shall they have the angle equal to the angle, contained by the equal lines. Which was to be demonstrated.

Advertisement.

Observe here and elsewhere, a constant propriety of Speech among the *Greek* Geometricians, that the same strait lines when they relate to a Figure, are ever called Sides, *πλάγαι*, but when referr'd to an angle contained by those sides, then they retain, or resume the appellation of strait lines, *εὐθειῶν γραμμῶν*, not *πλάγῶν*.

ANNOTATIONS.

This eighth Proposition is in effect the Converse of the fourth, in which two sides of a Triangle with the contained angle, were supposed equal to the like parts of an other

other Triangle; and from thence was proved the equality of their bases, &c. Now here two sides of a Triangle with the base, are supposed equal to the like parts of an other Triangle, and from thence is proved the equality of the angles contained by the equal sides. And to compleat the Converse, *Euclide* might have gone on, and said, also the whole Triangle is equal to the whole Triangle, &c. as in the 4th. Proposition. But this evidently following of it self, needed not to be repeated.

The demonstration of this Proposition, like as that of the 4th. depends upon the eighth Axiom of Congruous Magnitudes. And here also in this demonstration, as before in the fourth, may be clearly observed the difference between a Geometrical and a Mechanical Congruency; the Mechanical being manifested only by Sense, and the Geometrical only by the force of Reason. As in this 8th. Proposition the Triangles ABC , DEF , are proved to be Congruous, not from an evidence of Sense; but from the 7th. Proposition; which is a Theorem most rationally demonstrated: and indeed inserted among these Elementary Propositions chiefly for that purpose, there being hereafter no further use made thereof.

In the fifth and sixth Propositions, the parts of a single Triangle are compared to one another: First, in the fifth from two equal sides is proved an equality of their opposite angles: Then in the sixth, from two equal angles, is demonstrated an equality of their subtending sides.

In the fourth and eighth Propositions, are compared two Triangles to one another: and from three equal parts given in each Triangle, is demonstrated an equality between the two Triangles in the whole, and in every remaining part.

These four Propositions are the Fundamental Theorems of the Elements: And the Ground upon which they stand is the Axiom of Congruency; which Mathematical Congruency ought therefore to be rightly understood, according as we have before declared.

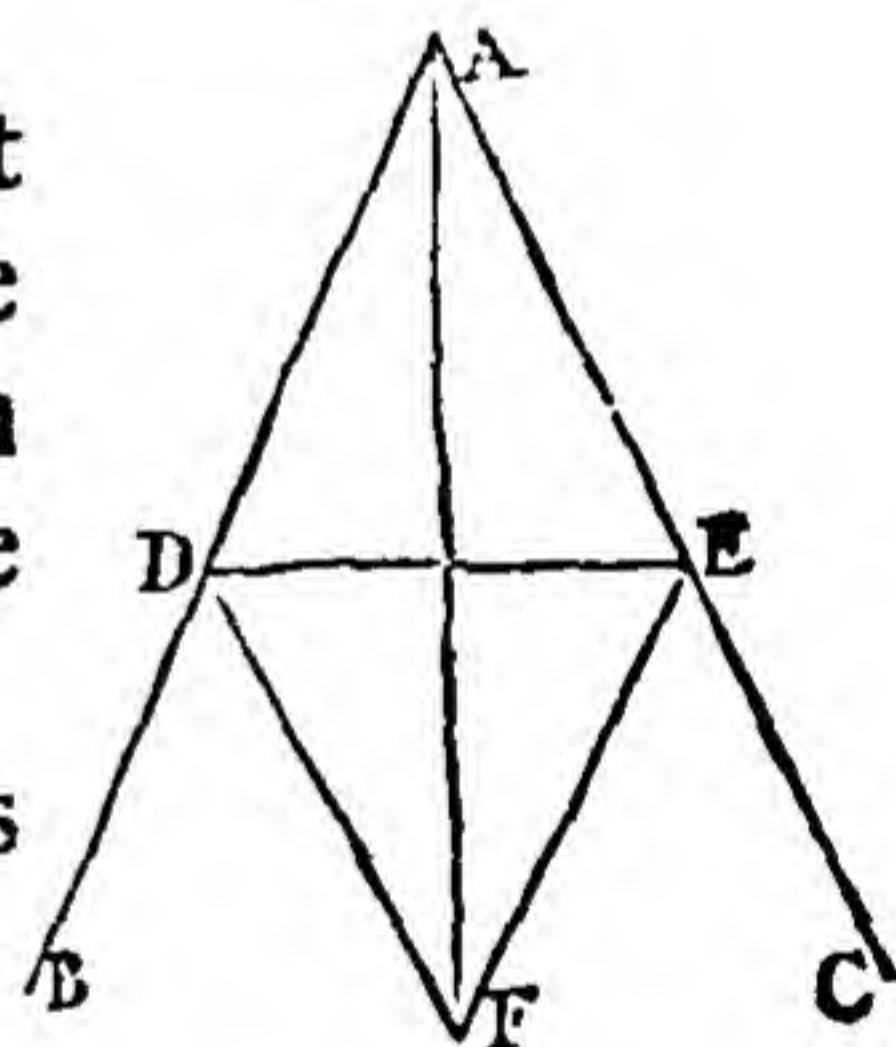
PROPOSITION IX.

TO cut a given strait-lin'd angle into halves.

Let the given strait lin'd angle be BAC .

It is required to cut the same into halves.

Let there be taken in the line AB , any point as D . And from the line AC , let the line AE be taken off equal to AD , and draw DE . Then on DE let be constituted an Equilateral Triangle DEF : and draw AF .



I say, that the angle BAC is cut into halves by the strait line AF .

For because AD is equal AE , and AF common, therefore there are the two lines DA , AF , equal to the two lines EA , AF , each to each; and the base DF , is equal to the base EF , therefore the angle DAF , is equal to the angle EAF . [Prop. 8.]

Wherefore the given strait-lin'd angle BAC , is cut into halves by the strait line AF . Which was to be done.

The Practice.

From the point A at any distance whatever in the lines AB , AC , take AD equal to AE ; then from the points D and E , with the same opening of the Compasses, let be described two Arches intersecting each other, suppose at the point F , having drawn the strait line AF , the angle BAC is divided into two equal parts.

PROPO-

PROPOSITION X.

TO cut a given finite strait line into halves.

Let the finite strait line given be CB .

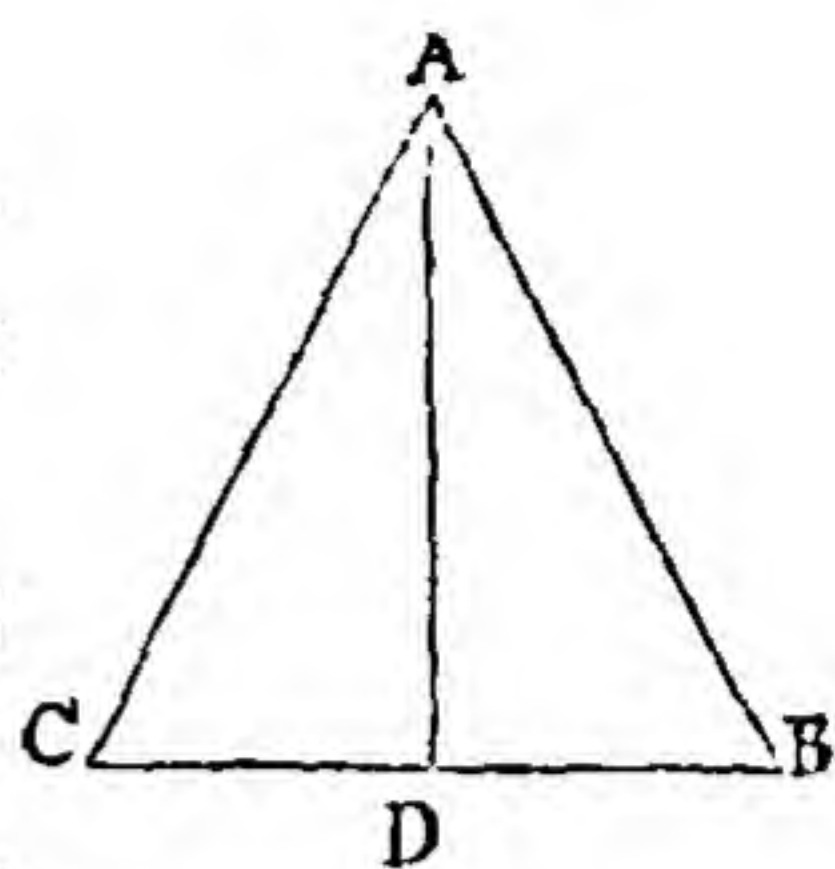
It is required to cut CB into halves.

Let there be constituted upon CB an Equilateral Triangle CAB , and let the angle CAB be cut into halves, by the line AD .

I say, that the strait line CB is cut into halves in the point D .

For because CA is equal to AB , and AD common, therefore there are the two lines CA , AD , equal to the two lines BA , AD , each to each, and the angle CAD , is equal to the angle BAD : Therefore the base CD , is equal to the base DB .

Wherefore the given finite strait line CB , is cut into halves in the point D . Which was to be done.



The Practice.

Opening the Compasses to any distance greater than half the line CB , from the points C and B , with the same opening of the Compasses let be described two Arches intersecting one another on each side the line CB , a strait line drawn betwixt the points of intersection will divide the line into two equal parts.

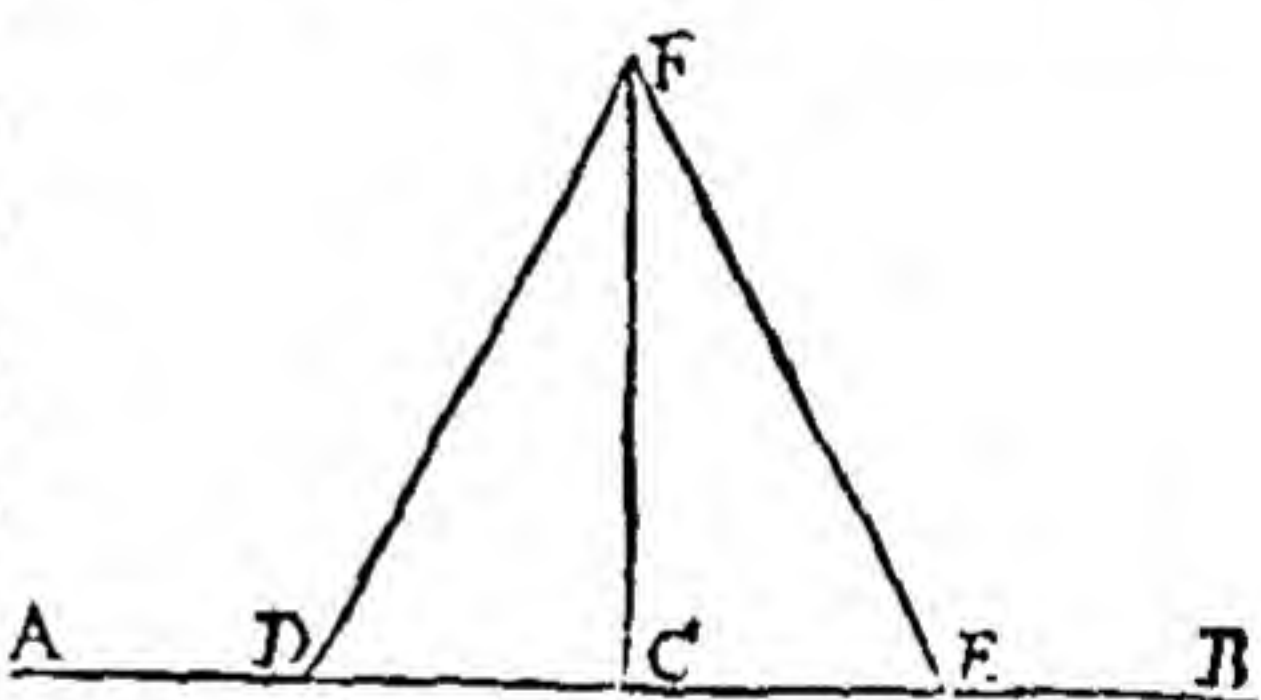
PROPOSITION XI.

TO a given strait line, from a point given in the same to draw a strait line at Right angles.

Let the given strait line be AB , and the point given in the same be c .

It is required from the point c unto AB to draw a strait line at Right angles.

Let there be taken in the line AC any point as D , and to CD let there be put an equal line CE : Then on DE let be constituted an Equilateral Triangle FDE . And draw the line CF .



I say, that to the given strait line AB from the point c given in the same, is drawn at Right angles the strait line CF .

For because CD is equal to CE , and CF common; therefore there are the two lines CD , CF , equal to the two lines CE , CF , each to each, and the base DF is equal to the base EF ; wherefore the angle DCF is equal to the angle ECF , and these are consequent angles. But when a strait line standing upon a strait line, makes the consequent angles equal to one another, then each of those equal angles is a Right angle. Wherefore each of the angles DCF , ECF , is a Right angle.

Therefore to the given strait line AB , from the point c given in the same, is drawn at Right angles the strait line CF . Which was to be done.

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The Practice.

On each side of the given point C , in the line AB , take CD and CE equal to one another, then from the points D and E , with any opening of the Compasses greater than CD or CE , let be described two Arches intersecting each other, suppose at the point F , the strait line EC drawn betwixt the points F and C , shall be Perpendicular to the line AB .

This is the practice with Ruler and Compasses, but the readiest way of drawing Perpendiculars both in this and the next Proposition, is by a Square.

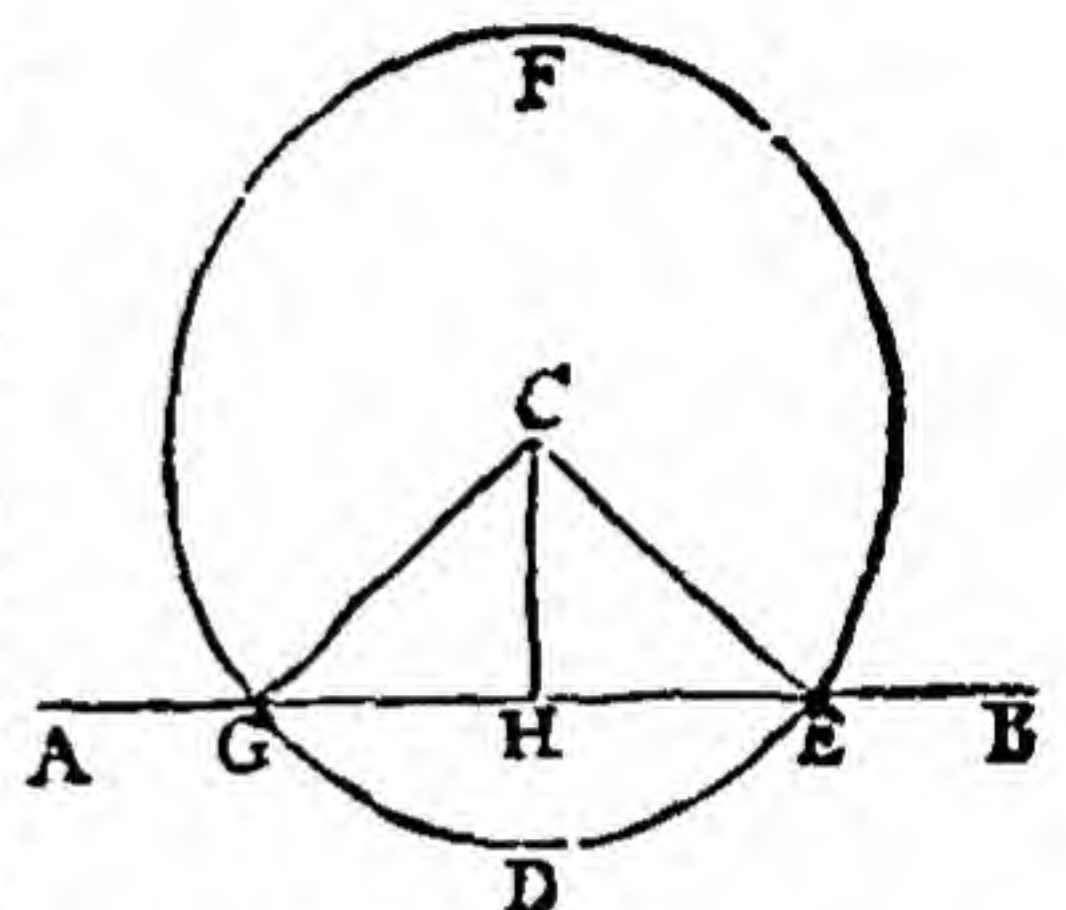
PROPOSITION XII.

TO a given infinite strait line. from a given point which is not in the same line, to draw a Perpendicular strait line.

Let the given infinite strait line be AB , and let the given point which is not in the same line be c .

It is required to AB , from the point c to draw a perpendicular strait line.

Let there be taken on the other side of the line AB , any point as D . Now the Center c , and the distance CD , let be described a Circle EFG . Then let the line EG be cut into halves in the point H , and let be joyned CG , CH , CE .



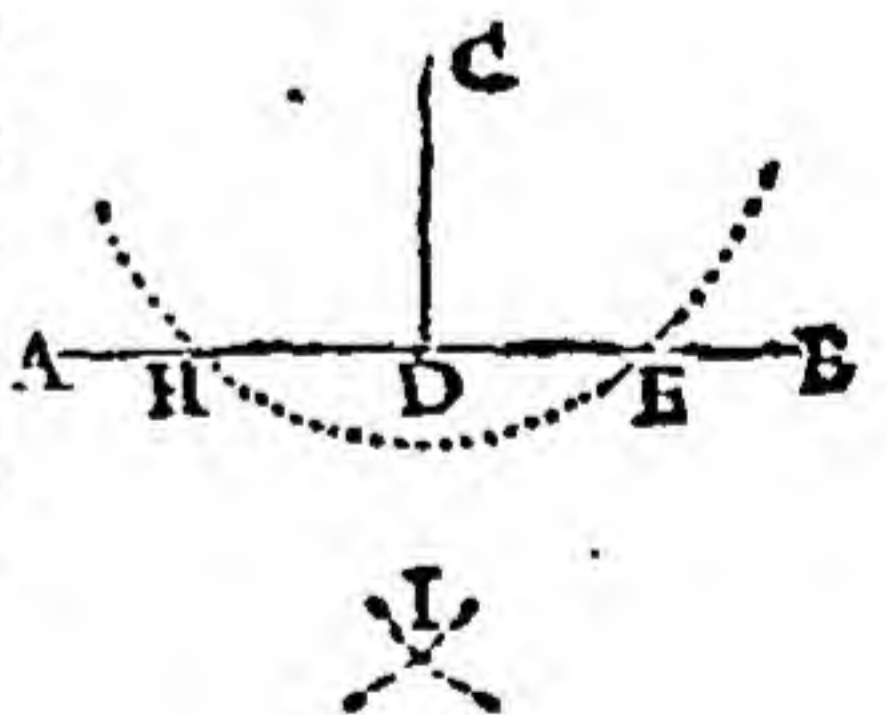
I say, that to AB , from the point c is drawn a perpendicular CH .

For because GH is equal to HE , and HC common; therefore there are the two lines GH , HC , equal to the two lines EH , HC , each to each; and the base CG is equal to the base CE : wherefore the angle CHG , is equal to the angle CHE , and these are consequent angles. But when a strait line standing upon a strait line makes the consequent angles equal to one another: then each of those equal angles is a Right angle, and the standing strait line is called a perpendicular to that, upon which it stands.

Therefore to the given infinite strait line AB from the given point c , which is not in the same line, is drawn the perpendicular CH . Which was to be done.

The Practice.

From the point C at any distance greater than the nearest CD , let be described an Arch, cutting the line AB at the points H and E , then from the points H and E , with the same opening of the Compasses, let be described two Arches intersecting each other at the point I , the Ruler being laid to the points I and C draw CD , it shall be perpendicular to AB from the point C .



PROPOSITION XIII.

IF a strait line standing any ways upon a strait line makes angles, it shall make either two Right angles, or angles equal to two Right.

For let a strait line AB standing upon a strait line CD , make angles,

gles, as CBA, ABD . I say, that the angles CBA, ABD , are either two Right angles, or equal to two Right.

For if the angle CBA , be equal to the angle ABD , then are they two Right angles, [by the 10th. Def]

But if not, let from the point B , be drawn BE at right angles to CD . Wherefore CBE, EBD , are two Right angles.

Now whereas the angle CBE , is equal to the two angles CBA, ABE , let the angle EBD be added in common. Wherefore the angles CBE, EBD , are equal to the three angles CBA, ABE, EBD .

Again, whereas the angle DBA , is equal to the two angles DBE, EBA , let the angle ABC, D be added in common. Wherefore the angles DBA, ABC , are equal to the three angles DBE, EBA, ABC .

But the angles CBE, EBD , were proved equal to the same three angles; And things equal to one and the same, are equal to one another. Therefore also the angles CBE, EBD , are equal to the angles DBA, ABC .

But the angles CBE, EBD , are two Right angles, wherefore also the angles DBA, ABC , are equal to two Right angles. Therefore if a strait line standing any ways upon a strait line makes angles it shall make either two Right angles, or angles equal to two Right. Which was to be demonstrated.

PROPOSITION XIV.

If to a strait line, and to a point in the same, two strait lines not lying the same way, do make the consequent angles equal to two Right angles, Those strait lines shall be directly placed to one another.

For to the strait line AB , and to a point in the same B , let two strait lines BC, BD , not lying the same way, make the consequent angles ABC, ABD , equal to two Right angles. I say, that BD is directly placed to BC .

For if BD be not directly placed to BC ; let BE be directly placed to BC .

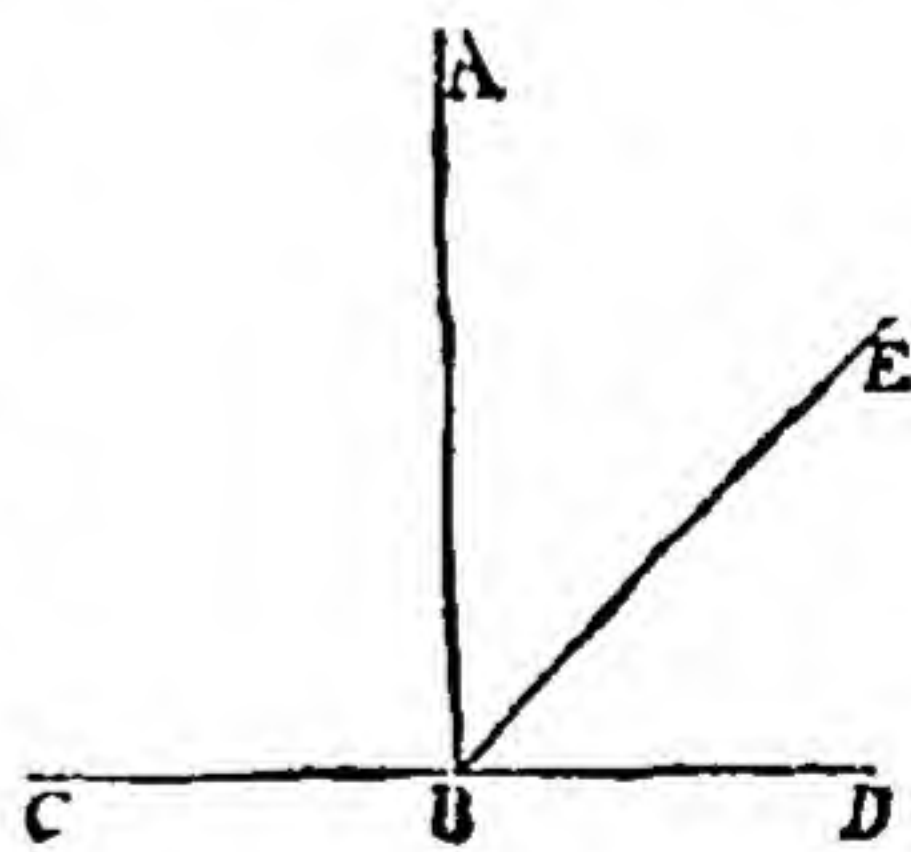
Now forasmuch as the strait line AB , stands upon the strait line CBE : therefore the angles ABC, ABE , are equal to two Right angles: [Prop. 13.]

But also the angles ABC, ABD , are equal to two Right angles [by Supposition] wherefore the angles CBA, ABE , are equal to the angles CBA, ABD .

Let the common angle CBA be taken away. Therefore the remaining angle ABE , is equal to the remaining angle ABD , the less to the greater: which is impossible. Therefore BE is not directly placed to BC .

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In like manner may we prove, that there is not any other line besides BD . Therefore BC is directly placed to BD .

If therefore to a strait line, and to a point in the same, two strait lines not lying the same way, do make the consequent angles equal to two Right angles, Those strait lines shall be directly placed one to the other. Which was to be demonstrated.

PROPOSITION XV.

If two strait lines cut each other, they shall make the Vertical angles equal to one another.

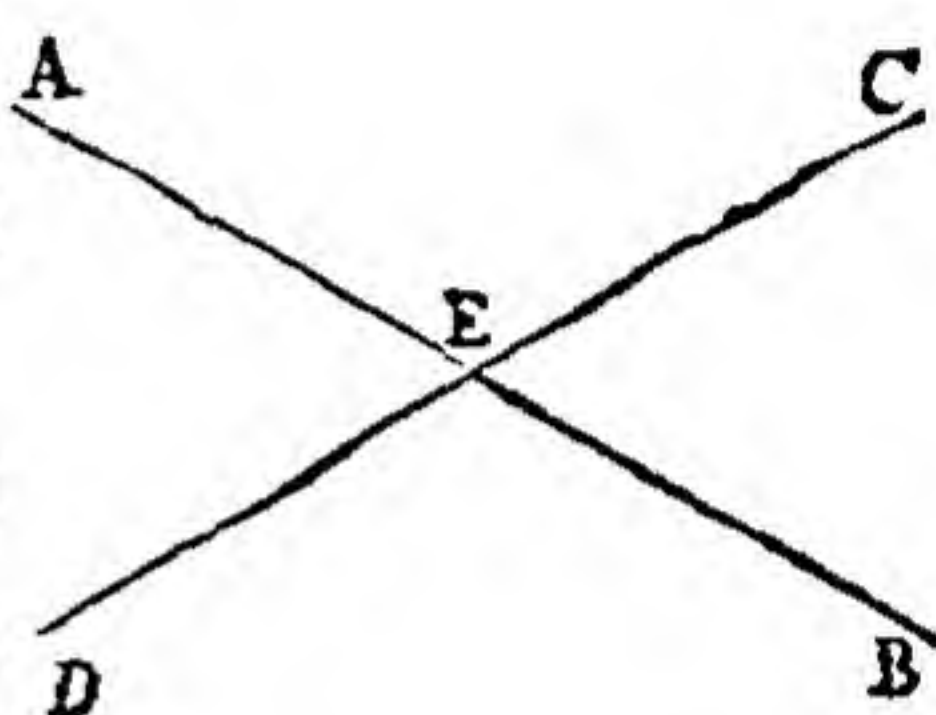
For let the two strait lines AB , CD , cut each other in the point E .

I say, that the angle AEC , is equal to the angle DEB : and the angle CEB , is equal to the angle AED .

Forasmuch as the strait line AE , stands upon the strait line CD , making the angles CEA , AED , therefore the angles CEA , AED , are equal to two Right angles.

Again, because the strait line DE stands upon the strait line AB , making the angles AED , DEB , therefore the angles AED , DEB , are equal to two Right angles.

But the angles CEA , AED , were proved equal to two Right angles. Wherefore the angles CEA , AED , are equal to the angles AED , DEB : let the common angle AED be taken away; then the remaining angle CEA , is equal to the remaining angle DEB . In like manner it may be proved, that the angles CEB , DEA , are equal. If therefore two strait lines cut each other, they shall make the Vertical angles equal to one another. Which was to be demonstrated.



Corollary.

From hence 'tis manifest, that if strait lines, how many soever, cut one another in the same point, they shall make the angles at the section equal to four Right angles.

PROPOSITION XVI.

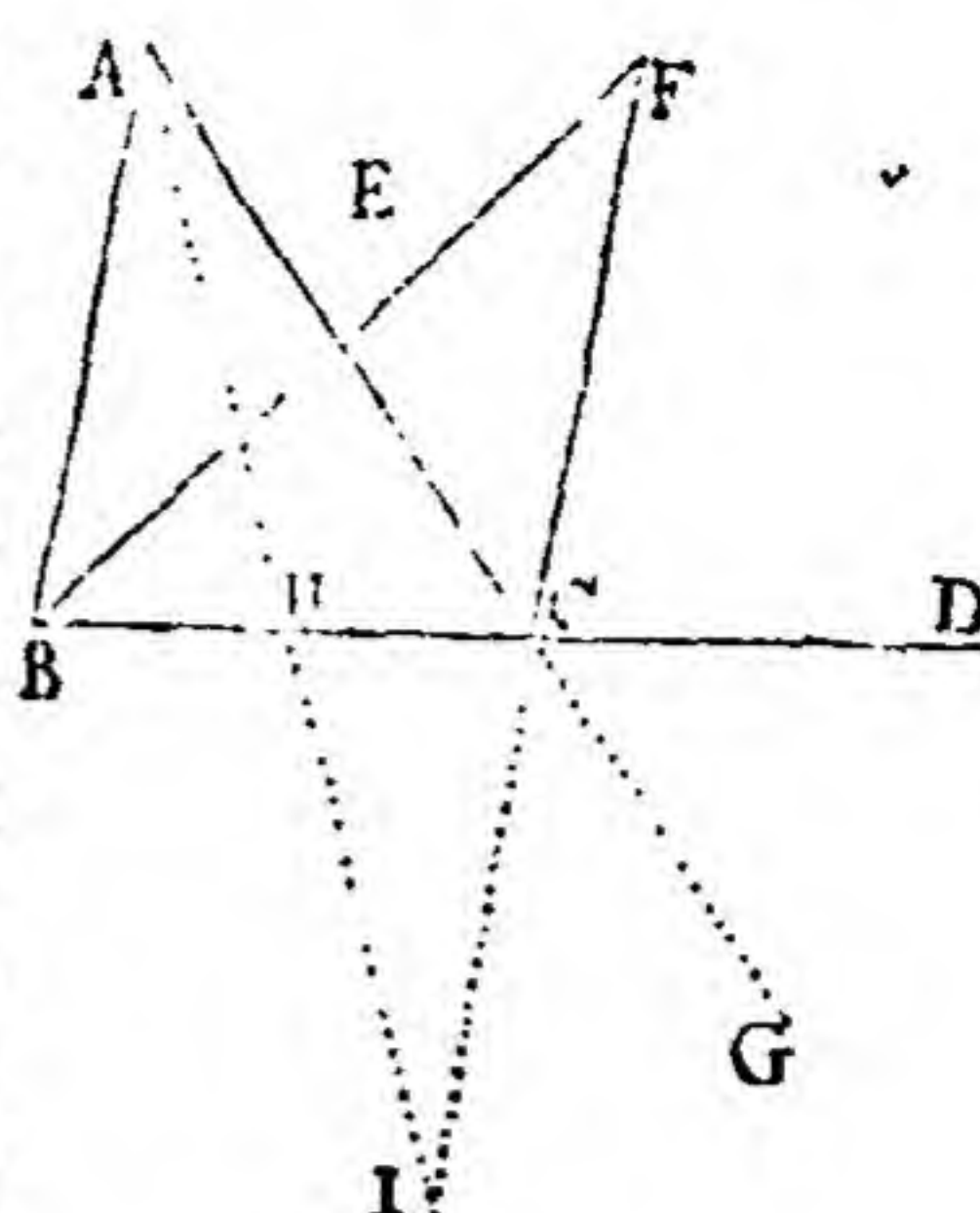
Of every Triangle, any one side being produced, the outward angle is greater than either of the inward, and opposite angles.

Let the Triangle be ABC , and let the side BC be produced to D .

I say, that the outward angle ACD , is greater than either of the inward, and opposite angles CBA , BAC .

Let AC be cut into halves in the point E ; and drawing BE let it be continued to F , so that EF be put equal to BE : And let be joyn'd FC . Forasmuch as AE is equal to EC , and BE , to EF : therefore there are the two lines AE , EB , equal to the two lines CE , EF , each to each, and the angle AEB , is equal to the angle FEC (for they are Vertical angles)

angles) Therefore the base AB , is equal to the base FC , and the Triangle ABE is equal to the Triangle FEC ; and the remaining angles are equal to the remaining angles, each to each, under which are subtended equal sides, [Prop. 4.] therefore the angle BAE , is equal to the angle ECF . But the angle ECD , is greater than the angle ECF . Therefore the angle ECD is greater than the angle BAE : That is, the outward angle ECD , or ACD , is greater than the inward and opposite angle BAC .



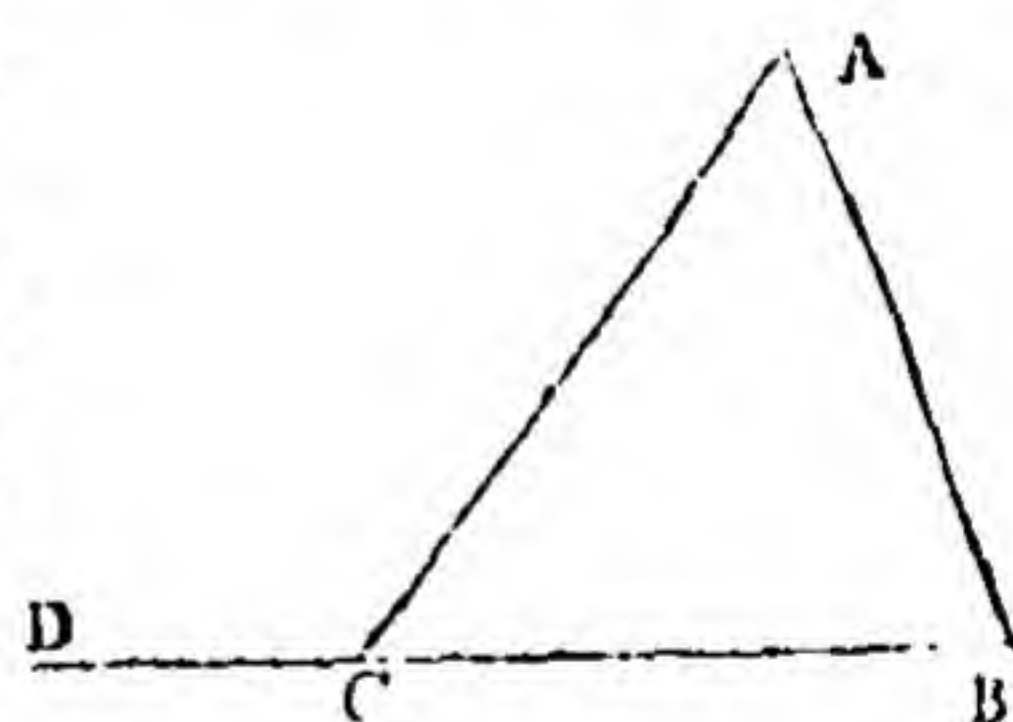
In like manner, the side AC being produced to G , and the side BC being cut into halves in the point H : and drawing AH , let it be continued to I , so that HI be put equal to AH : and let be joyned IC ; It will be demonstrated as before, that the outward angle BCG , is greater than the inward and opposite CBA . But the angle BCG , is equal to the Vertical angle ACD ; therefore the angle ACD , is also greater than the angle CBA : And it has been proved greater than the angle BAC . Therefore of every Triangle any one side being produced, the outward angle is greater than either of the inward and opposite angles. Which was to be demonstrated.

PROPOSITION XVII.

Of every Triangle two angles taken together every way, are less than two Right angles.

Let the Triangle be ABC . I say, that any two angles of the Triangle ABC , are less than two Right angles.

For let BC be produced to D : And because of the Triangle ABC the outward angle ACD , is greater than the inward and opposite angle ABC ; let be added in common the angle ACB : therefore the angles ACD , ACB , are greater than the angles ABC , BCA . But the angles ACD , ACB , are equal to two Right angles; therefore the angles ABC , BCA , are less than two Right angles. In like manner we may demonstrate, that the angles BAC , ACB , are less than two Right angles: And also that the angles CAB , ABC , are less than two Right angles. Therefore of every Triangle two angles taken together every way, are less than two Right angles. Which was to be demonstrated.



PROPOSITION XVIII.

Of every Triangle the greater side subtends the greater angle.

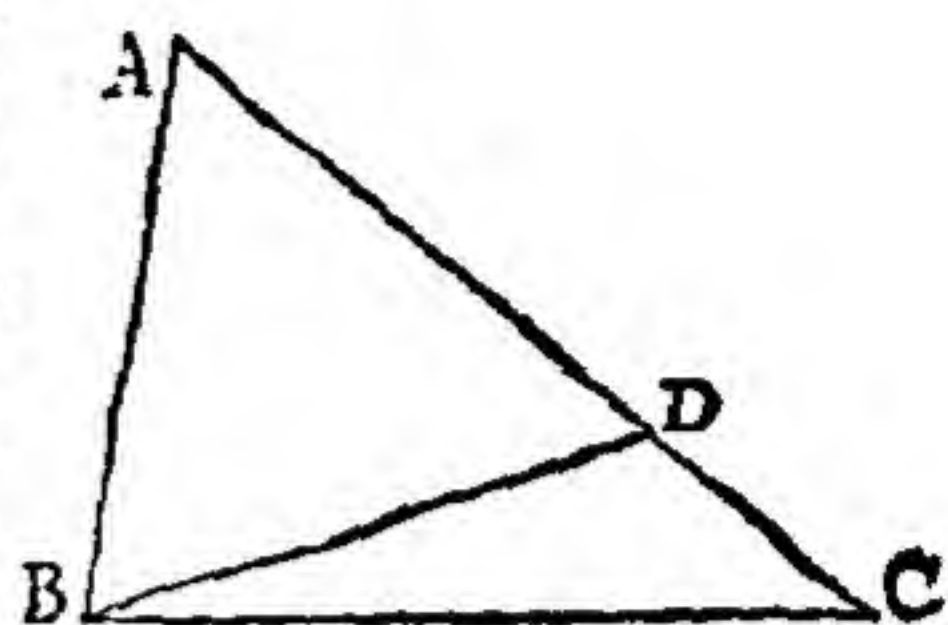
Let the Triangle be ABC , having the side AC greater than the side

AB. I say, that also the angle ABC , is greater than the angle BCA .

For because AC is greater than AB , let AD be put equal to AB ; and let be joyned BD .

Now forasmuch as of the Triangle BDC , the outward angle ADB is greater than the inward and opposite angle DCB ; and that the angle ADB , is equal to the angle ABD , (because the side AB is equal to the side AD) Therefore the angle ABD , is greater than the angle ACB .

But the angle ABC is greater than the angle ABD , wherefore it is much greater than the angle ACB . Therefore of every Triangle the greater side subtends the greater angle. Which was to be demonstrated.

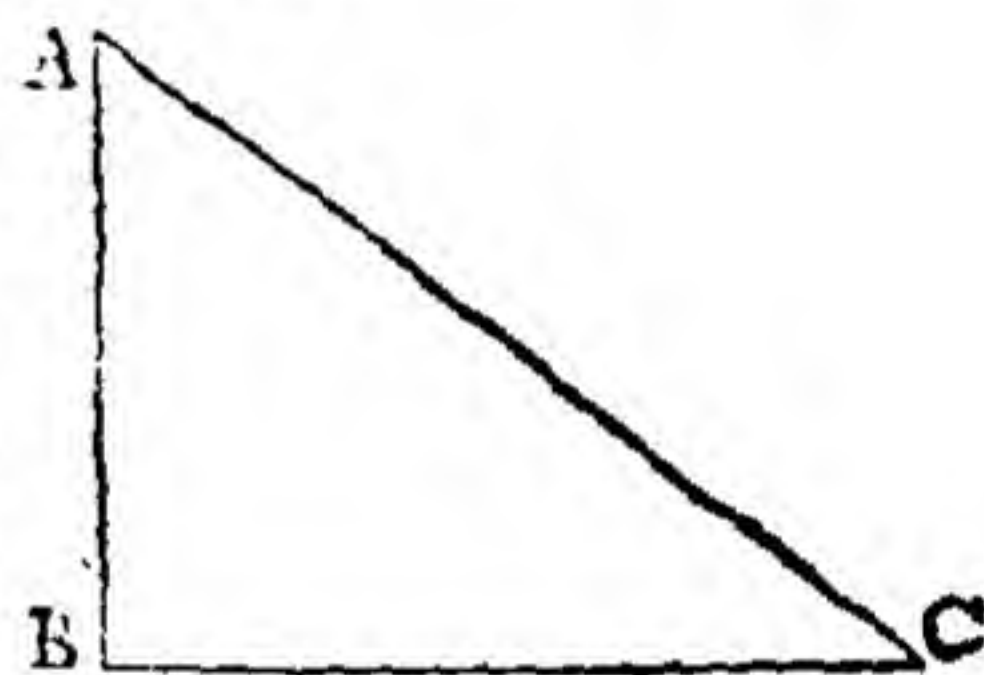


PROPOSITION XIX.

Of every Triangle under the greater angle; is subtended the greater side.

Let the Triangle be ABC , having the angle ABC , greater than the angle BCA . I say, that the side AC is greater than the side AB . For if not, then AC is either equal to AB , or less than it.

But AC is not equal to AB ; for that then the angle ABC should be equal to the angle ACB . But it is not equal [by Supposition] therefore AC is not equal to AB . Neither yet is AC less than AB , for then also the angle ABC should be less than the angle ACB . But it is not less [by Supposition.] therefore AC is not less than AB . And it has been demonstrated, that it is not equal: therefore the side AC is greater than the side AB . Wherefore of every Triangle under the greater angle is subtended the greater side. Which was to be demonstrated.

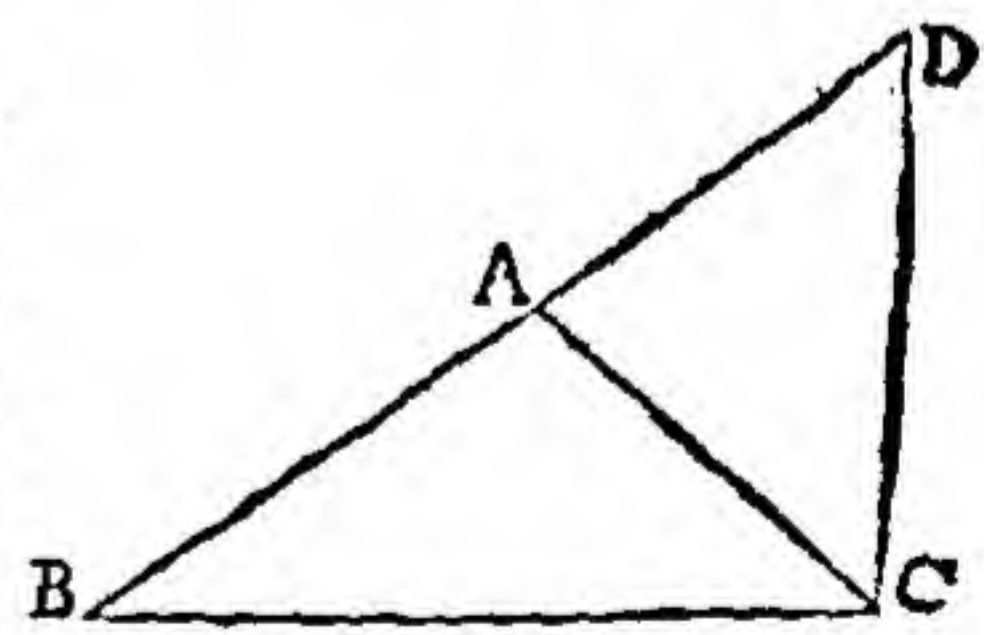


PROPOSITION XX.

Of every Triangle two sides taken together any way, are greater than the remaining side.

Let the Triangle be ABC . I say, that two sides of the Triangle ABC taken together any way, are greater than the remaining side. Namely BA, AC , than BC : and AB, BC , than CA : and BC, CA , than AB . For let BA be produced to the point D , and to CA let AD be put equal, then let be joyned DC .

Now forasmuch as DA is equal to AC , therefore the angle ADC is equal to the angle ACD . But the angle BCD is greater than the angle ACD : therefore the angle BCD is greater than the angle ADC . And because DCB is a Triangle having the angle BCD greater than the angle BDC ; and that under the



the greater angle is subtended the greater side: therefore DB is greater than BC . But DB is equal to BA, AC : therefore BA, AC , are greater than BC .

After the same manner shall we demonstrate, that AB, BC , are greater than CA ; And BC, CA , than AB . Therefore of every Triangle two sides taken together every way, are greater than the remaining side. Which was to be demonstrated.

PROPOSITION XXI.

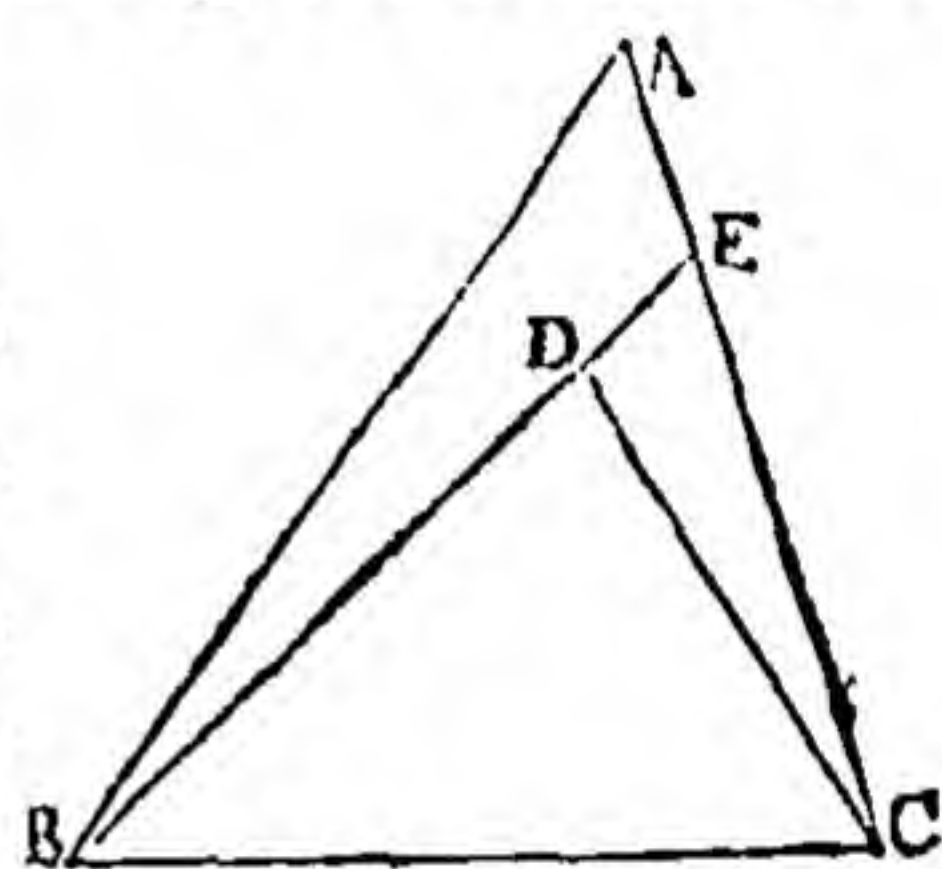
I*F from the Ends of any one side of a Triangle be constituted two strait lines within the Triangle: The constituted lines shall be less than the two remaining sides of the Triangle: But shall contain a greater angle.*

For on one of the sides BC , of the Triangle ABC , from the ends B, C , let be constituted within, two strait lines BD, DC .

I say, that BD, DC , are less than BA, AC , the two remaining sides of the Triangle ABC : but do contain an angle BDC greater than the angle BAC .

For let BD be produced to E .

Now forasmuch as of every Triangle two sides are greater than the remaining side; therefore of the triangle BAE the two sides BA, AE , are greater than BE . Let EC be added in common, therefore BA, AC , are greater than BE, EC .



Again, because of the Triangle CED , the two sides CE, ED , are greater than CD : let DB be added in common, therefore CE, EB , are greater than CD, DB . But BA, AC , have been proved greater than BE, EC : therefore BA, AC , are much greater than BD, DC .

Again, forasmuch as of every Triangle the outward angle is greater than the inward and opposite: therefore of the Triangle CED the outward angle BDC , is greater than DEC . And by the same reason, of the Triangle BAE , the outward angle BEC , is greater than BAE . But BDC has been prov'd greater than DEC , that is, BEC ; therefore the angle BDC is much greater than the angle BAC .

If therefore upon any one side of a Triangle, be from the ends thereof constituted two strait lines within the Triangle, the constituted lines are less than the two remaining sides of the Triangle: but shall contain a greater angle. Which was to be demonstrated.

ANNOTATIONS.

This Proposition is of much use in *Optics* concerning Visual Rays, and Angles; in that the same Object shall appear greater, or lesser, as upon various distances, the angles received in the Eye are greater, or lesser. And therefore in general 'tis to be noted that every Thing appears to us less, than it really is in magnitude.

The like use of this Proposition is made in *Perspective*, *Picture*, and *Architecture*: where Images, Statues, Columns, &c. are proportioned according to their heights

heights and distances. *Phidias* his Statue of *Minerva* was very famous in this point, which seemed near hand so monstrous to the vulgar: but seated in that part of the Temple where it was design'd to be placed, it appear'd most beautiful. We have the like example of Pictures in the banqueting Room at *White Hall*. Thus many Geometrical Propositions, which seem trivial, have excellent Uses.

PROPOSITION XXII.

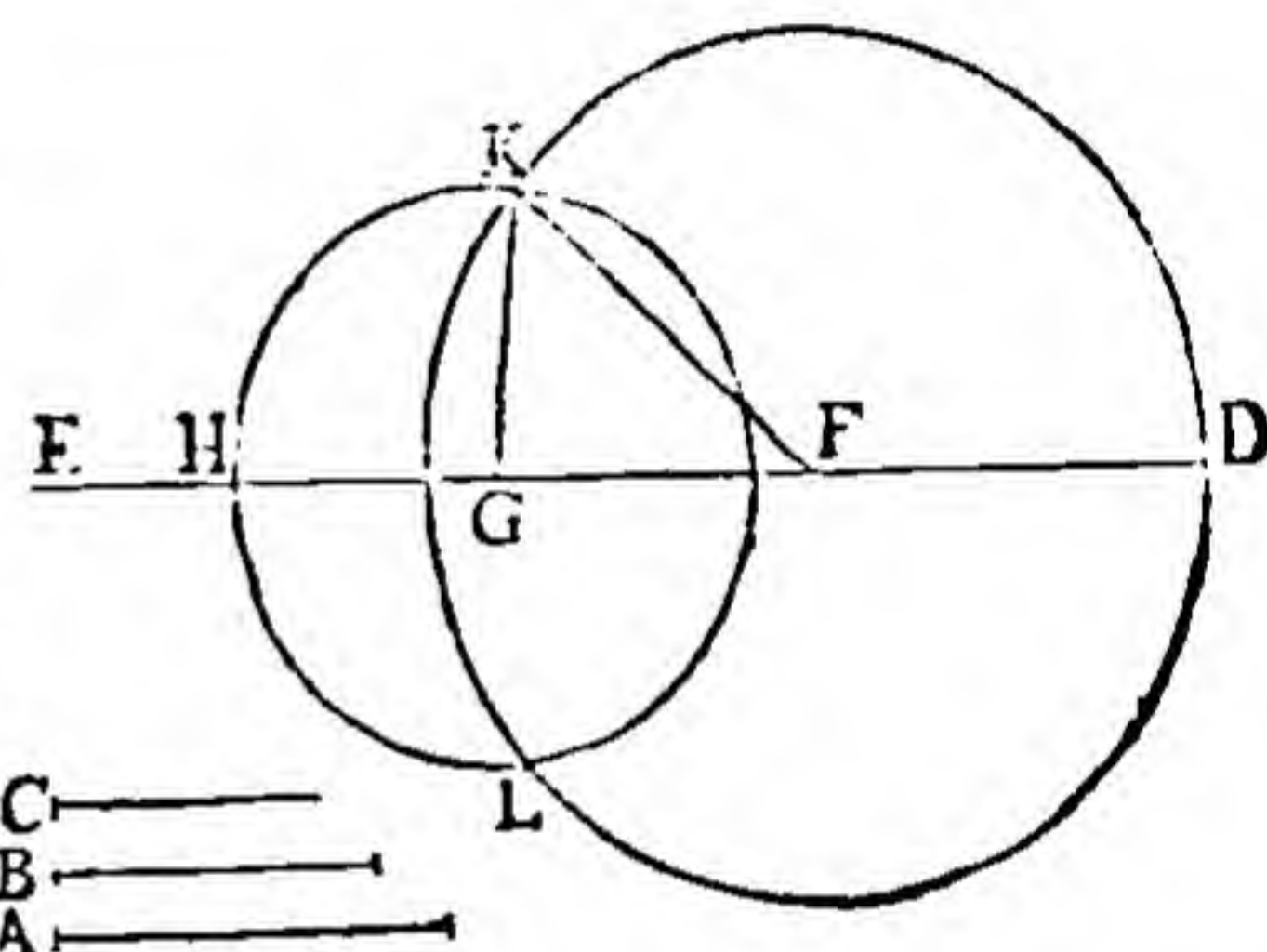
OF three strait lines, which are equal to three given strait lines, to constitute a Triangle.

But any two taken every way, ought to be greater than the remaining line.

Let three given strait lines be A, B, C , of which let any two taken every way, be greater than the remaining line: namely A, B , than C . A, C , than B , and B, C , than A . It is required to constitute a Triangle of strait lines equal to A, B, C . Let be put a strait line DE , terminated at D , but interminate towards E ; and let DF be put equal to A , and FG equal to B , and GH to C . Now the Center F , and distance FD , let be described the circle DKL : And again, the Center G , and distance GH , let be described the circle HLK : and let be joyned KF, KG . I say, that the Triangle KFG , is constituted of three strait lines equal to A, B, C .

For because the point F is the Center of the circle DKL , therefore FD is equal to FK ; but FD is equal to A , wherefore also FK is equal to A .

Again, because the point G is the Center of the circle HLK , therefore GH is equal to GK : but GH is equal to C : wherefore also GK is equal to C ; and FG also is equal to B . Wherefore the three strait lines KF, FG, GK , are equal to the three strait lines A, B, C . Therefore of the three strait lines KF, FG, GK , which are equal to the three given strait lines A, B, C , is constituted the Triangle KFG . Which was to be done.

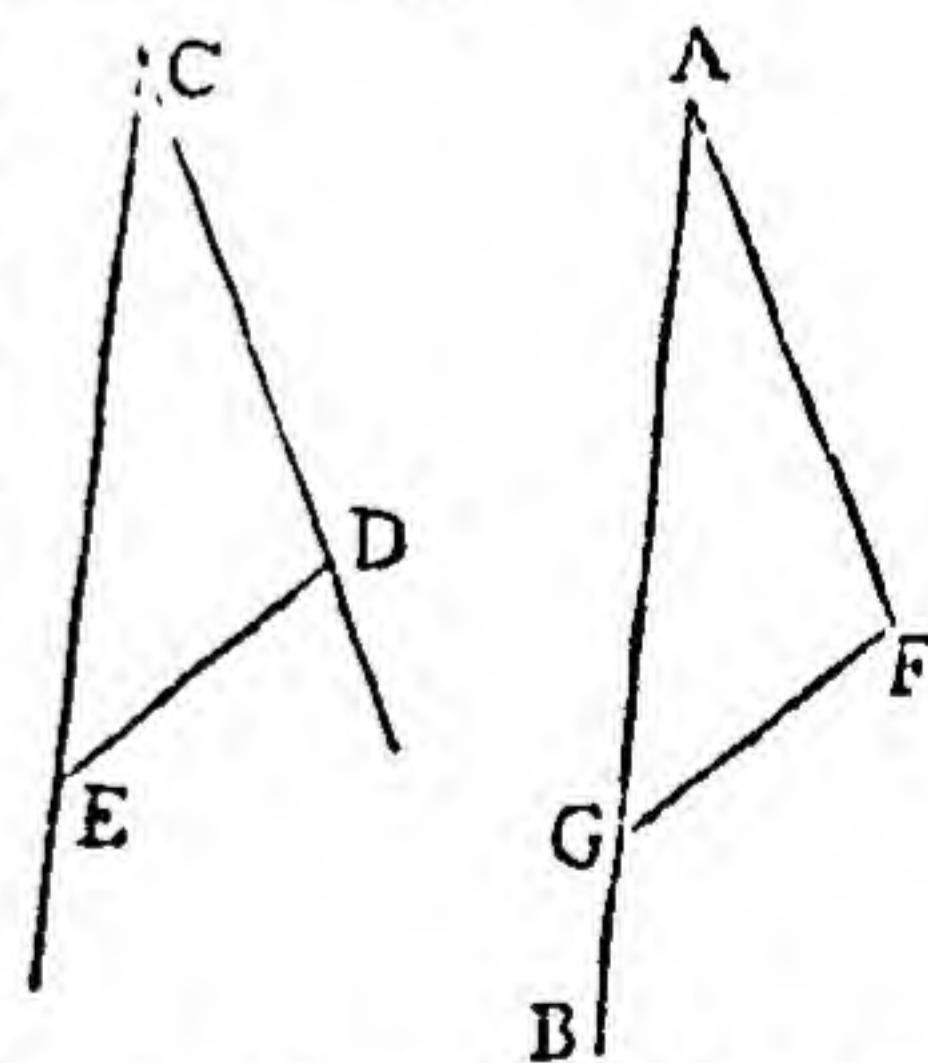


PROPOSITION XXIII.

TO a given strait line, and to a point in the same, to constitute a strait-lin'd angle equal to a given strait-lin'd angle.

Let the given strait line be AB , and in the same let the point be A ; and let the given strait lin'd angle be DCE . It is required to the given strait line AB , and in it to the point A , to constitute a strait-lin'd angle equal to the given strait-lin'd angle DCE .

Let there be taken in each line CD, CE , any points; as D, E ; and let be drawn the strait line DE . Now of three strait lines which are equal to the three strait lines $CD,$



CD, DE, CE; let be constituted a Triangle AFG, so that CD be equal to AF; and CE to AG; as also DE to FG.

Then forasmuch as the two strait lines DC, CE, are equal to the two strait lines FA, AG, each to each, and the base DE is equal to the base FG: therefore the angle DCE is equal to the angle FAG. [Prop. 8.] Therefore to a given strait line AB, and to a point in the same A, is constituted a strait-lined angle FAG, equal to a given strait-lined angle DCE. Which was to be done.

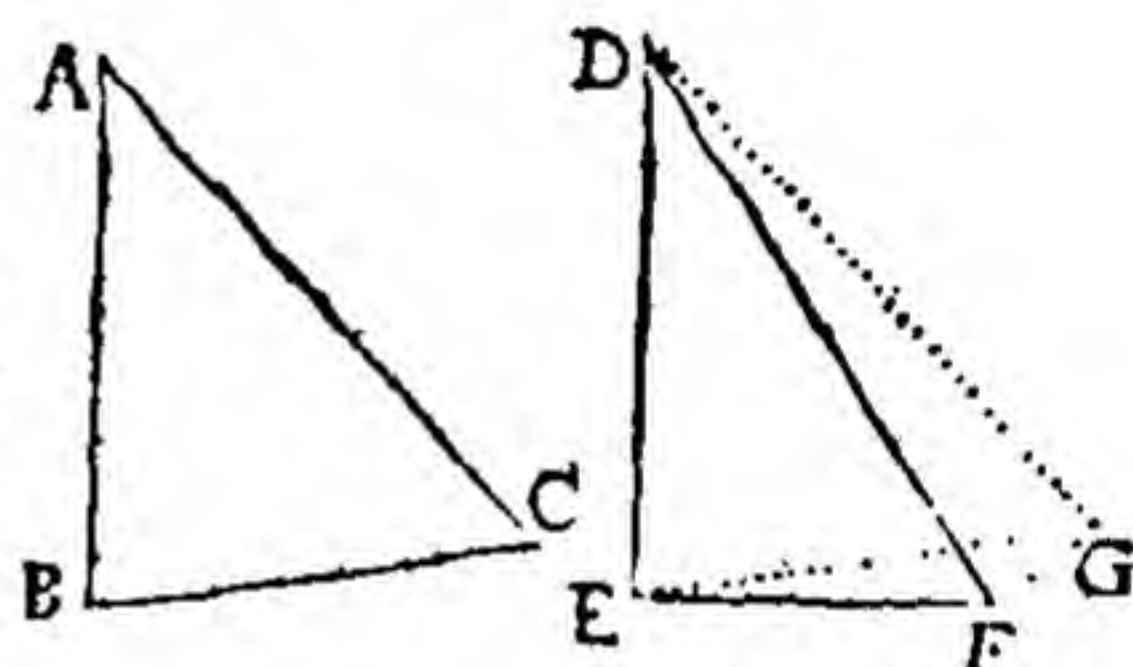
PROPOSITION XXIV.

If two Triangles have two sides equal to two sides, each to each: And have the angle greater than the angle, which is contained by the equal lines: They shall also have the base greater than the base.

Let the two Triangles be ABC, DEF, having two sides AB, AC, equal to two sides DE, DF, each to each, that is, AB to DE; AC to DF: And let the angle BAC be greater than the angle EDF: I say, that the base BC is greater than the base EF.

Forasmuch as the angle BAC is greater than the angle EDF, let to the strait line DE, and to a point in the same D, be constituted the angle EDG, equal to the angle BAC. And to either of the lines AC, DF, let DG be put equal, and let be joyned GE, GF.

Now because AB is equal to DE, and AC to DG; therefore there are the two lines BA, AC, equal to the two lines ED, DG, each to each; and the angle BAC, is equal to the angle EDG, therefore the base BC, is equal to the base EG. Again, because DG is equal to DF, therefore the angle DFG is equal to the angle DGF, [Prop. 5.] But DGF is greater than its part EGF, therefore also DFG is greater than EGF. But EFG is greater than its part DFG, therefore EFG is much greater than EGF.



And because there is the Triangle EGF, having the angle EFG greater than the angle EGF, and that under the greater angle is subtended the greater side: therefore the side EG is greater than EF. But EG is equal to BC: wherefore BC is greater than EF.

If therefore two Triangles have two sides equal to two sides, each to each: and have the angle greater than the angle, which is contained by the equal lines: They shall also have the base greater than the base. Which was to be demonstrated.

PROPOSITION XXV.

If two Triangles have two sides equal to two sides, each to each, and have the base greater than the base: They shall also have the angle greater than the angle, which is contained by the equal lines.

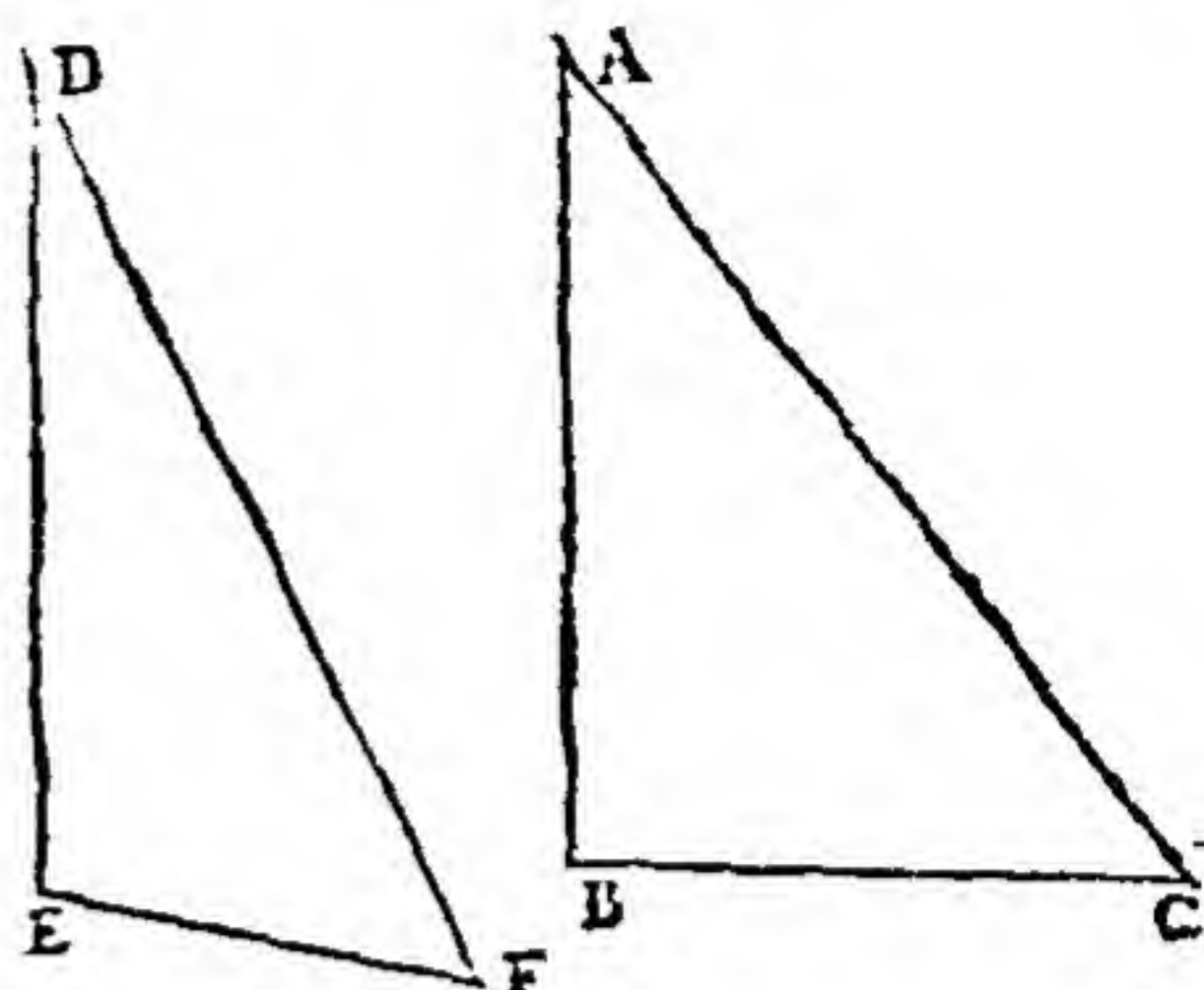
Let the two Triangles be ABC, DEF, having two sides AB, AC, equal
O to

to two sides DE , EF , each to each, that is, AB to DE ; AC to DF : And let the base BC be greater than the base EF : I say, that the angle BAC is greater than the angle EDF .

For if not, then BAC is either equal to EDF , or less. But the angle BAC is not equal to the angle EDF : For then the base BC should be equal to the base EF . [Prop. 4.] But it is not so [by Supposition] therefore the angle BAC is not equal to the angle EDF .

But neither is it less. For then the base BC should be less than the base EF [Prop. 24.] But it is not so [by Supposition] therefore the angle BAC is not less than the angle EDF ; and it has been proved, that it is not equal. Therefore the angle BAC is greater than the angle EDF .

If therefore two Triangles have two sides equal to two sides, each to each, and have the base greater than the base: They shall also have the angle greater than the angle, which is contained by the equal lines. Which was to be demonstrated.



PROPOSITION XXVI.

If two Triangles have two angles equal to two angles, each to each, and one side equal to one side, either *THAT* which is between the equal angles, or *THAT*, which is subtended under one of the equal angles: They shall also have the other sides equal to the other sides, each to each, and the remaining angle equal to the remaining angle.

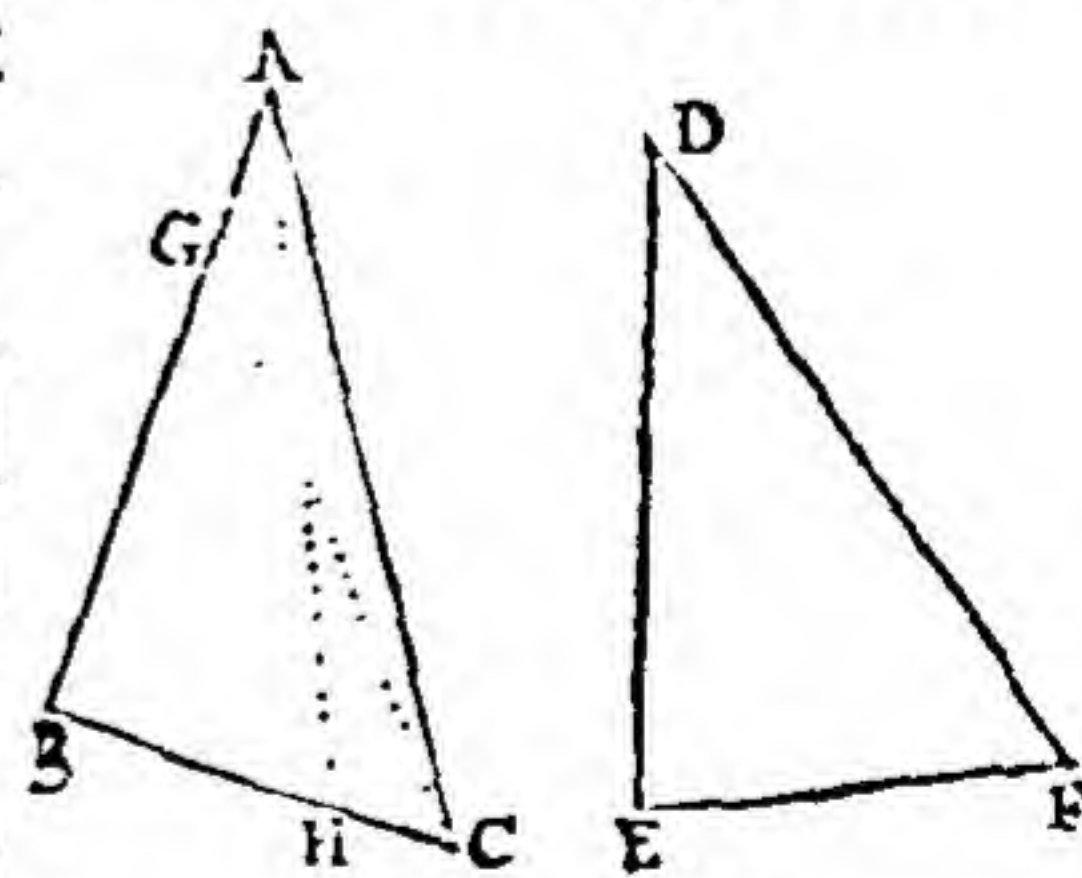
Let the two Triangles be ABC , DEF , having two angles ABC , BCA , equal to two angles DEF , EFD , each to each, that is, ABC to DEF , and BCA to EFD .

And let them have one side equal to one side.

First, the side between the equal angles: that is, BC equal to EF . I say, That they shall have the other sides equal to the other sides, each to each, namely AB to DE ; and AC to DF : and the remaining angle BAC , equal to the remaining angle EDF .

For if AB be unequal to DE , one of them shall be the greater.

Let AB be the greater, and let BG be put equal to ED , and let be joyned GC . Now forasmuch as BG is equal to ED , and BC to EF , therefore there are the two lines BG , BC , equal to the two lines ED , EF , each to each, and the angle GBC is equal to the angle DEF ; therefore the base GC , is equal to the base



DF . [Prop. 4.] And the Triangle GBC shall be equal to the Triangle DEF , and the remaining angles shall be equal to the remaining angles,

angles, each to each, under which are subtended equal sides;

Therefore the angle BCG , is equal to the angle EFD . But EFD was supposed equal to BCA ; wherefore BCG is equal to BCA : The less to the greater; which is impossible. Therefore AB is not unequal to DE , equal therefore it is.

But also BC is equal to EF ; therefore there are the two lines AB, BC , equal to the two lines DE, EF , each to each, and the angle ABC is equal to the angle DEF , wherefore the base AC is equal to the base DF : and the remaining angle BAC is equal to the remaining angle EDF .

Now again, let the sides subtended under the equal angles be equal, as AB to DE ; I say, that also the other sides shall be equal to the other sides, that is, AC to DF , and BC to EF : And also the remaining angle BAC , shall be equal to the remaining angle EDF . For if BC be unequal to EF , one of them is the greater.

Let BC (if possible) be the greater, and let BH be put equal to EF , and let be joyned AH .

Now forasmuch as BH is equal to EF , and AB to DE : therefore there are the two lines AB, BH , equal to the two lines DE, EF , each to each, and they contain equal angles: wherefore the base AH is equal to the base DF , and the Triangle ABH : is equal to the Triangle DEF . And the remaining angles shall be equal to the remaining angles, each to each, under which are subtended equal sides; therefore the angle BHA is equal to EFD . But EFD is equal to BCA , therefore also BHA is equal to BCA : that is, of the Triangle AHC , the outward angle BHA , is equal to the inward and opposite BCA : which is impossible. Therefore BC is not unequal to EF , equal therefore it is. But also AB is equal to DE : therefore there are the two lines AB, BC , equal to the two lines DE, EF , each to each, and they contain equal angles: wherefore the base AC , is equal to the base DF , and the Triangle ABC , is equal to the Triangle DEF , and the remaining angle BAC , is equal to the remaining angle EDF .

If therefore two Triangles have two angles, equal to two angles, each to each, and one side equal to one side, either that which is between the equal angles, or that which is subtended under one of the equal angles: they shall &c. Which was to be demonstrated.

PROPOSITION XXVII.

I*f a strait line falling on two strait lines, makes the alternate angles equal to one another, the strait lines shall be Parallels one to the other.*

For let the strait line EF , falling on the two strait lines AB, CD , make the Alternate angles AEF, EFD , equal to one another: I say, that AB is Parallel to CD .

For if not, then AB, CD , being produced shall meet either on the

parts of B, D ; or on the parts of A, C . Let them be produced and meet on the parts of B, D , in the point G . Therefore of the Triangle GEF , the outward angle AEF is greater than the inward and opposite angle EFG . [Prop. 16.] But it is also equal [by Supposition] which is impossible; therefore AB, CD , being produced shall not meet on the parts of B, D .

After the same manner shall be demonstrated, that they meet not on the parts of A, C . But meeting in neither part, they are Parallels, [Def. 35.] therefore AB is Parallel to CD .

Wherefore if a straight line falling on two straight lines, makes the Alternate angles equal to one another, the straight lines shall be Parallels, one to the other. Which was to be demonstrated.

PROPOSITION XXVIII.

If a straight line falling on two straight lines, makes the outward angle equal to the inward and opposite on the same parts: Or the inward angles on the same parts equal to two Right angles; the straight lines shall be Parallels one to the other.

For let the straight line EF falling on the two straight lines AB, CD , make the outward angle EGB , equal to the inward and opposite, and on the same parts, namely to the angle GHD : or the inward angles on the same parts, namely BGH, GHD , equal to two Right angles. I say, that AB is Parallel to CD .

For because EGB is equal to GHD [by Supposition] and EGB is equal to the Vertical angle AGH ; therefore also AGH is equal to GHD , and they are Alternate, therefore AB is Parallel to CD .

Again, because BGH, GHD , are equal to two Right angles [by Supposition.] and also AGH, BGH , are equal to two Right angles; [Prop. 13.] therefore AGH, BGH , are equal to BGH, GHD . Let BGH common be taken away, then the remaining angle AGH is equal to the remaining angle GHD . And they are Alternate, therefore AB is Parallel to CD .

If therefore a straight line falling on two straight lines, makes the outward angle equal to the inward and opposite on the same parts: or the inward angles on the same parts equal to two Right angles: the straight lines shall be Parallels one to the other. Which was to be demonstrated.

PROPO-

PROPOSITION XXIX.

ON Parallel Lines a strait line falling doth make the Alternate angles equal to one another.

And the Outward angle equal to the Inward, and Opposite on the same parts.

And the Inward angles on the same parts equal to two Right.

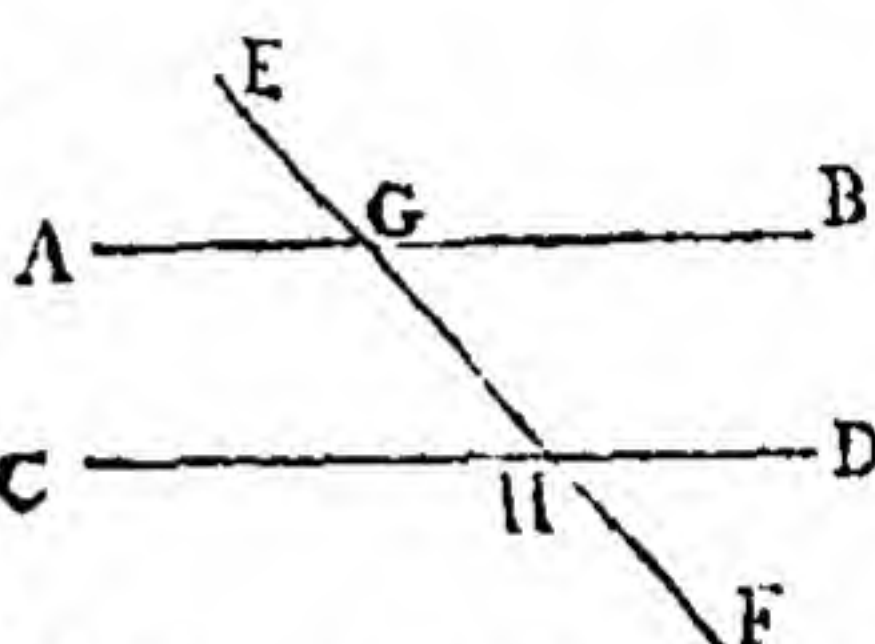
For on the Parallels AB , CD , let the strait line EF fall, I say that it makes the Alternate angles AGH , GHD equal.

And the Outward angle EGB equal to GHD the Inward and Opposite angle on the same parts.

And the Inward angles on the same parts EGH , GHD , equal to two Right angles.

For if the angle AGH be unequal to GHD ; one of them is the greater. Let the greater be AGH .

Now forasmuch as AGH is greater than GHD , let be added in common BGH . Therefore AGH , BGH are greater than BGH , GHD [Ax. 4.] But AGH , BGH are equal to two Right angles [Prop. 13.]; therefore BGH , GHD are less than two Right angles. But strait lines infinitely produced from Angles less than two Right, do meet together, [Ax. 11.] therefore AB , CD , infinitely produced shall meet together. But meet they do not; for that they are supposed Parallels; therefore AGH is not unequal to GHD : equal therefore it is; And AGH , GHD , are Alternate angles.



But again, AGH is equal to the Vertical angle EGB , [Prop. 15.] therefore also EGB is equal to GHD . The Outward angle equal to the Inward and Opposite.

Let now be added in common BGH : therefore EGB , BGH are equal to BGH , GHD . But EGB , BGH are equal to Two Right angles [Prop. 13.] therefore also BGH , GHD , The Inward angles, are equal to two Right.

Wherefore a strait falling on Parallel lines does make the Alternate angles equal to one another: And the Outward angle equal to the Inward, and Opposite on the same parts: And the Inward angles on the same parts equal to two Right. Which was to be demonstrated.

ANNOTATIONS.

This Proposition is the Converse of the two preceding; and in the three last Propositions is comprised the Fundamental Doctrine of Parallelism; wherein three Specificative and Convertible Properties of Parallels are laid down.

First, From the Equality of the Alternate angles the Lines are proved Parallels, in Prop. 27.

Secondly, From the Equality of the Outward angle to the Inward and Opposite: And then next,

Thirdly, From the Equality of the two Inward angles to Two Right, the Lines are also proved Parallels, in Prop. 28.

So that in these two Propositions is demonstrated, that all strait lines having any one of these three properties are PARALLELS, that is, NON-CONCURRING STRAIT LINES.

Now the following 29th. Proposition is the Converse of the two preceding, and demonstrates, that all Parallels have all these three properties.

But in this 29th. Proposition, the first and main part of the Demonstration depends wholly upon the 11th. Axiom, which tho' it be certainly true; yet for that it is lyable to dubitation, and some Objections may be made against it; this Demonstration hath not clearly passed without some reprehension.

For besides what hath been said before in the Annotations upon this 11th. Axiom, *Euclide* himself in the 17th. Proposition, doth in effect demonstrate, That *two strait lines meeting together, being cut by a strait line, are drawn from angles less than two Right*. And say they, it might be as reasonably required of *Euclide*, to have demonstrated the Converse, That *two strait lines drawn from angles less than two Right, shall meet together*, which is the 11th. Axiom, and assumed for a Principle without any Demonstration.

Again, in the 28th. Proposition it is demonstrated, That *if the two inward angles be equal to two Right, then the lines are Parallels*. But also it seems as requisite and reasonable to have demonstrated, That *if the two inward angles be less than two Right, then the lines are not Parallels*; but at length shall meet together: This Supposition having no more Natural evidence then the other. There have been in all Ages several Attempts made to remove this stumbling block: But too tedious they are to be here examined. You may peruse what *Proclus* has ventured at in his Commentaries on Prop. 29. and what *Clavius* has laboriously perform'd. What a strange notion of Parallels *Borellus* has fram'd in his *Euclides Restitutus*, at Prop. 14. Lib. 1. and what others have endeavoured herein. There are likewise two Translations of *Euclide* into Arabic, one of *Nasaradinus* printed at *Rome*. The other of ——— never Printed, a Copy whereof is in the *Oxford Library*. In both of them much Labour is taken to clear this Matter.

After these great Geometricians, we shall with pardon adventure upon this Matter; and in lieu of *Euclid's* 11th. Axiom bring into the Elements the consideration of the DISTANCE OF PARALLELS, and their EQUIDISTANCES toward one another. For altho' in our Annotations upon the Definition of Parallels, we have shewn, that the name *Parallels* ought not in *Euclid's* Sense to be Translated *Equidistant lines*; or by that name should be conceived *Equidistant strait lines*, but only *Nonconcurring strait lines*: yet we do not so wholly exclude the Notion of Equidistancy in the doctrine of Parallelism, but that there may be a just use made thereof; tho' Equidistancy be not taken into the Definition of Parallels.

First then it is observable, that vulgarly Parallels are conceived to be Equidistant strait lines; altho' the Geometrician puts only the notion of Nonconcurrency into the Definition, without any regard had to the Equidistancy of Parallels; and this is done upon very good reason. For a *Nonconcurrency* in some strait lines is a Notion generally useful throughout all Geometry: therefore *Euclide* among the rest of his Definitions proper to his first Element, has laid down this Notion of Nonconcurrency under the name of Parallels. So that Parallels and Nonconcurring strait lines may be substituted indifferently for one another in any demonstration, as the *Definitum* and *Definition* ought to be. But Parallels and Equidistant strait lines cannot be so indifferently taken and used; notwithstanding the vulgar conception of them. Yet some particular use may be made in Geometry of the Equidistancy of Parallels, as we shall shew; if according to the vulgar conception it be admitted among the other common Notions, that Parallels are equidistant strait lines: And so this to be received for a Maxim from *Euclid's* Definition of Parallels, as he has from the Definition of a Right angle put for an Axiom, that all Right angles are equal to one another.

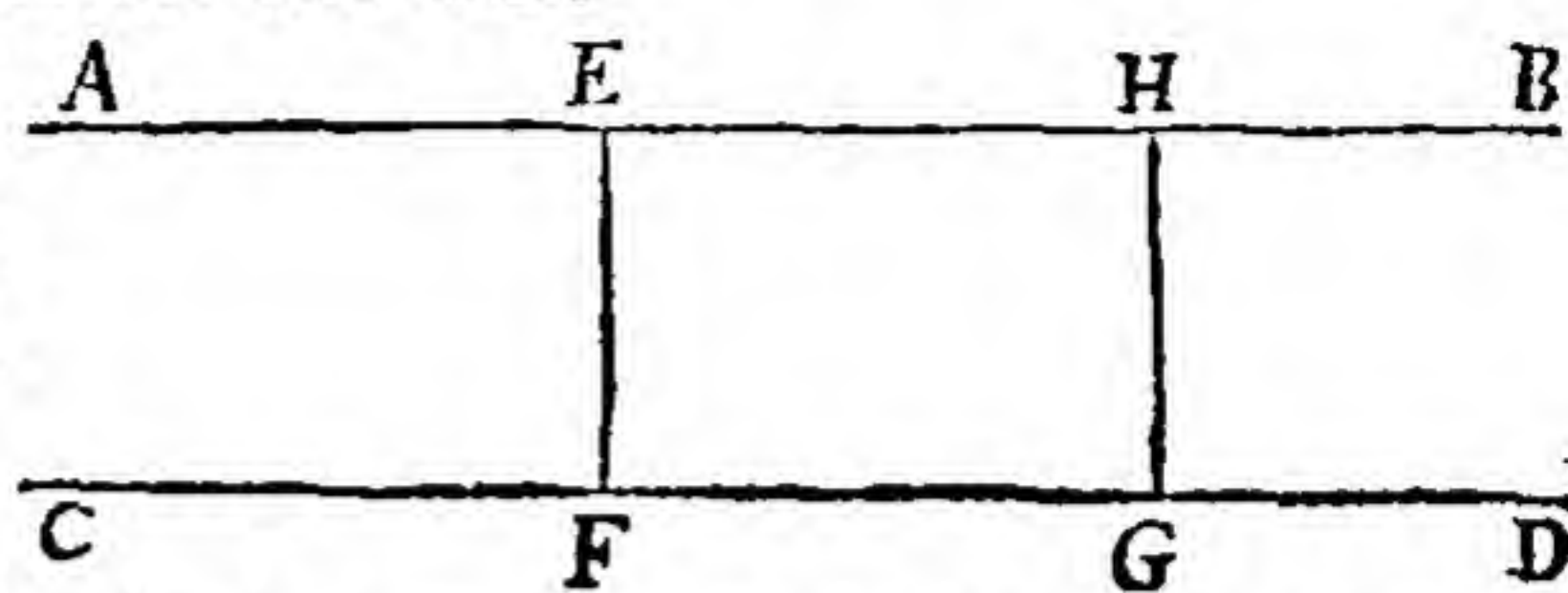
To proceed then in this matter, we shall as aforesaid, add to *Euclide* only a Definition of the distance of Parallels, and instead of his 11th. Axiom assume their Equidistancy as a common Notion.

DEFINITION XXXV.

The distance of Parallels is a strait line, drawn from any point in either Parallel, perpendicular to the other.

As of the Parallels AB, CD, the distance is the strait line EF, drawn from the point E in the line AB, perpendicular to the line CD.

And again, the distance of the same Parallels is the line GH drawn from the point G in the line CD, perpendicular to the line AB, and so forth infinitely.



This Notion, or Definition of distance is agreeable to the 4th. Definition of the third Element, and to the 4th. Definition of the sixth Element.

AXIOM XII.

Parallels are every where equally distant from one another.

That is, the Perpendiculars drawn from any point in either of the Parallels to the other, are equal to one another. As in the Parallels AB, CD, the line EF perpendicular to CD is equal to GH perpendicular to AB. So every where from any points in *the One*, the perpendiculars to *the Other*, are mutually equal to one another.

We have formerly shewn how *Posidonius* has defined Parallels from the equality of their perpendiculars; yet we find not what advantage was further made of that Definition, toward the amendment of *Euclid's* demonstration, or for any other use he makes thereof in Geometry. But according to *Euclid's* Definition, the Notion of two strait lines in the same plane produced both ways infinitely, which shall never meet, is as proper and common a subject of Geometry, as Angles and Figures are, and of as general an extent.

Yet furthermore we acknowledge, that the Equidistance of these strait lines is a Notion concomitant with that of Nonconcurrency, and that they mutually put one another, as a cause puts the effect, and an effect puts the cause. So that in Parallels Artificers do in Architecture, and other the like matters, respect their Equidistancy, as best suiting with their business: whereas the Geometrician makes use only of their Nonconcurrency. And our great Geometrician the Famous Savilian Professor of Geometry in Oxford Dr *Wallis* says, *Parallelismus & Equidistantia vel idem sunt, vel certe se mutuo comitantur.*

Seeing therefore that these Notions are naturally, and in common Sense immediately conjoyn'd, we do retain *Euclid's* Definition of Parallels, and have assumed for a Geometrical Axiom their Equidistancy.

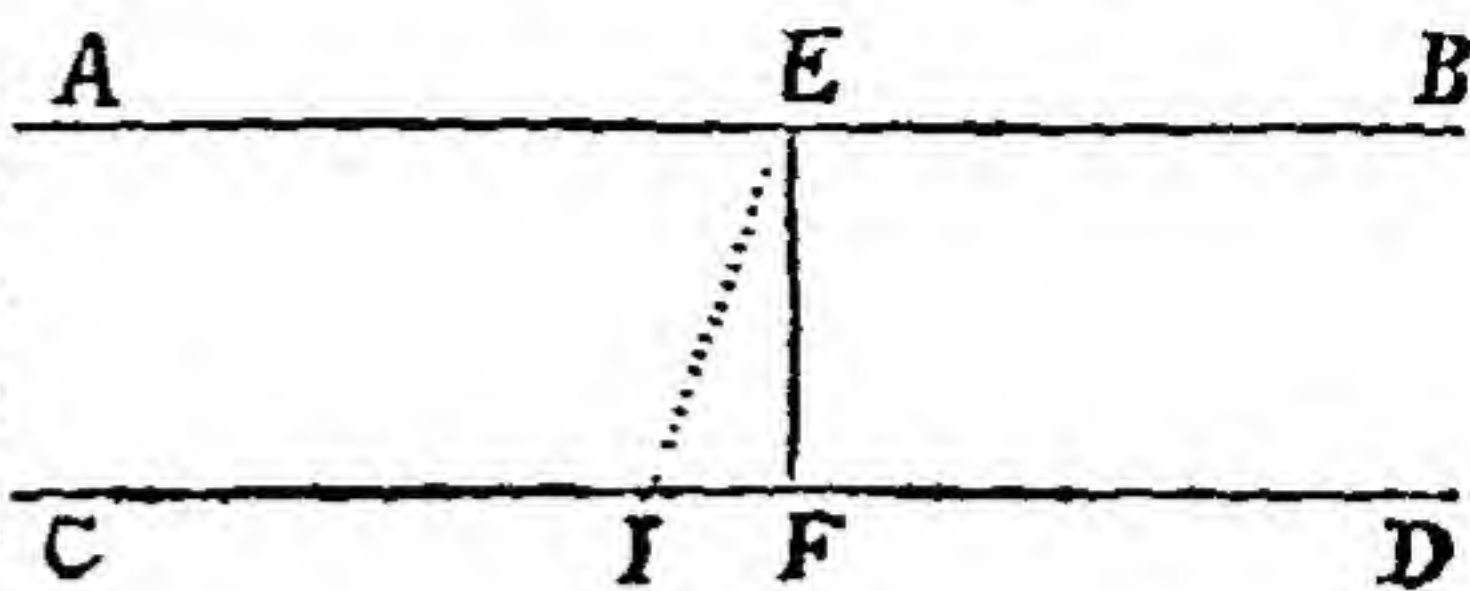
If this may be so allowed, or at least admitted, as a more clear and obvious Notion than the 11th. Ax. of *Euclide*, then shall we briefly demonstrate that troublesome part of the 29th. Proposition, concerning the equality of the Alternate angles in Parallels, without any use of the 11th. Axiom.

The Demonstration of the equality of the Alternate Angles in Parallels.

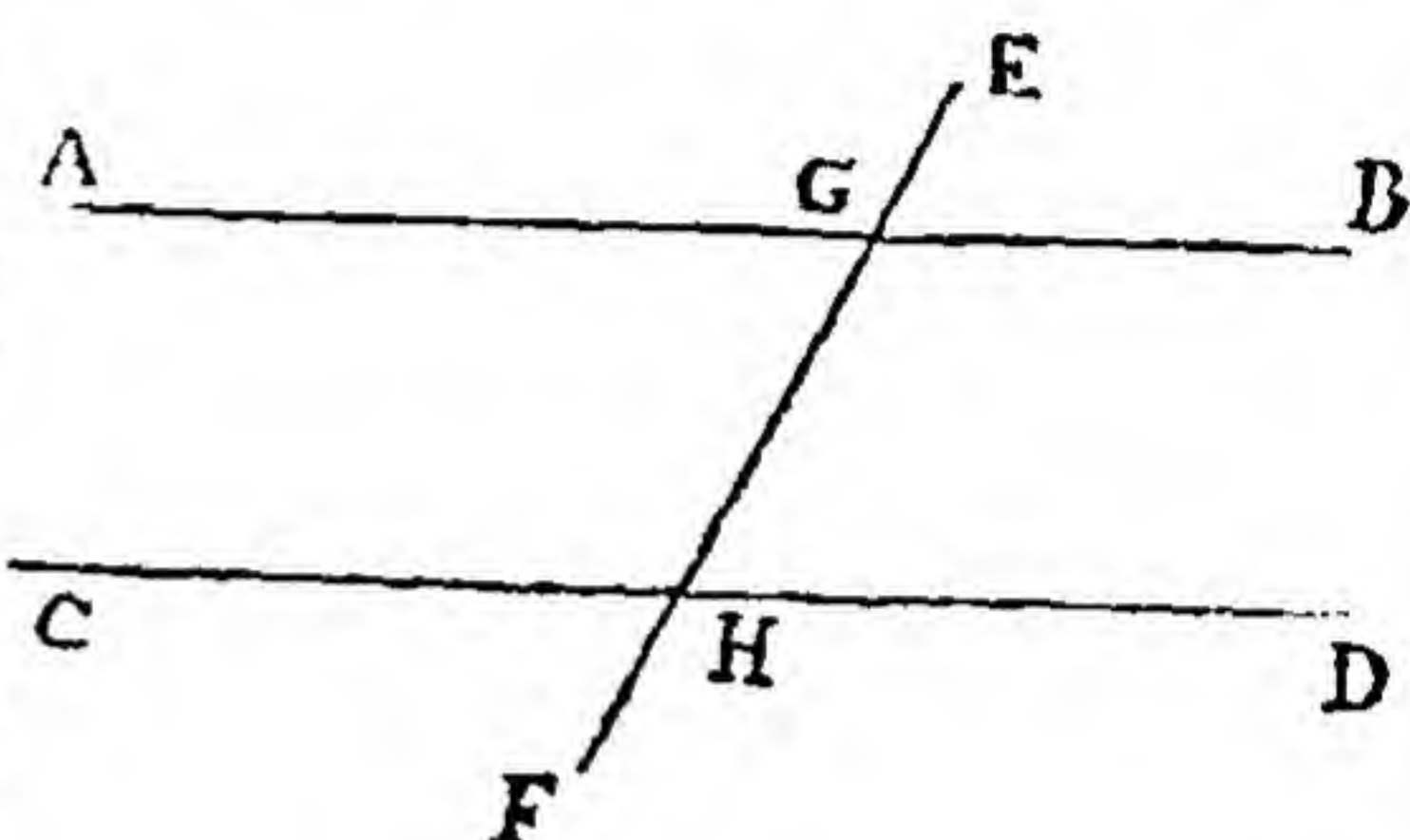
On the Parallels AB, CD, let first the strait line EF fall on AB at Right angles.

I say, that EF likewise falls on CD at Right angles; and therefore makes the Alternate angles equal, and the outward angle equal to the inward and opposite, and the two inward angles equal to two Right. For if EF falls not at Right angles

gles on CD , let EI fall at Right angles on CD , [by Prop. 12.] therefore EI is the distance of the Parallels AB , CD , [Def. 35.] Likewise FE falling on AB at Right angles, [by Supposition] is the distance also of the same Parallels. Wherefore FE , EI , are equal [by Ax. 12. that Parallels are Equidistant strait lines.] And because of the Triangle EFI , the sides EF , EI , are equal, therefore the angles at the base EFI , EIF are equal. But EIF is by Construction a Right angle, wherefore EFI is a Right angle, so that the angles at the base are equal to two Right. But they are less [by Prop. 17.] therefore EI is not at Right angles to CD , and by the same reason no other can be drawn from the point E besides EF . Wherefore EF is at Right angles to CD , and also it is at Right angles to AB [by Supposition.] therefore all the angles at E and F are Right, and equal to one another.



Again, on the Parallels AB , CD , let the strait line EF fall otherwise at adventure. I say, that it makes the Alternate angles AGH , GHD , equal to one another.

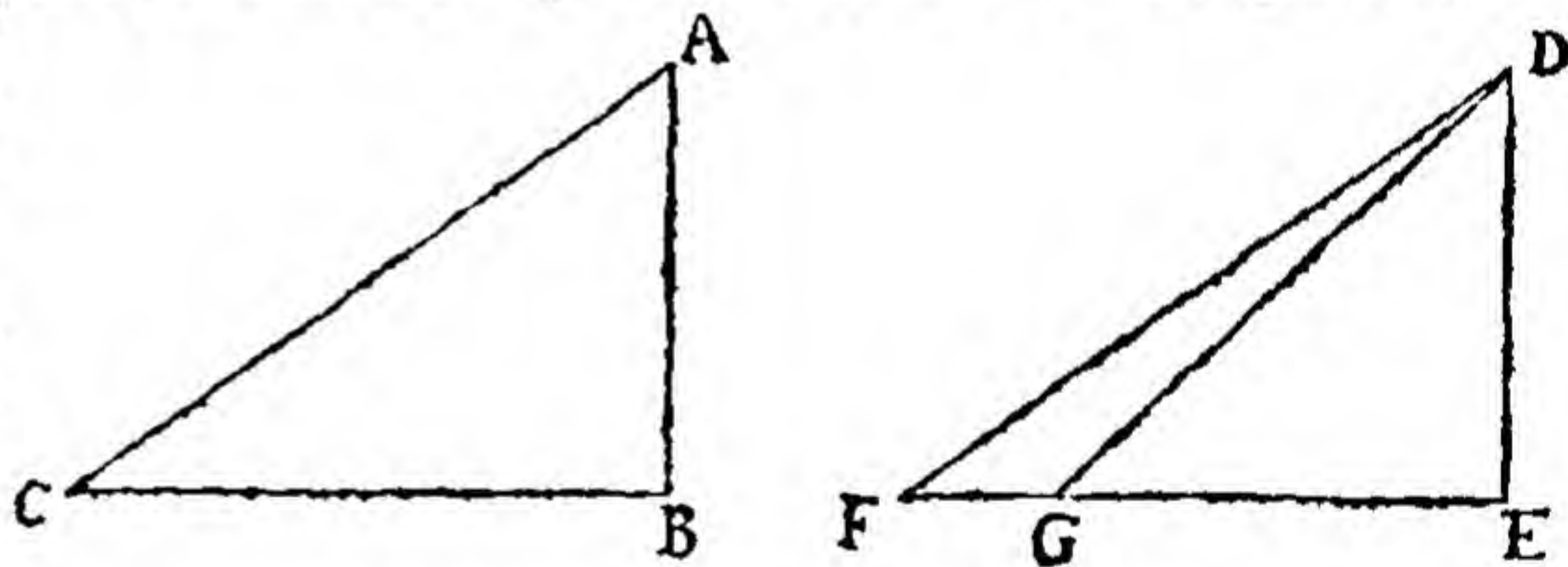


Now for the demonstration thereof we shall premise this Lemma.

A Lemma is a Proposition taken in by the by, to make way for the proof of some Principal Proposition.

L E M M A.

If two Right angled Triangles ABC , DEF , have the side AC , subtending the right angle B , equal to the side DE , subtending the right angle E : and a side AB about the right angle B , equal to the side DE , about the right angle E , then shall they have the remaining side BC , equal to the remaining side EF . For if BC be not equal to EF , then one of them is the greater. Let EF be the greater, and from the greater EF take the line EG equal to BC the less; and let be drawn GD .



Forasmuch then as EG is made equal to BC , and ED is equal to BA , [by Supposition] and they contain

right angles at E and B , therefore the base DG shall be equal to base AC [Prop. 4.] But AC is equal to DE [by Supposition] therefore DG is equal to DE : so that in the Equicrural Triangle DGE , the angles at the base DGE , DEG , are equal to one another. But DGE is greater then the right angle DEG , (*the outward greater then the inward and opposite*, by Prop. 16.) therefore the angles DGE , DEG , are greater than two Right: which is impossible [by Prop. 17.] therefore the line BC is not unequal to EF , equal therefore they are to one another. Which was to be demonstrated.

After the demonstration of this Lemma, we thus further proceed to prove the equality of the Alternate angles in any Oblique Section.

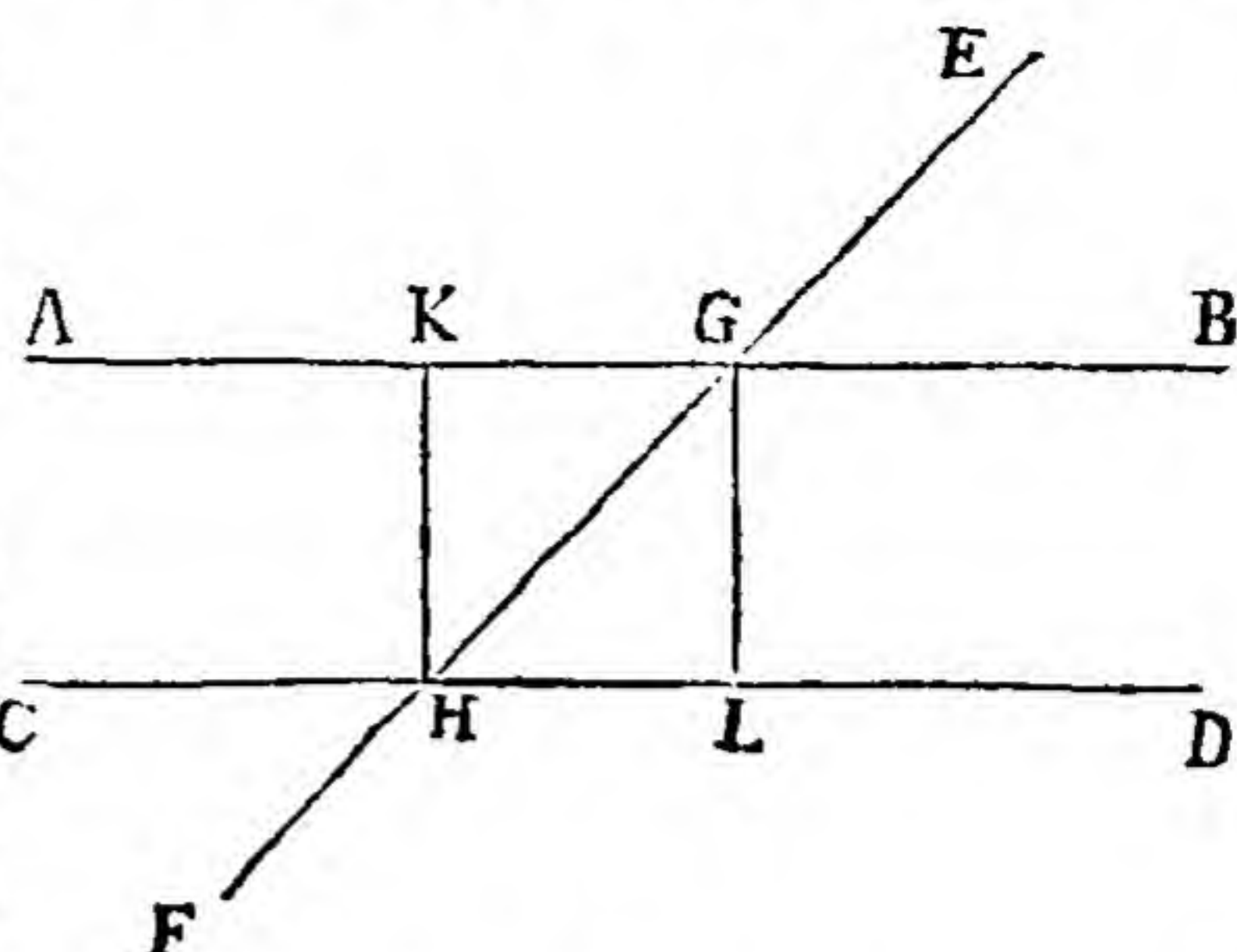
On the Parallels AB , CD , let the strait line EF fall at adventure. I say, that it makes the Alternate angles AGH , GHD , equal to one another: and also the Alternate angles CHG , HGB , equal to one another.

For from the point H to the line AB , draw a perpendicular HK [by Prop. 12.] Again, from the point G to the line CD draw a perpendicular GL .

Now forasmuch as in the right angled Triangles HKG , GLH , the line HK is equal to the line GL , for that each is the distance of the same Parallels, [Def.

35. and Ax. 12.] and HG , subtending the right angles at K and L , common, therefore the remaining side KG , is equal to the remaining side LH (by the precedent Lemma). Wherefore there are the two lines HK, KG , equal to the two lines GL, LH , each to each, and they contain equal angles, namely Right; therefore [by Prop. 4.] the angles KGH, GHL , are equal; that is, in the Parallels AB, CD , the Alternate angles AGH, GHD , are equal to one another.

Again, because the angles AGH, HGB , are equal to two Right [Prop. 13.]; and likewise CHG, GHD , are equal to two Right, therefore AGH, CHG , are equal to CHG, GHD . Taking therefore away the equal Alternate angles AGH, GHD , the remaining Alternate angles CHG, HGB , are equal to one another.



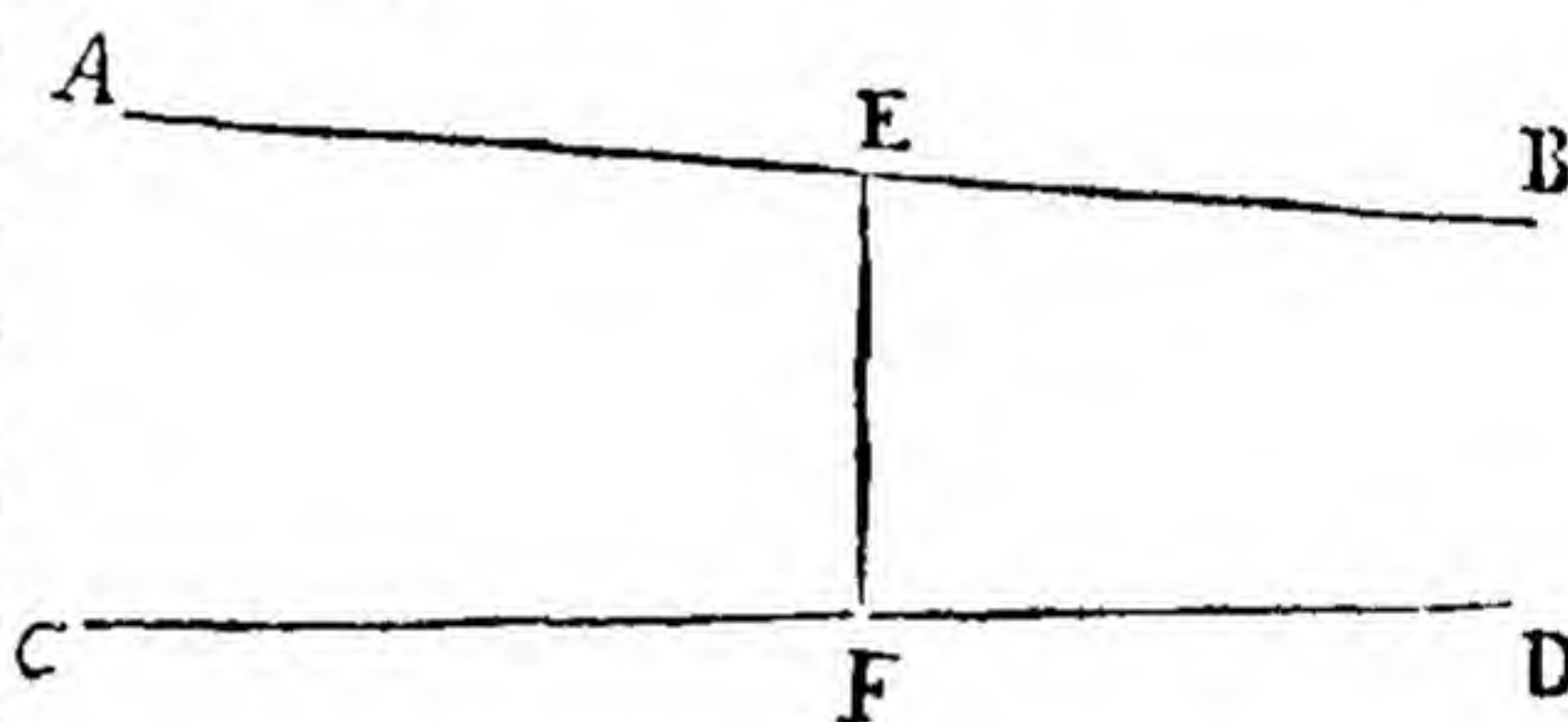
Wherefore on Parallel lines a strait line falling, doth make the Alternate angles equal to one another, &c. Which was to be demonstrated. Now what follows in *Euclide* is without exception.

Having thus demonstrated this 29th. Proposition without the help of the 11th. Axiom; we shall next demonstrate that 11th. Axiom.

A Demonstration of the Eleventh Axiom of *Euclide*.

If on two strait lines AB, CD , a strait line EF falling, doth make the inward angles toward the same parts EFD, FEB , less than two Right: I say, that the lines AB, CD , being infinitely produced toward the parts of B, D , where the angles are less than two Right, shall meet together.

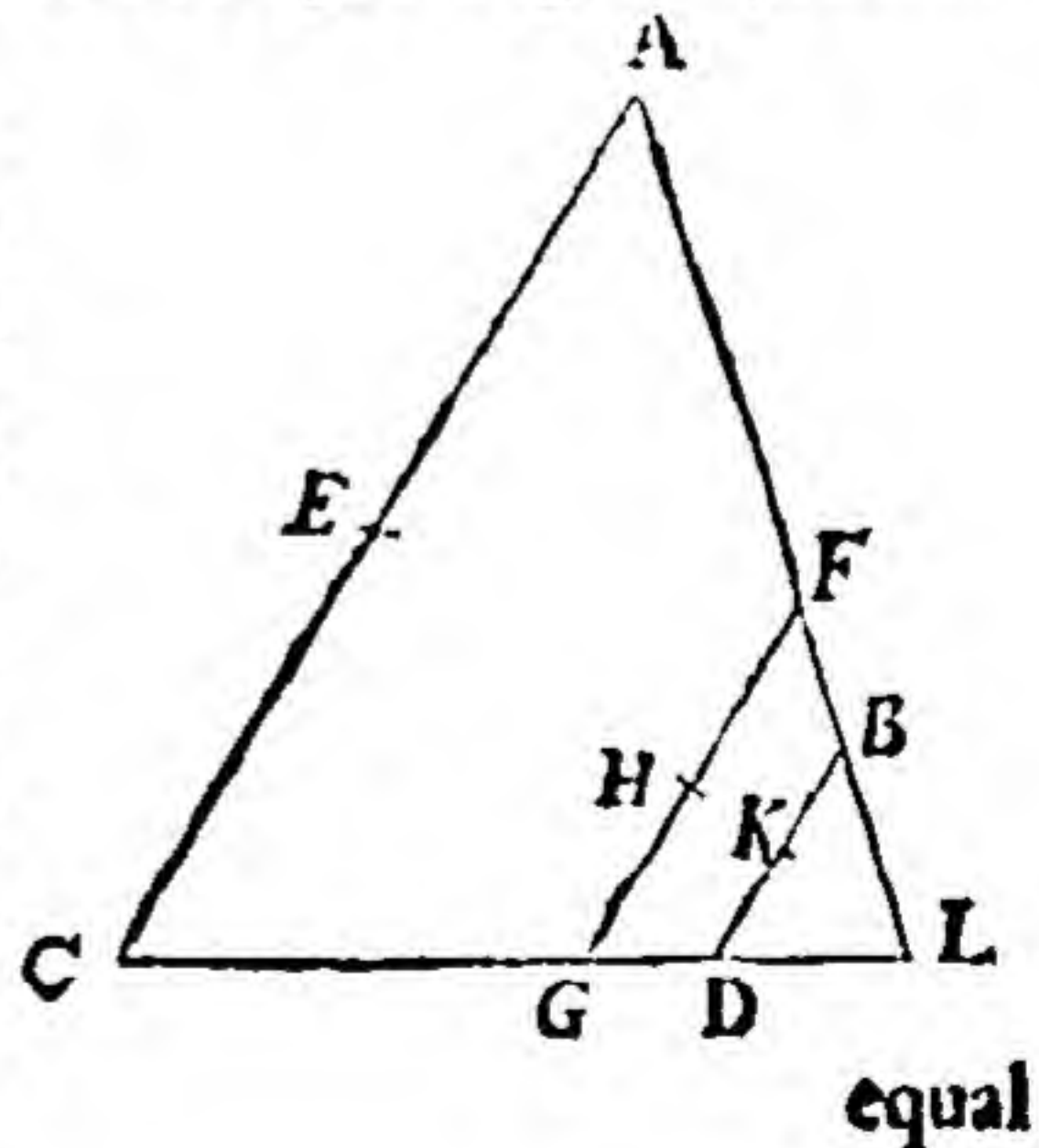
Forasmuch as the angles AEF, FEB , are equal to two Right [Prop. 13.]; and CFE, EFD , are equal to two Right, therefore these four are equal to four Right angles. But EFD, FEB , are less than two Right [by Supposition]; therefore the angles EFC, FEA , are greater than two Right. Wherefore the lines BA, DC ,



being infinitely produced toward the parts of A, C , shall that way never meet [by Prop. 17.]. If now they meet not toward the parts of B, D , then the lines AB, CD , are Parallels [Def. 34.]; and Parallels have the two inward angles toward the same parts equal to two Right [by Prop. 29.]. But the angles EFD, FEB , are supposed less; and to be less and equal to the same is impossible; Therefore the lines AB, CD , being infinitely produced toward the parts of B, D , shall meet together. Which was to be demonstrated.

Notwithstanding this, it is demonstrable that two strait lines drawn from angles less than two Right, may in some manner be for ever prolonged; yet shall they never meet together.

For let the strait lines AB, CD , be cut by AC making the inward angles BAC, DCA , less than two Right. Now let AC be cut into halves, or otherwise in E : and equal to EA let be put AF , and to EC , CG ; then draw FG . Again, let FG be cut in H , and equal to HF , let be put FB , and to HG , GD ; then draw BD . I say, that the lines AB, CD , may for ever be *thus* prolong'd, yet never shall they meet together. For if possible, let them meet in the point L ; therefore BD being cut in K , the line KB shall be



P

equal

equal to BL , and KD to DL . Wherefore of the Triangle BDL the sides DL , LB , shall be equal to the third side BD , which is impossible by the 20th. Proposition. Therefore the lines AB , CD , drawn from angles less than two Right, may for ever be prolonged, and never meet together. Which was to be demonstrated.

From hence it is manifest, that Magnitude is infinitely divisible: and that an infinite progress may be made in a finite Space.

And moreover for the better understanding of *Euclide* in this matter, we are to distinguish between a production of lines *ἐς ἀπείρου*, in *infinitum*, Infinitely, and *ἀπείρους*, *Infinities*, Infinite Times. The former is an unlimited, free course of prolongation, such as Geometricians always understand by *ἐς ἀπείρου*. The other here in this Instance is a limited and restrained prolongation, made step by step, and in such a manner as that the steps are shorter, and shorter made continually, and the lines are approaching nearer and nearer; yet so as never to meet together.

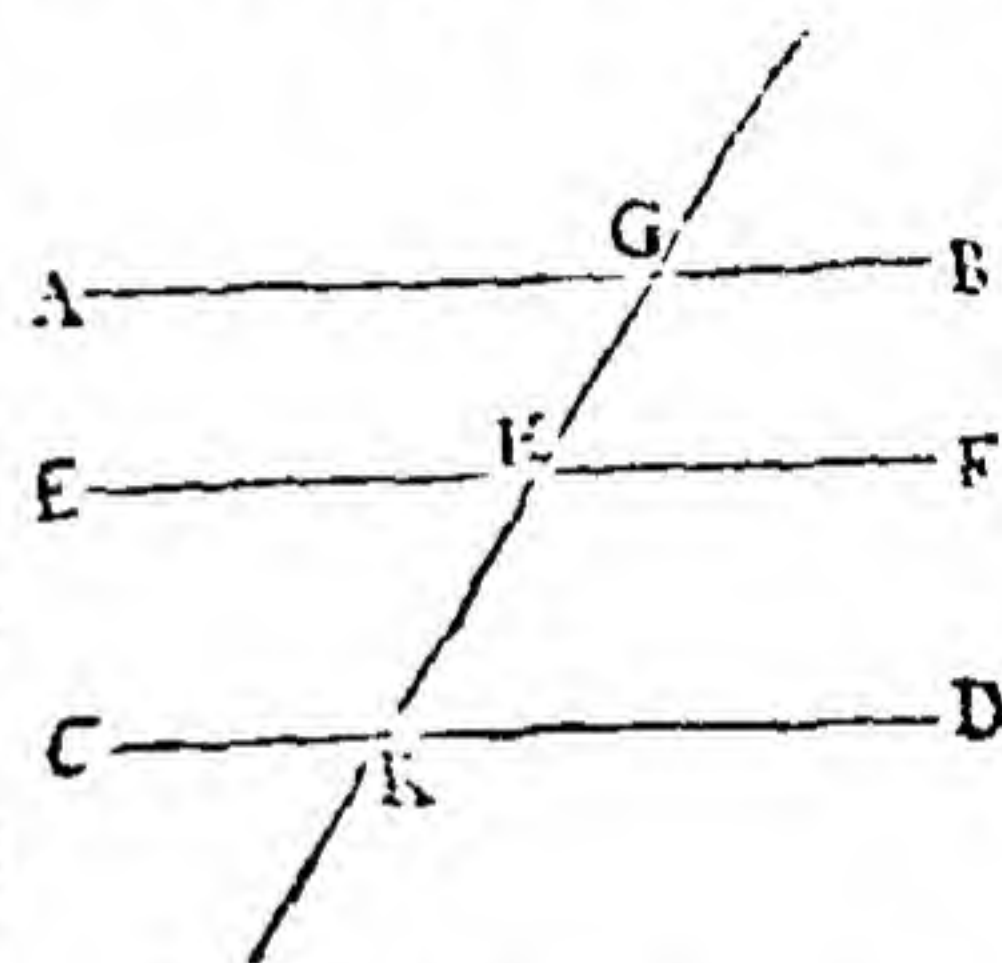
PROPOSITION XXX.

Strait lines Parallel to the same strait line, are also Parallel to one another.

Let each of the lines AB , CD , be parallel to EF : I say, that AB is parallel to CD .

For let a strait line GK fall upon them.

Now forasmuch as the strait line GK falls on the parallels AB , EF ; therefore the angle AGH is equal to the Alternate angle GHE , [Prop. 29.]



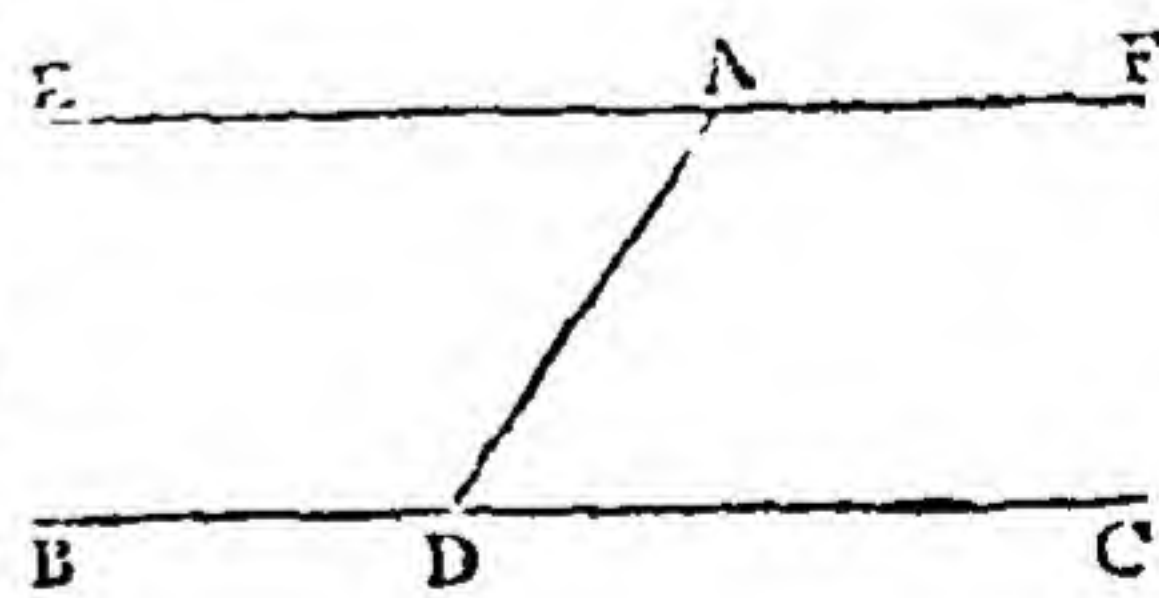
Again, because the strait line GK falls on the parallels EF , CD ; therefore the outward angle GHE is equal to the inward and opposite GKD , [Prop. 29.]

But the angle AGH , that is, AGK has been proved equal to GHE . Therefore AGK is also equal to GKD : and they are Alternate angles; wherefore AB is parallel to CD [Prop. 27.]. Therefore strait lines parallel to the same strait line, are parallel to one another. Which was to be demonstrated.

PROPOSITION XXXI.

By a given point to draw a strait line parallel to a strait line given.

Let the given point be A , and the given strait line be BC . It is required by the point A , to draw a strait line parallel to BC . In the line BC , let be taken any point as D , and let be joyn'd AD : then to the strait line DA , and to the point in the same A , let be constituted the angle DAE , equal to the angle ADC , [by Prop. 23.] and to the strait line EA , let directly be produced the line AF .



Now forasmuch as on the lines BC , EF , the strait line AD falling, hath made the Alternate angles EAD , ADC , equal to one another; therefore EF is parallel to BC , [Prop. 27.]; wherefore by the given point

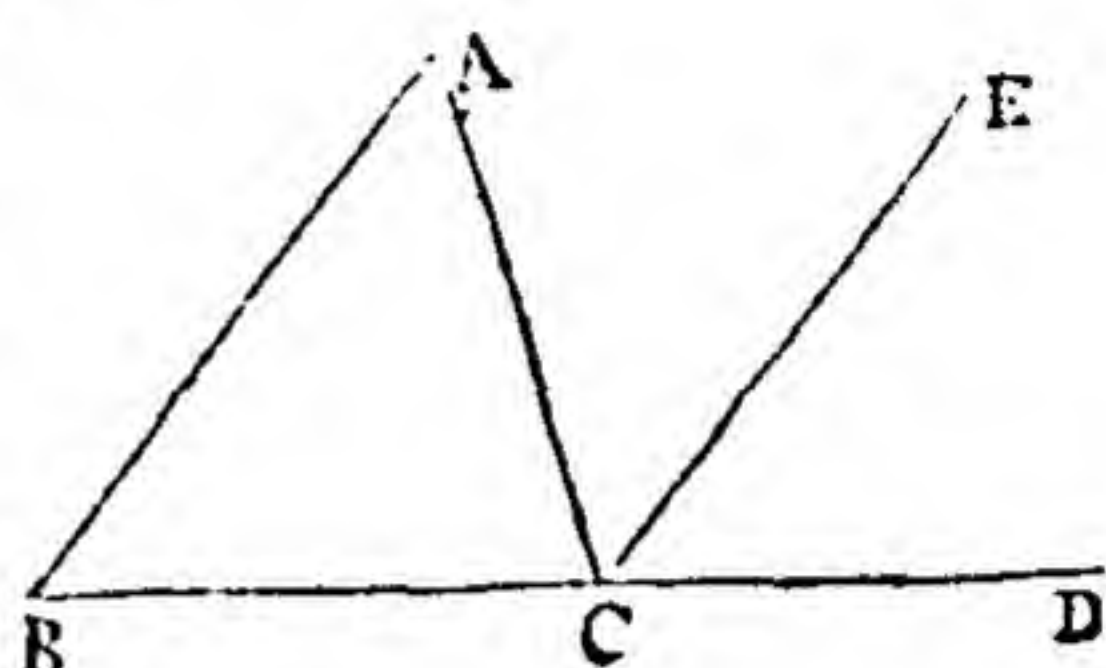
point ^A is drawn the strait line EA , parallel to the given strait line BC . Which was to be done.

PROPOSITION XXXII.

Of every Triangle one of the sides being produced, the outward angle is equal to the inward and opposite.
And the three inward angles of a Triangle are equal to two Right.

Let the Triangle be ABC , and one of the sides BC , be produced to D .
I say, that the outward angle ACD is equal to the two inward and opposite CAB, ABC . And of that Triangle the three inward angles ABC, BCA, CAB , are equal to two Right.

For by the point c , let CE be drawn parallel to AB . [Prop. 31.] Now forasmuch as AB is parallel to CE , and on them falls AC ; the Alternate angles BAC, ACE , are equal to one another. [Prop. 29.] Again, because AB is parallel to CE , and on them falls the strait line BD ; the outward angle ECD , is equal to the inward and opposite ABC . But it hath been prov'd that ACE is equal to BAC ; therefore the whole outward angle ACD is equal to the two inward and opposite BAC, ABC .



Let the angle ACB be added in common, therefore the angles ACD, ACB , are equal to the three angles ABC, BAC, ACB : But the angles ACD, ACB , are equal to two Right; [Prop. 13.] therefore ABC, BAC, ACB , are also equal to two Right.

Therefore of every Triangle one of the sides being produced, the outward angle is equal to the two inward and opposite.

And the three inward angles of a Triangle, are equal to two Right. Which was to be demonstrated.

Corollaries.

1. Of an Equilateral Triangle all the three angles are given.

For each angle is a third part of two Right angles, that is, 60 Degrees of 180; or two third parts of one Right angle, that is 60 of 90 Degrees.

2. Of an Equicrural Triangle if one angle be given, the other two are also given.

For the angles at the base are equal, and the third angle compleats, or makes up two Right angles, that is, 180 Degrees, or twice 90.

3. Of a Scalene Triangle, if two angles be given, the third is also given; and if one angle be given, the sum of the other two is also given.

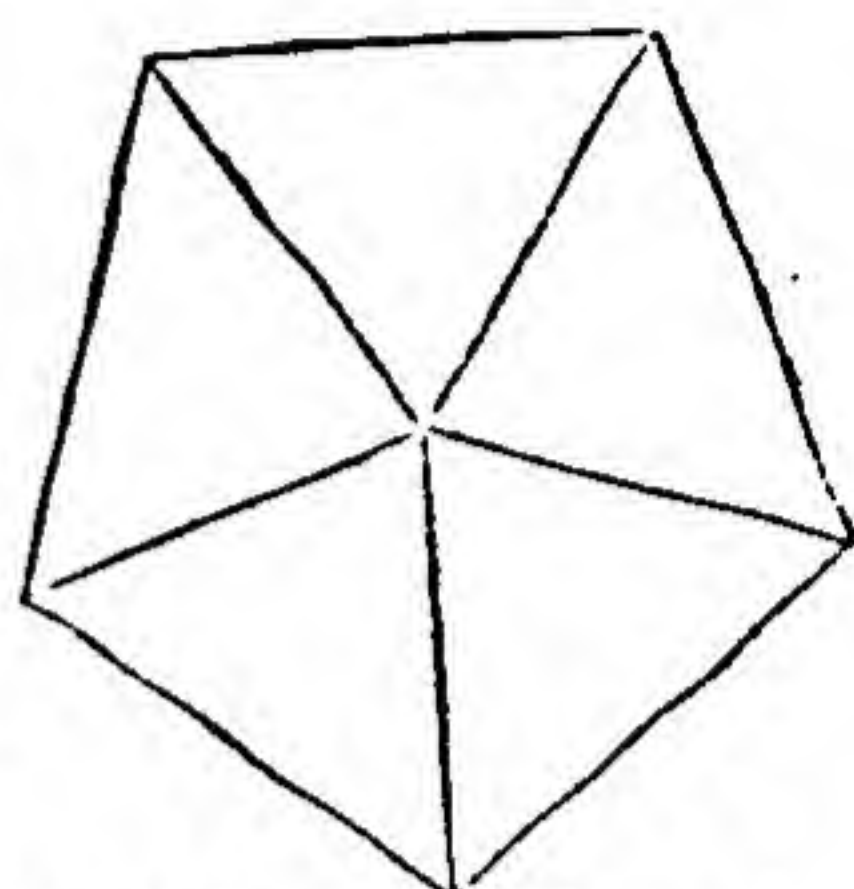
For these angles added to the given angle, compleat two Right angles. As if the given angle be 60, the sum of the other two is 120, which together make 180, or two Right angles.

4. Of a Scalene Right-angl'd Triangle, if one of the acute angles be given, the other is also given.

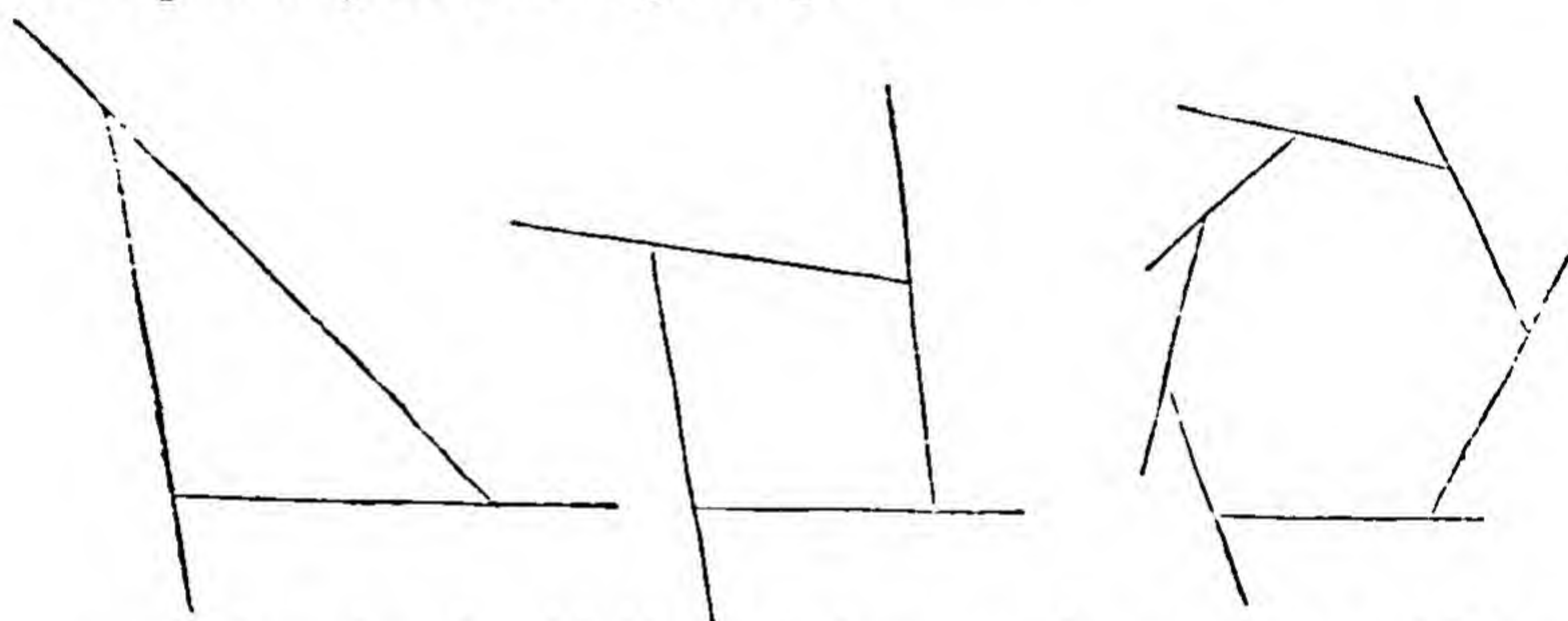
For each of the Acute angles is the Complement of the other to a Right angle. As if one be 60 Degrees, the other is 30, which together Compleat 90 Degrees, or a Right angle.

5. Of every Multilateral Figure the inward Angles are equal to twice so many Right Angles, less by four, as is the number of their Sides.

For from any point within the Multilateral Figure, let strait lines be drawn to every angle, then shall there be made so many Triangles as is the number of the Sides. As in a Figure of five Sides, there shall be five Triangles, which contain twice five, or ten Right angles. And of these Triangles their Vertical angles about the point within, are always equal only to four Right angles: wherefore the remaining angles are equal to six Right angles, that is, to twice five, less by four. And the like in all other Multilateral Figures.



6. Of every Multilateral Figure, the outward angles are altogether equal only to four Right angles.



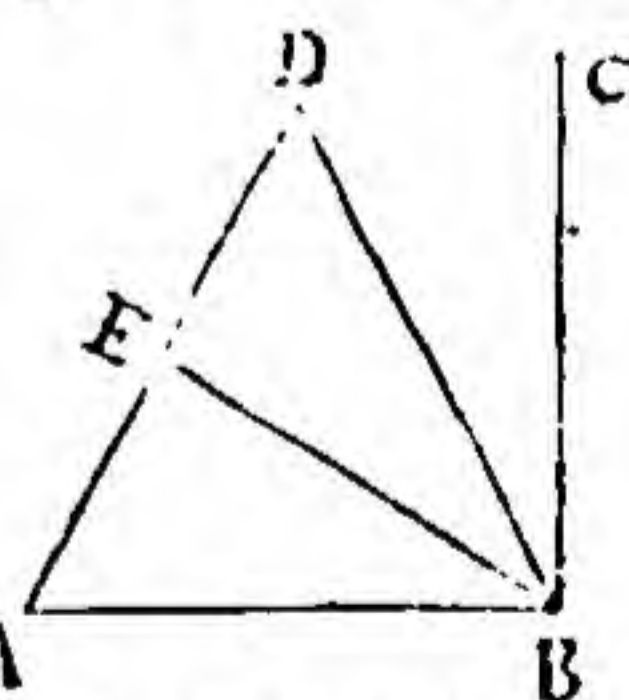
For each inward angle with it's outward, are together equal to two Right angles; and all the inward angles are equal to twice so many Right angles, less by four as is the number of their Sides: Therefore all the outward angles are equal only to four Right angles. The same is likewise manifest in all Quadrilateral and Trilateral Figures.

ANNOTATIONS.

A Problem.

To divide a Right angle into three equal angles.

From hence 'tis manifest, how to trisect a Right angle. For let ABC be a Right angle, and on AB let be constituted an Equilateral Triangle ABD . Now because the angle ABD is two third parts of the Right angle ABC , [by the first Corollary]; therefore the angle DBC is one third of the same Right angle. Again, let the angle ABD be bisected by the line BE , then shall each angle ABE , EBD , be a third part of the Right angle. Wherefore the Right angle ABC is divided into three equal angles ABE , EBD , DBC .

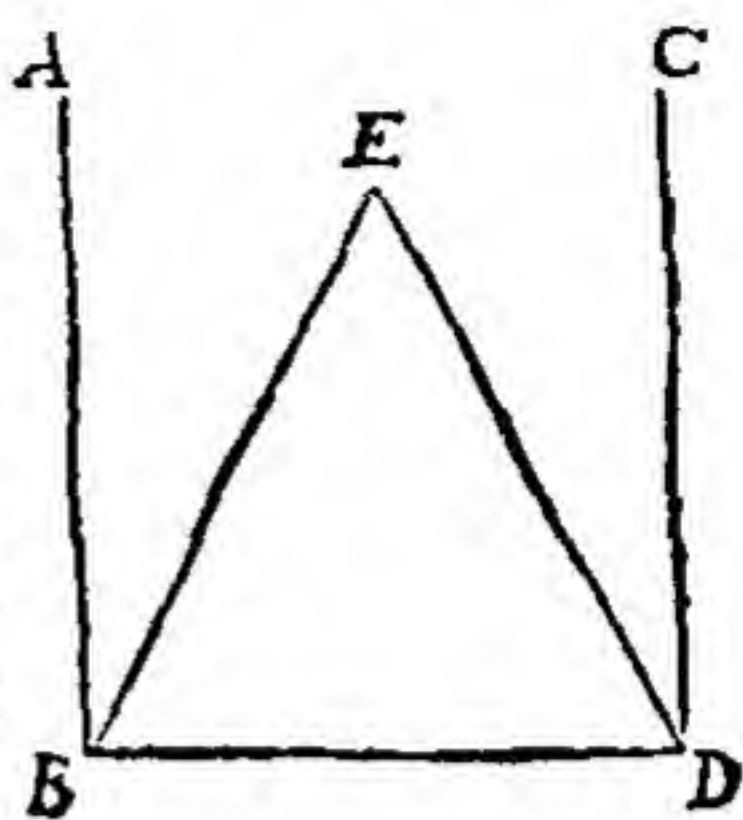


Archimedes lays the foundation of his mensuration of a Circle upon the Trisection of a Right angle, and the division of an Equilateral Triangle into two Right-angl'd Triangles; in each of which one Acute angle is known to be the double of the other, one to be 60, the other 30 Degrees: and the Side subtending the Right angle to be also double of the Side subtending the least angle, that is, AB to be double of AE . Upon which grounds he demonstratively proceeds to his Immortal Glory.

If

If every strait lin'd angle, Obtuse and Acute, could likewise be Geometrically Trisected, it would also be of excellent use. But this lyes in the same obscurity with the Quadrature of a Circle, and the Duplication of a Cube; and the pretenders to the Solutions of these Problems have all hitherto shamefully miscarry'd in their vain attempts, and overweening opinion of themselves.

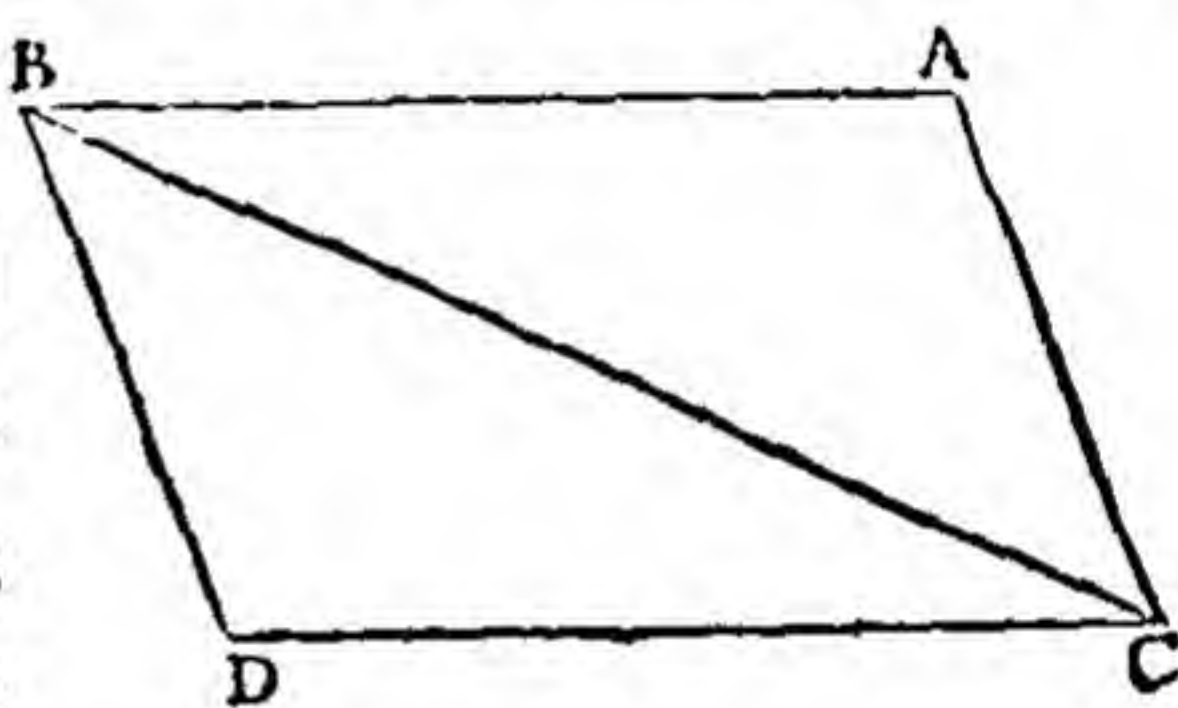
Lastly, to look into the Physical reason, why the three angles of a Triangle are equal to two Right, it may thus plainly appear. For let the lines AB, CD , be at Right angles to BD . If they be supposed to incline toward each other till they meet in the point E ; then what is by this inclination diminished from the Right angles ABD, CDB , the same is again restored in the angle BED ; so that the three angles EBD, BDE, BED , are equal to the two Right angles ABD, CDB .



PROPOSITION XXXIII.

*S*trait lines, which * the same way joyn equal and parallel lines, they also are equal and parallel.

Let the equal and parallel lines be AB, CD , and the strait lines, which the same way joyn them be AC, BD . I say, that AC, BD , are also equal and parallel: For let be drawn BC . Now forasmuch as AB is parallel to CD , and on them falls BC , the Alternate angles ABC, BCD , are equal to one another; and because AB is equal to CD , and BC common: therefore the two lines AB, BC , are equal to the two lines BC, CD , and the angle ABC is equal to the angle BCD , therefore the base AC is equal to the base BD , and the Triangle ABC is equal to the Triangle BCD , and the remaining angles shall be equal to the remaining angles, under which are subtended equal sides: therefore the angle ACB is equal to the angle CBD . And because on the two strait lines AC, BD , the strait line BC falling, hath made the Alternate angles ACB, CBD , equal; therefore AC is parallel to BD , [Prop. 27.]; and it hath been proved to be also equal to the same. Therefore strait lines, which the same way joyn equal and parallel lines, they also are equal and parallel. Which was to be demonstrated.



ANNOTATIONS.

* Which the same way,] That is, from the point A to the point C , and from the point B to the point D : not cross-ways from A to D , and from B to C .

Because the two strait lines, which joyn equal, and parallel lines are here prov'd to be equal, and parallel to one another, therefore the comprehended superficies now found to be bounded by parallel lines, is called a *Parallelogram space*: as follows in the next Proposition. Therefore it is not properly said to be a *Parallelogram Figure*, but a *Parallelogram Space*, as inclosed by parallel lines, which Space, or Area, is the thing considered in all *Euclid's* Propositions concerning *Parallelograms*. And a strange oversight it was in *Clavius* (otherwise a most faithful *Expositor*) to give a particular definition of a *Parallelogram*, as a distinct Figure, after *Euclide* had defined all the kinds of *Quadrilateral Figures*. *Quandoque bonus dormitat Homerus*.

This Theorem plainly discovers the natural Origin and Genesis of *Parallelogram spaces*, from two equal and parallel lines conjoyn'd by two other strait lines. A notion very remarkable.

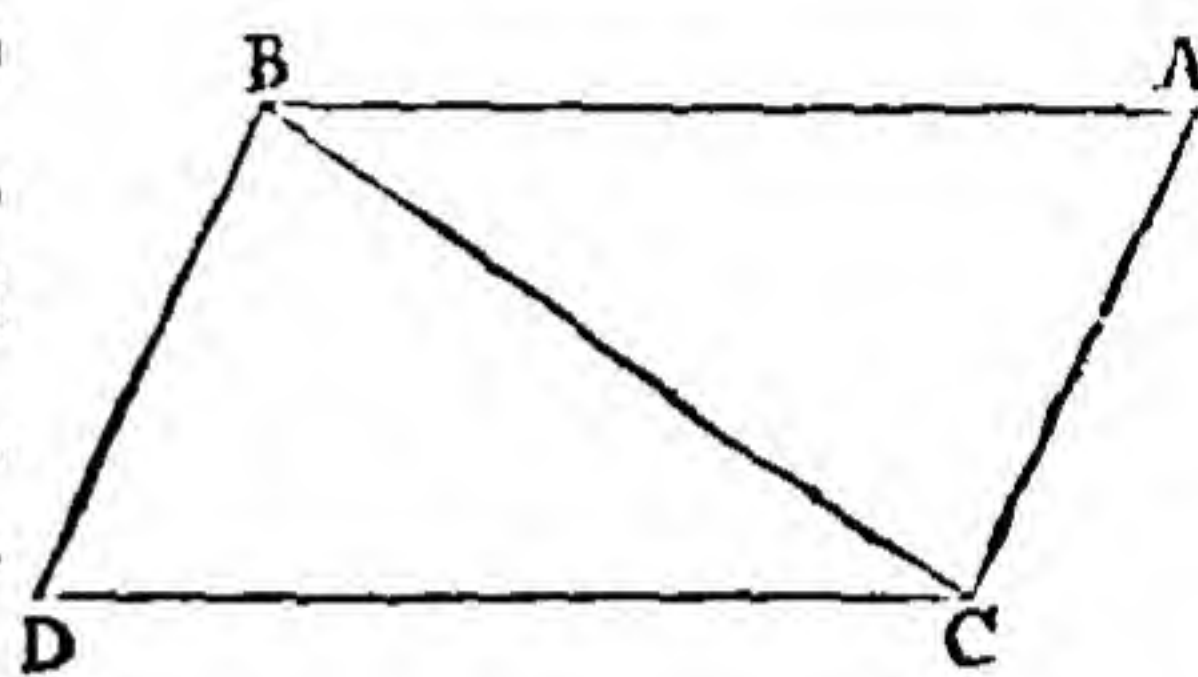
PROPOSITION XXXIV.

OF Parallelogram Spaces the opposite sides, and also the opposite angles, are equal to one another.

And the Diameter cuts the same into halves.

Let the Parallelogram be $ACDB$, and the Diameter thereof BC . I say, that of the Parallelogram $ACDB$, the opposite Sides, and also the opposite Angles, are equal to one another. And the Diameter BC , cuts the same into halves. Forasmuch as AB is parallel to CD , and on them falls the strait line BC , therefore the Alternate angles ABC , BCD , are equal to one another. Again, because the line AC is parallel to the line BD , and on them falls the strait line BC , therefore the Alternate angles ACB , CBD , are equal to one another.

There are then the two Triangles ABC , CBD , having the two angles ABC , BCA , equal to the two angles BCD , CBD , each to each; and one side equal to one side, that is, the side adjacent to the equal angles, namely BC common to both. Therefore [by Prop. 26.] they shall have the remaining sides equal to the remaining sides, each to each, and the remaining angle equal to the remaining angle: wherefore the side AB is equal to the side CD , and AC to BD : and the angle BAC to the angle BDC . And because the angle ABC is equal to the angle BCD , and the angle CBD to the angle ACB ; therefore the whole angle ABD , is equal to the whole angle ACD : and it is proved, that the angle BAC , is equal to the angle BDC . Therefore of Parallelogram Spaces the opposite Sides and also the opposite Angles, are equal to one another.



I say also that the Diameter cuts the same into halves.

Forasmuch as AB is equal to CD , and BC common, therefore there are two lines AB , BC , equal to the two lines BC , CD , each to each; and the angle ABC , is equal to the angle BCD , wherefore also the Base AC is equal to the Base BD , and therefore the Triangle ABC is equal to the Triangle BCD : wherefore the Diameter BC cuts the Parallelogram $ACDB$ into halves. Which was to be demonstrated.

ANNOTATIONS.

The name of Parallelogram Spaces, we have noted to be literally formed (as in common speech) from the termination of Planes made by parallel lines; and this name extends only to the Square, Oblong, Rhombus, and Rhomboid: wherefore after the Definitions of these four Quadrilateral Figures, *Euclid* defines not a Parallelogram; for that he had then inartificially defin'd anew, what was before defined. But now upon this common affection here demonstrated, he does comprehend under that one name the Square, Oblong, Rhombus, and Rhomboid: so that what Properties at any time are demonstrated upon Parallelograms in general, that is Parallelogram spaces, do alike belong to all these four Figures.

Euclid proceeds after the same manner in Solids, at Prop. 24. and 25. El. XI. where having laid down in distinct words a Solid comprehended by parallel Planes, he

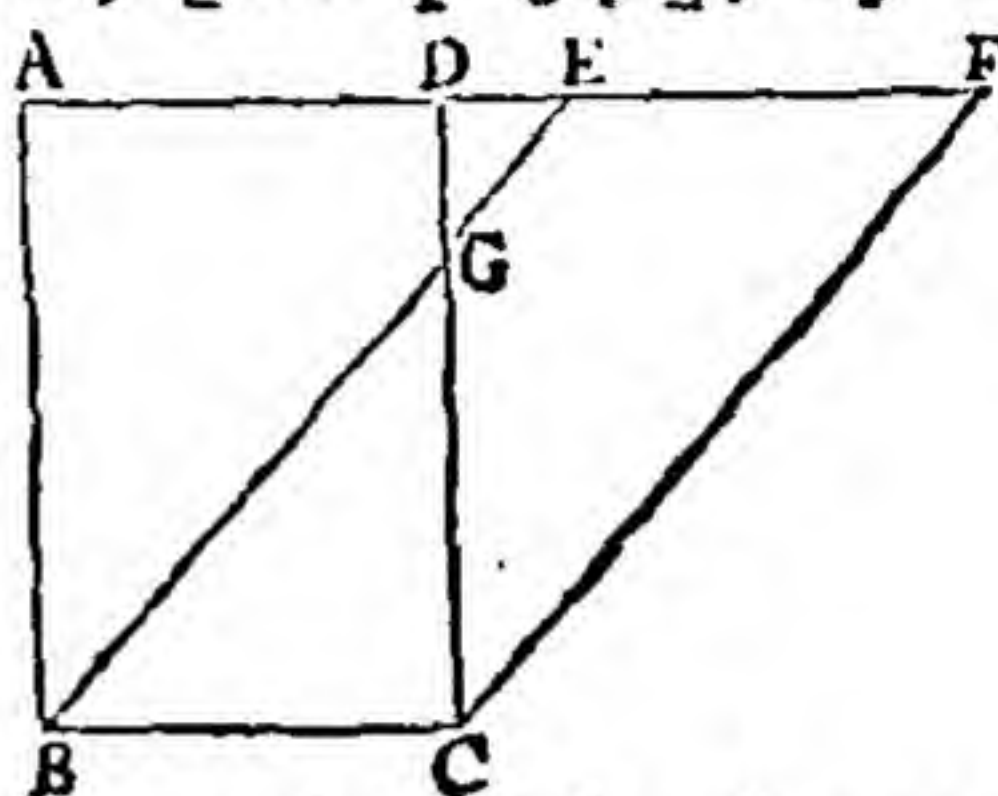
he next after this in one compound word, calls the same Solid a *Parallelepipedon*, without premising any Definition.

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PROPOSITION XXXV.

Parallelograms on the same Base, and in the same Parallels, are equal to one another.

Let the Parallelograms be $ABCD$, $EBCF$, on the same Base BC , and in the same Parallels AF , BC . I say, that the Parallelogram $ABCD$, is equal to the Parallelogram $EBCF$. Forasmuch as $ABCD$ is a Parallelogram, therefore AD is equal to BC , [Prop. 34.]; by the same reason also EF is equal to BC : so that AD is equal to EF , and DE is common; therefore the whole AE is equal to the whole DF ; but also AB is equal to DC . Wherefore the two lines EA , AB , are equal to the two lines FD , DC , each to each, and the angle FDC is equal to the angle EAB , the outward to the inward; therefore the Base EB is equal to the Base FC , and the Triangle EAB , is equal to the Triangle FDC . Let DGE common to both be taken away: then shall the Trapezium $ABGD$ be equal to the Trapezium $EGCF$. Let the Triangle GBC be added in common: therefore the whole Parallelogram $ABCD$, is equal to the whole Parallelogram $EBCF$: wherefore Parallelograms on the same Base, and in the same Parallels, are equal to one another. Which was to be demonstrated.



ANNOTATIONS.

Of Geometrical Places.

When in Theorems, or Problems, the same thing may be alike in several places *Indeterminately*, then is this call'd the *Geometrical Place* of that Theorem, or Problem, and these kind of Propositions are call'd *Local Theorems*, and *Local Problems*. As in this 35. Prop. it evidently appears, that to the Parallelogram $ABCD$, there may be infinite other equal Parallelograms, on the same base AB , in the same parallel lines: so that of one of the parallels the whole line, as AF , infinitely produced, is the common Place of this Equality in Parallelograms seated on the same base: The like also is on equal bases. And moreover in Triangles on the same, or equal bases; as it is demonstrated in the 36, 37, 38, and 41. following Propositions; This is said to be *Locus planus ad lineam rectam*. Likewise there are Geometrical Plane Places of the same nature, found in the Circumference of a Circle. As if it be required to draw from the ends of a strait line two strait lines, which shall contain a Right angle; 'tis evident by Prop. 21. and 31. El. III. that in a Semicircle every one of the angles is a Right angle, so that the Circumference of a Semicircle, is the Geometrical Place of a Right angle. This is said to be *Locus planus ad Circumferentiam circuli*, and the Problem called a plane Problem, or a Problem *in loco plano*.

Besides these plane Places in strait lines, and the Circumference of a Circle, there are also *Loci solidi*, Geometrical solid Places, which admit of such solid Problems. These are found in the *Conic Sections*, namely, the *Parabola*, *Hyperbola*, and *Ellipsis*.

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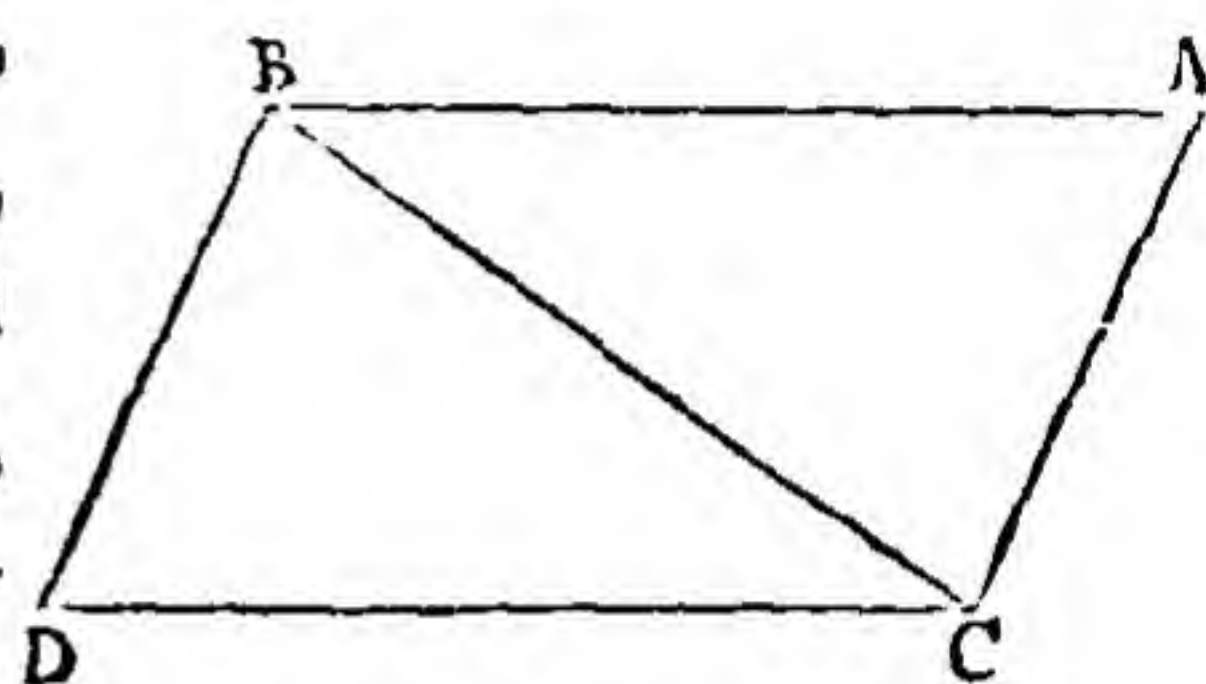
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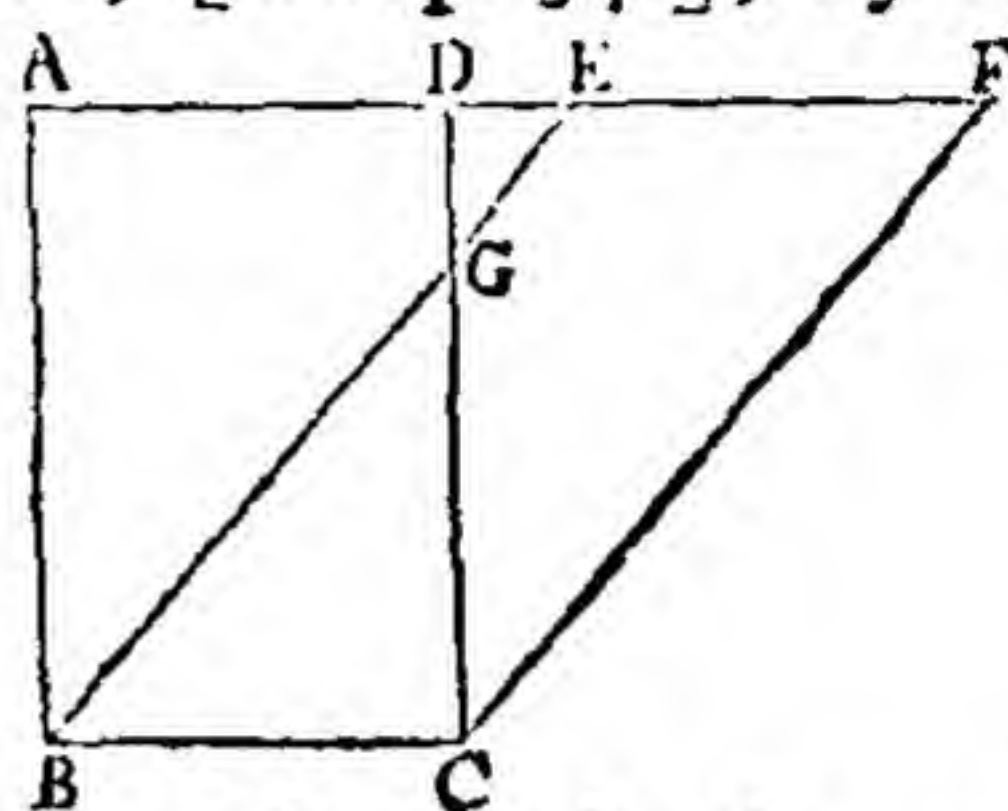
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They

They are called solid Places, and solid Problems, notwithstanding that these Figures lye in a plain superficies, because they have their Origin in a solid Figure, as the CONE: and are made by the cutting of a Conical Superficies with a Plane: as the Conic Elements of *Apollonius* shew, how these Figures are seated and created in a Conic Body. *Prop.* 11, 12, 13. *Lib.* I.

There are also Linear Problems differing much from these Solid, and Plain Problems, tho' they be described in a simple plain superficies; but not by a simple motion, as is the strait line, and Circle. They are therefore in a special manner called *Linear*, because their solutions are effected by certain lines arising from compounded, and involved motions. Such is the Helix or Spiral line of *Archimedes*, the Conchoid of *Nicomedes*, the *Lineæ tetragonizantes*, or *Quadratrices*, with divers others described by the Ancients, and Moderns. See *Pappus* after *Prop.* 4. *Lib.* III. and *Prop.* 30. *Lib.* IV.

Geodæsia, or the Mensuration of Plain Figures.

Elementary Annotations.

Upon this Proposition, and some of the next following, is grounded the Doctrine of the Mensuration of all Plain Figures, as to their superficial Content, or *Area*; which is one sort of practical Geometry deduced from these Speculative Elements; and of a necessary use in many human Affairs. This Doctrine is commonly named *Geodæsia*, from the Partition and Distribution of Lands; it being one of the most valuable Matters handled in this part of Geometry: And with us particularly called the *Art of Surveying*. But the use of the word *Geodæsia*, like as the word *Geometria*, is enlarged beyond its original signification, and extended to the general Doctrine of the Mensuration of all sorts of Figures in a plain superficies. And to this use fully answers the name *Epipedometria*, or *Planometria*, an easier word, tho' Critically not so proper, as being compounded of Latin and Greek.

Now in all kind of Mensuration, whatsoever is taken for a measure whereby to estimate and value any proposed quantity, the same must be certain and determined. In *Discrete* quantity it is an *Unite*, which naturally measures all Numbers. In *Continued* quantity, as Magnitude, it must be a *supposed Unite* to measure Magnitudes. I say *supposed*; for that Magnitude being a quantity infinitely divisible, has no indivisible unite in it self, whereby to measure Magnitudes, as Number has an indivisible unite to measure Numbers. But instead thereof we make to our selves by mutual agreement some certain measures, as an Inch, or Foot, in every kind of Magnitude, which as a Geometrical unite may answer to an unite in Numbers, so that in Magnitudes the Geometrical measure is only a *supposed Unite taken by Consent*.

As *some one* strait line is put to measure Lengths: And let this measure be called the *Lineal Unite*.

Some one plain Figure to measure plain Figures: And let this be called the *Superficial Unite*.

Some one Solid Figure to measure Solids: And let this be called the *Solid Unite*.

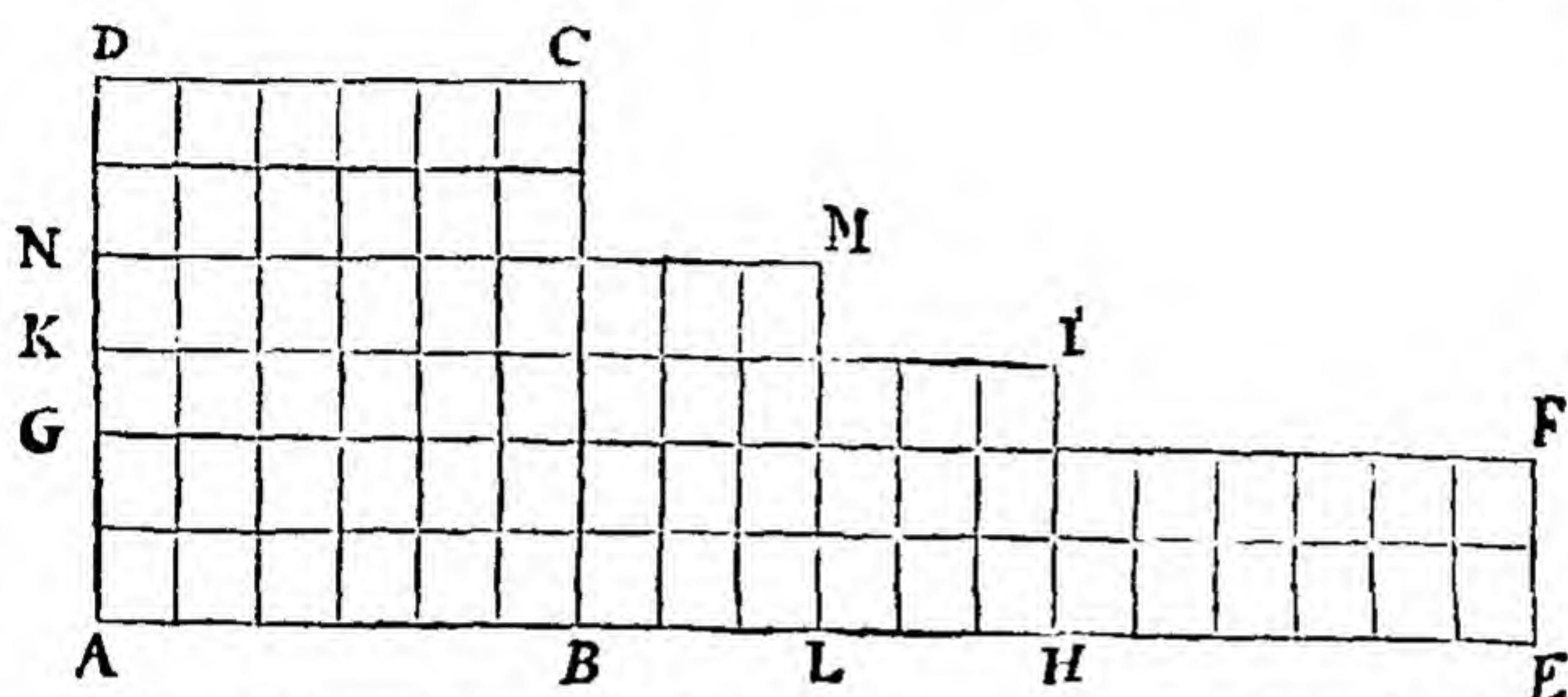
The value then, or estimate of any Magnitude is made from the multitude or number of the Geometrical measuring Unites, which that Magnitude shall contain: Be they Lineal, Superficial, or Solid Unites, according to the species of the Magnitude, as it is either a length, a superficies, or a solid, which is proposed by some certain measure to be estimated.

In the Mensuration of Lengths, there is no other trouble than to agree upon what *known Length* the Lineal Unite, or measuring Line shall be. Whether Inch, Foot, Yard, Pearch, or any other Civil and Political measure, according to the custom of the Place.

But in the Mensuration of Planes, which is according to Length and Breadth, it is not only a known superficial quantity to be agreed upon, but also *what Figure of a known superficial quantity*, is most proper to be the common measure of all plain Figures.

This matter requires some Artifice, in regard of divers mistakes that may arise
in

As for Example, let the Figure ABCD be right angl'd, and have the side AB 6 inches in length, and the side BC as much in breadth: and accordingly let the Figure be divided by parallel lines. So now it is easily demonstrated from the Diagram, that the four sides of this Figure (which are its Perimeter) shall be 24 inches; and the whole Area shall contain 36 square inches: As is also found by multiplying 6 into 6; that is, by drawing the length AB into the breadth BC; which is the general Rule of all superficial Mensurations; for that every figurate superficies is to be measured by the two dimensions of length and breadth.

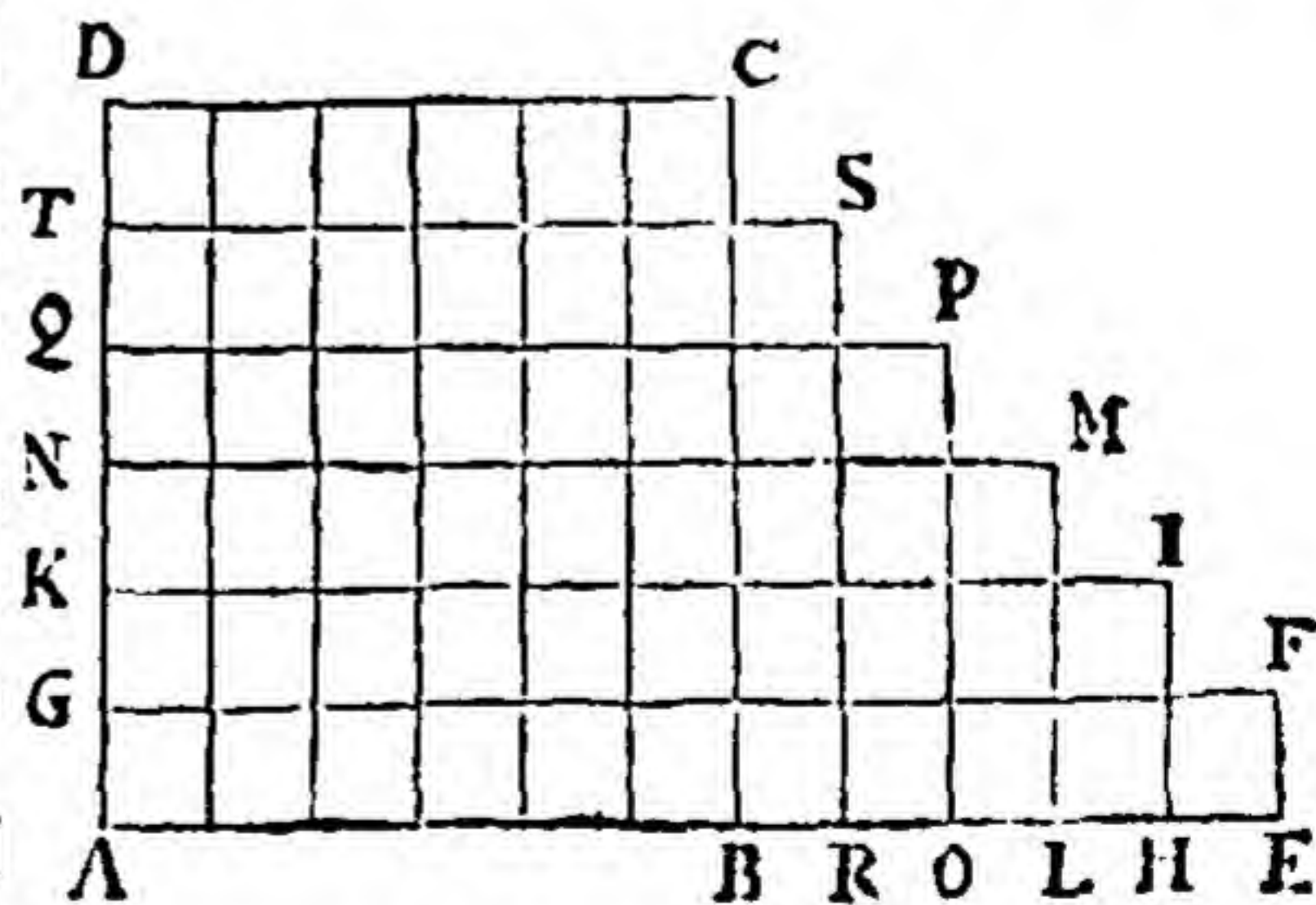


Again, let the Oblong A LMN have the side AL 9 inches, and the side LM 4 inches, then the Perimeter shall be 26 inches; and the Area as before 36 square inches.

| | | | | |
|---------------------|-----|----|-----------------|---------------------------|
| Areas equal all 36. | A B | 6 | } Perimeter 24 | } Perimeters all unequal. |
| | B C | 6 | | |
| | A E | 18 | } Perimeter 40. | |
| | E F | 2 | | |
| | A H | 12 | } Perimeter 30. | |
| | H I | 3 | | |
| | A L | 9 | } Perimeter 26. | |
| | L M | 4 | | |

In these Figures we have the Areas equal, and the Perimeters unequal: But the Perimeter of the square is the least; and in Oblongs of equal Areas with the square, where they differ most from a square Figure, that is, where the difference between the length and breadth is the most, there the Perimeter is the greatest; and as the difference becomes less and less, so the Perimeter is less and less, till in the square it is the least of all.

Likewise let the Oblong AHIK have the side AH 10 inches, and the side HI 2 inches; the Perimeter is again 24 inches, but the Area 20 square inches.



So farther, let the Oblong A L M N have the side A L 9 inches, and the side L M 3 inches: wherefore the Perimeter is also 24 inches; but the Area is 27 Square inches.

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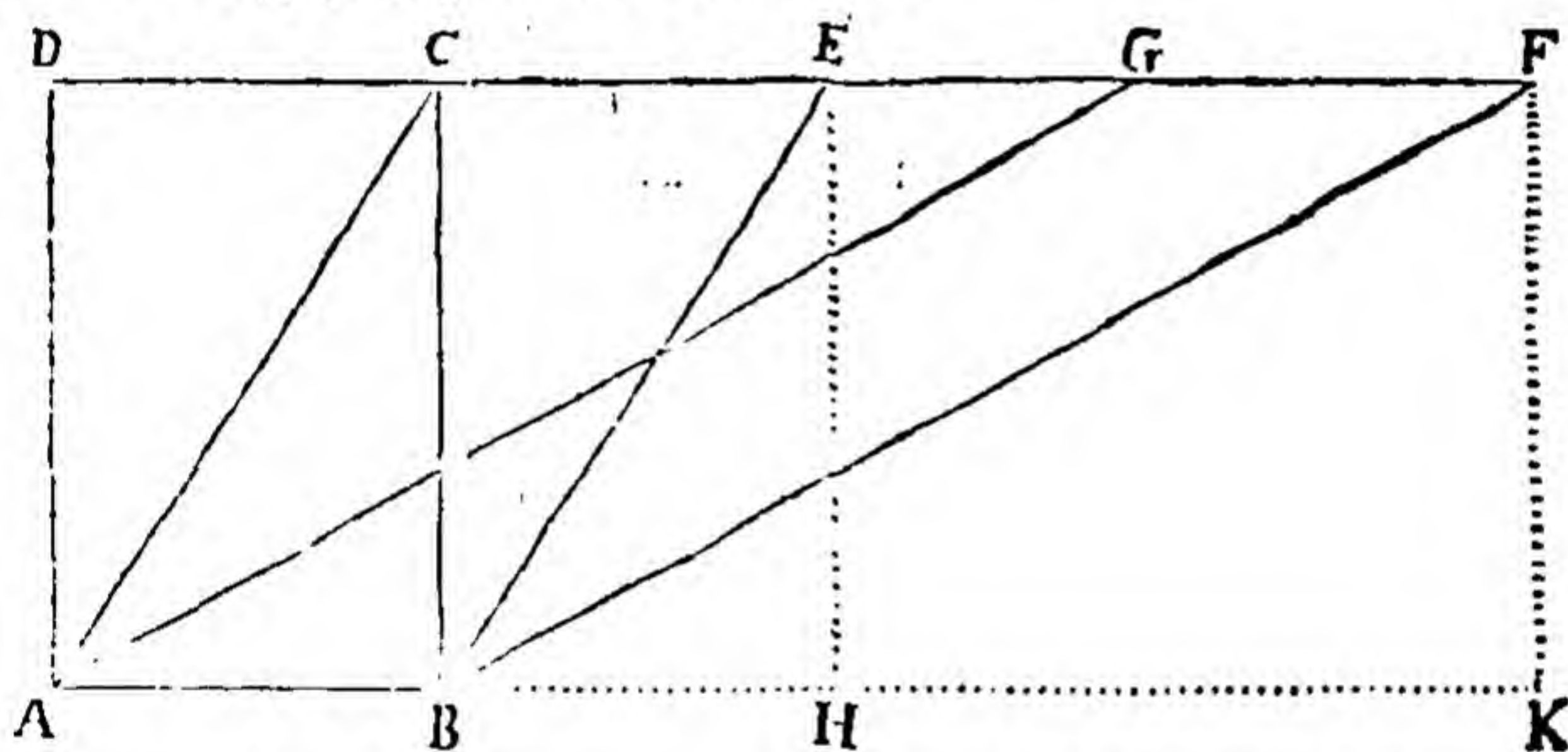
Again.

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side BE ought not to be esteemed its proper measuring breadth. For besides that this falls into the former erroneous way of measuring by the Perimeter, the absurdity is also farther made thus manifest. If of the Parallelogram space ABE, the side BE be its proper breadth, then likewise in the Parallelogram ABFG, having the same length AB, the side BF should as well be accounted the proper breadth thereof. Now because these Parallelogram spaces are equal to one another, and have the same length AB, therefore their breadths BE, BF, should also be equal to one another: but BE, BF, are easily demonstrated to be unequal; therefore their proper breadths are not to be esteemed by the sides BE, BF. In general therefore the oblique position of length to breadth, is a way altogether uncertain and undeterminable, as being infinitely variable, and so unfit for any Rule or common Practice in these superficial Mensurations.

Forasmuch then that neither from the Perimeter, nor from length and breadth taken obliquely towards one another, can be formed a Rule for the Mensuration of Parallelogram spaces; it necessarily follows that length and breadth are only to be taken at Right angles each to other, a way one and the same unalterable, commonly known, and easily practiced. And hereupon 'tis manifest, that the Right angl'd Parallelogram on the same base, and in the same parallels, is the *Standard* unto which all the other oblique angl'd Parallelograms are to be referred for their Mensurations. As the Areas of ABE, ABFG, &c. are all to be known from the Area of the Rectangle ABCD; for that any two of its sides, which contain an angle, as AB, BC, or AD, DC, being at Right angles to one another, are the very proper length and breadth of this Parallelogram; one whereof being drawn into the other, brings forth the Area, which in this 35th. Prop. is demonstrated to be equal to all possible oblique angl'd Parallelograms on the same base, and in the same parallels.

Therefore for the Mensuration of an oblique angl'd Parallelogram, it must be reduced to its Equivalent Rectangle: And this is done by drawing from any one side a perpendicular to the opposite, produced if need be.

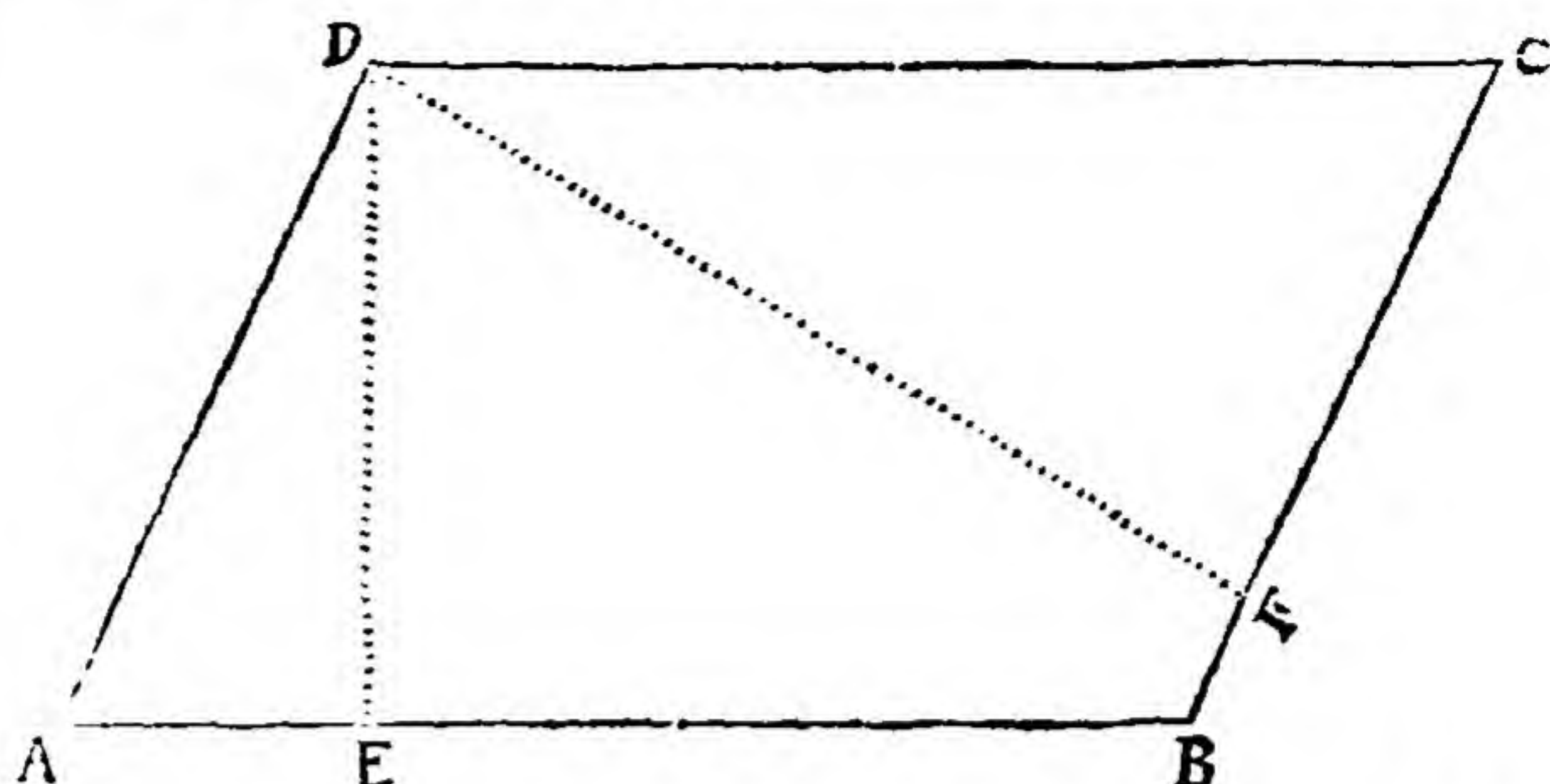


As in the oblique Parallelogram ABE, from the side CE, let be drawn EH perpendicular to the opposite side AB the base produced. Here then EH is equal to CB, or DA, the opposite sides of the Rectangle ABCD, for that they are parallel by Prop. 28. and therefore equal by Prop. 34. So that the perpendicular EH is the proper breadth of ABE. And as the Rectangle ABCD is measured according to its proper length and breadth, by the base AB drawn into the perpendicular BC, so is the Rhomboid ABE measured by the same base AB, drawn into the perpendicular HE equal to BC, or AD: Likewise ABFG is measured by AB into KF. Thus the perpendicular is the only true, and common breadth of all Parallelograms on the same base, and in the same parallels. Therefore the Rule for the Mensuration of oblique angl'd Parallelograms is this.

In oblique angl'd Parallelograms, the base, and a perpendicular to the base, from the opposite side, drawn into one another, give the Area of the Parallelogram.

By the base is meant any one side of the Parallelogram taken two ways; either

by letting fall a perpendicular from the longer side upon its opposite as a base, or from the shorter side upon its opposite as a base. For in the Rhomboid $ABCD$ the perpendicular may be DE upon the base AB , or the perpendicular DF upon the base BC . From these two different cadencies of the perpendicular upon the base, the Rectangles are changed both in base and perpendicular: yet each Rectangle is equal to the same Rhomboid by this 35th. Prop. And the Rhomboid is indifferently estimated by either Rectangle.

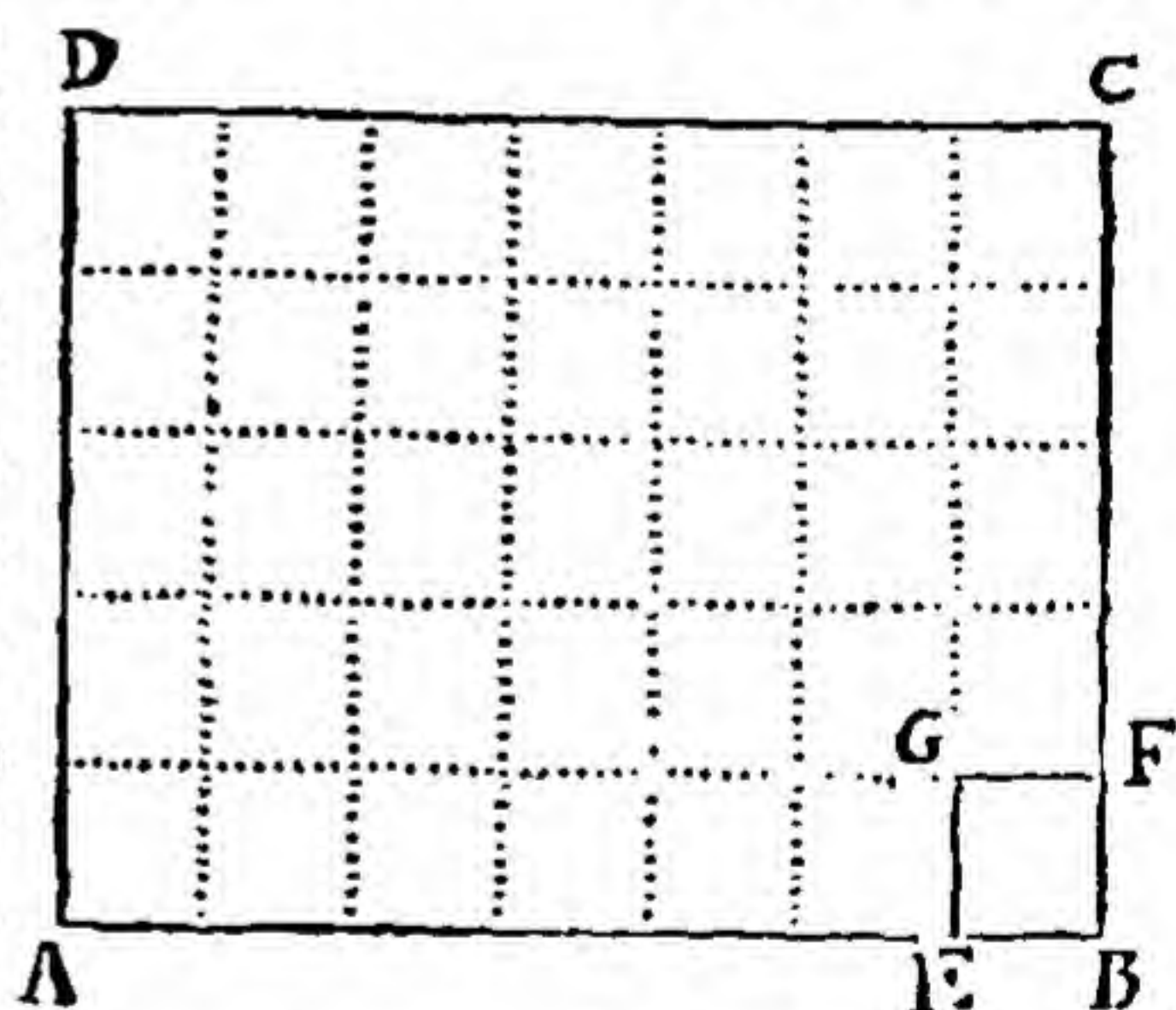


The perpendicular with its base; as DE with AB , or DF with BC , are called the *Laterarecta*, or the upright sides of the oblique Parallelogram, because they make a right angled Parallelogram equal to the oblique.

Now for the Mensuration of a Rectangle, we are first to know the quantity of its length and breadth, that is, the distinct quantity of two sides containing any of its angles. As for Instance of the Rectangle $ABCD$, we are to know the distinct quantity of the lines AB and BC .

To find therefore the quantity of any proposed length, we must have recourse to some certain and known measure of lengths, for a *Lineal unite* to make an account by the same. As in the side AB , let BE be the Lineal unite, whether Inch, Foot, or Yard, &c. put by consent for the Mensuration of AB . The same measure then is still to be used in the same matter, to avoid the confusion of measures; and therefore the very same is to be taken in the side BC for the Mensuration thereof; and let it be BF equal to BE ; and let the Figure compleated by lines parallel to EB , BF , be $EBFG$, or else conceive EB to pass transversely the length BF , equal to BE , then shall likewise be traced forth the same superficial unite or measuring Plane $EBFG$, whereby to estimate the whole Parallelogram space of $ABCD$.

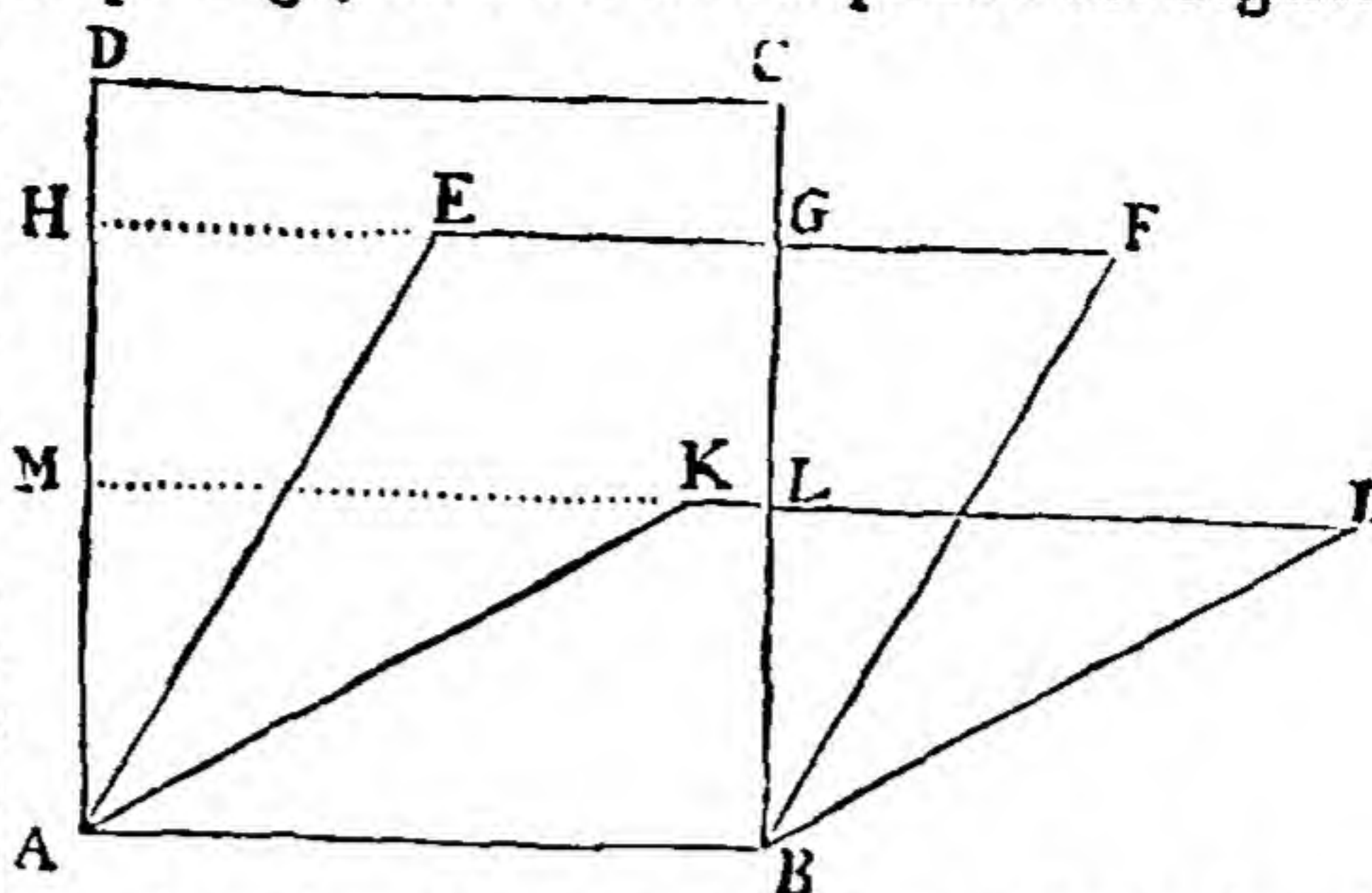
Whereas then EB , BF , are equal to one another, therefore the Figure $EBFG$ is a square: and if the line EB be put an inch, then $EBFG$ is in its Area an inch square; so that of the rectangle $ABCD$, if the side AB contain EB seven times, that is, seven Lineal units, as 7 inches, and according to the same measure the side BC 5 inches, then the whole space shall contain 35 superficial units, or square inches, each of them equal to $EBFG$. And this at once is found by Multiplying 7 into 5; that is, by drawing the length AB into the breadth BC : The general ground of all superficial Mensurations.



Thus have we shewn how an oblique Parallelogram space, is first to be reduced to Rectangle for its Mensuration; and then how all Rectangular Spaces, Squares, and Oblongs, must be measured by some certain Square space, whose side is a known measure of lengths.

Lastly, therefore to confirm this matter, let there be put some known measure of lengths, which we call the *LINEAL UNITE*, and let it be the line AB , suppose a foot: and to AB let be put AD equal and at *Right angles*. Again, let AE be

be put equal to AB , and at an *oblique angle*, then let be compleated the Figures $ABCD$, $ABFE$. Therefore $ABCD$ is a square, and $ABFE$ a Rhombus, having every side equal to AB the *Lineal unite*, or supposed foot. Yet the Rhombus $ABFE$, is not equal in Area to the Square $ABCD$, but only to a part thereof, namely, to the Oblong $ABGH$ by this 35th. Prop. And in this case the varieties are endless; as in the Rhombus $ABIK$, which is still a less part of $ABCD$, and but equal to the Oblong $ABLM$, and so forth infinitely; therefore there is no certainty but in the square Figure.



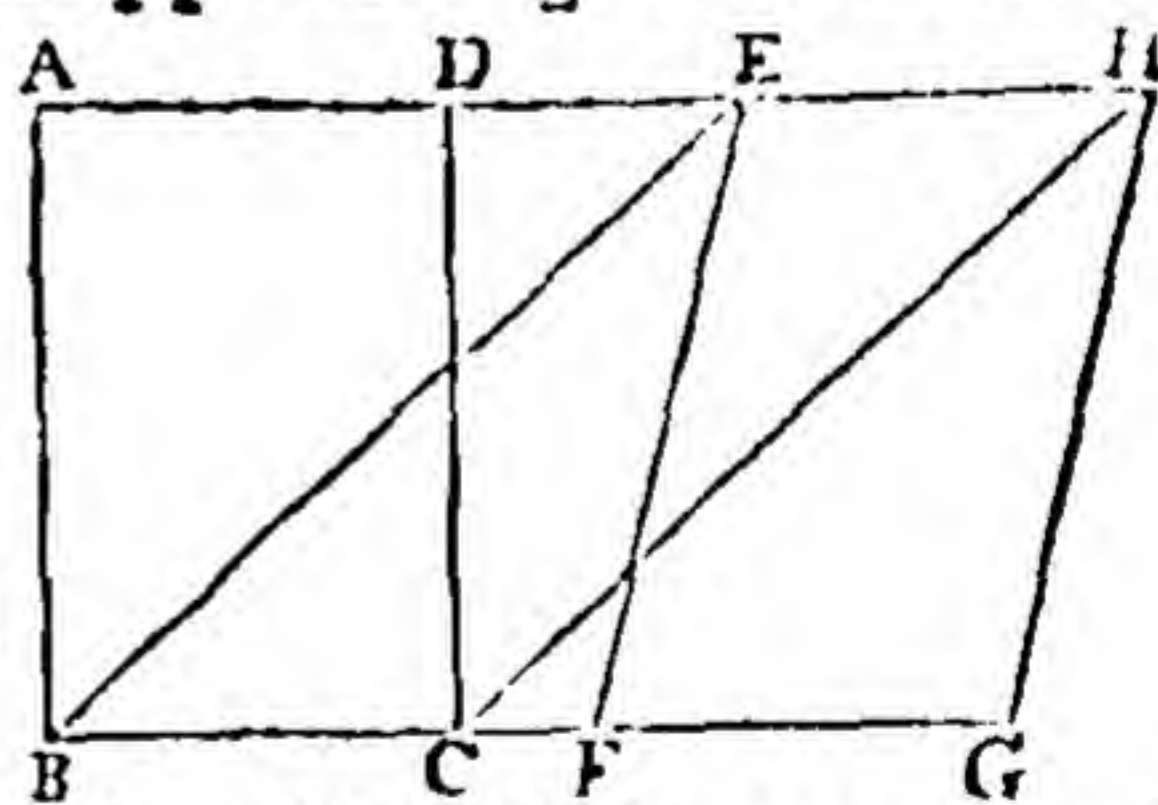
And as the *Square* is used in the Mensuration of *Planes*, so upon the like reasons the *Cube* is used in the Mensuration of *Solids*. And in general it is to be observed, that whatever measure is at first put for lengths, the same passeth for length, breadth, and depth, and forms the measuring Square or Cube; be the *Lineal unite* either Inch, Foot, Yard, Perch, &c.

Thus for the Mensuration of all kinds of Magnitudes, as they are of *one*, or *two*, or *three* Dimensions, there is in common practice constituted some certain measure conformable to each Dimension. And (as we must begin with the most simple Dimension) for lengths there is first made an agreement upon some *Lineal unite*: next, to continue in a certainty of measure, from the same *Lineal unite* is to arise the *Square unite* for Planes; and from the Square unite the *Cubic unite* for Solids. As to Instance in particular, a *Lineal Inch*, or *Foot*, &c. from this a *Square Inch*, or *Foot*, &c. then lastly, a *Cubic Inch*, or *Foot*, &c. to be the Measuring unite according to the Dimensions of the Magnitude, which is to be estimated by such or such a measure, Inch, or Foot, &c. suitable to its proper Dimension.

PROPOSITION XXXVI.

Parallelograms on equal bases, and in the same parallels are equal to one another.

Let the Parallelograms be $ABCD$, $EFGH$, on equal bases BC , FG , and in the same parallels AH , BG . I say, that the Parallelogram $ABCD$ is equal to the Parallelogram $EFGH$. For let be joyn'd BE , CH . Now forasmuch as BC is equal to FG [by Supposition]: and also FG is equal to EH [by Prop. 34.]; therefore BC is equal to EH ; but also they are parallels by Supposition, and BE , CH , joyn the same. Now lines which the same way joyn equals and parallels, are also equal and parallel [Prop. 33.]. Therefore EB , CH , are equal and parallel: therefore $EBCH$ is a Parallelogram, and is equal to $ABCD$; for it hath the same base BC , and is in the same parallels BG , AH . By the same reason, $EFGH$ is equal to the same $EBCH$; so that also the Parallelogram $ABCD$, is equal to the Parallelogram $EFGH$.



Therefore Parallelograms on equal bases, and in the same parallels, are equal to one another. Which was to be demonstrated.

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Corollary.

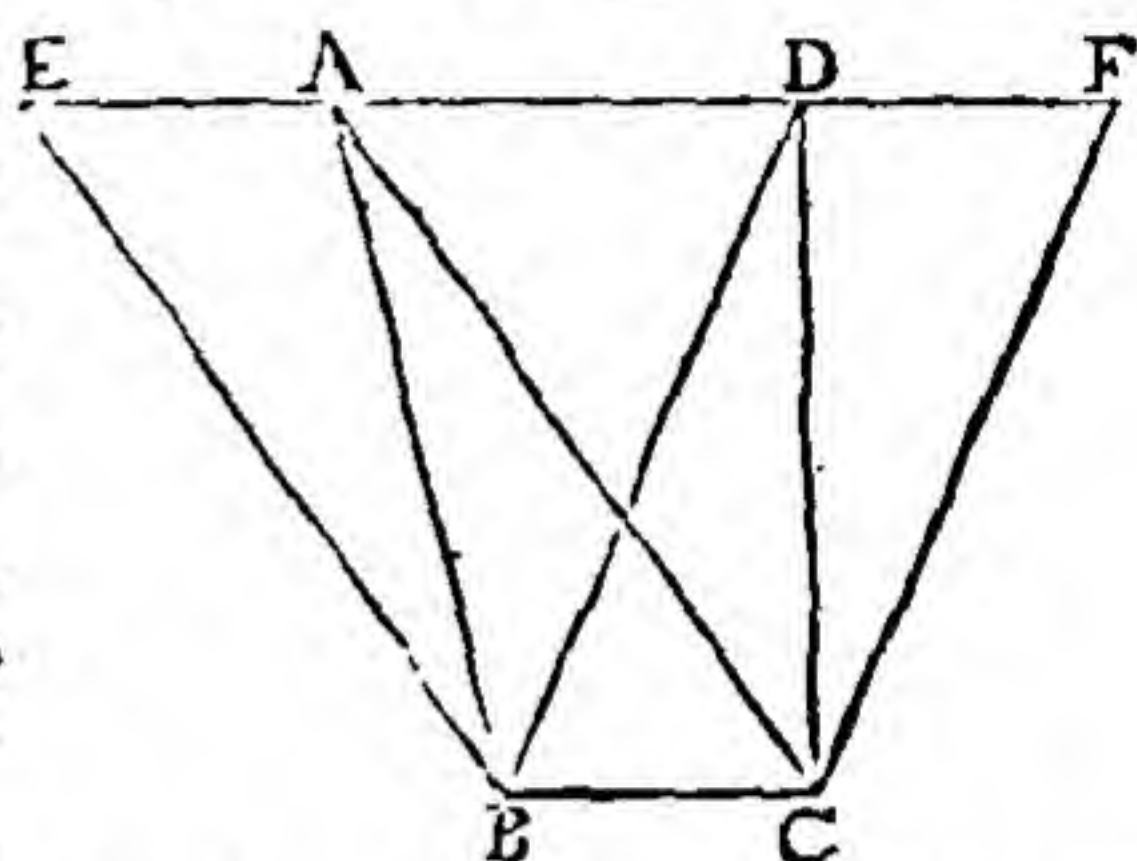
Corollary.

From hence 'tis manifest, that Parallelograms on unequal bases, and in the same parallels are unequal to one another: on the greater base the greater Parallelogram; on the less the less.

PROPOSITION XXXVII.

T *Triangles on the same base, and in the same parallels are equal to one another.*

Let the Triangles be ABC , DBC , on the same base BC , and in the same parallels AD , BC . I say, that the Triangle ABC , is equal to the Triangle DBC . Let AD be produced both ways to the points E , F , and by B let be drawn BE parallel to CA , and by C , CF parallel to BD : therefore each of these, $EBCA$, $DBCF$, is a Parallelogram, and $EBCA$ is equal to $DBCF$: for they are on the same base BC , and in the same parallels BC , EF . And the Triangle ABC is half of the Parallelogram $EBCA$; for the Diameter AB cuts the same into halves. And the Triangle DBC is half of the Parallelogram $DBCF$; for the Diameter DC , cuts the same into halves: but the halves of equals are equal to one another: wherefore the Triangle ABC is equal to the Triangle DBC .

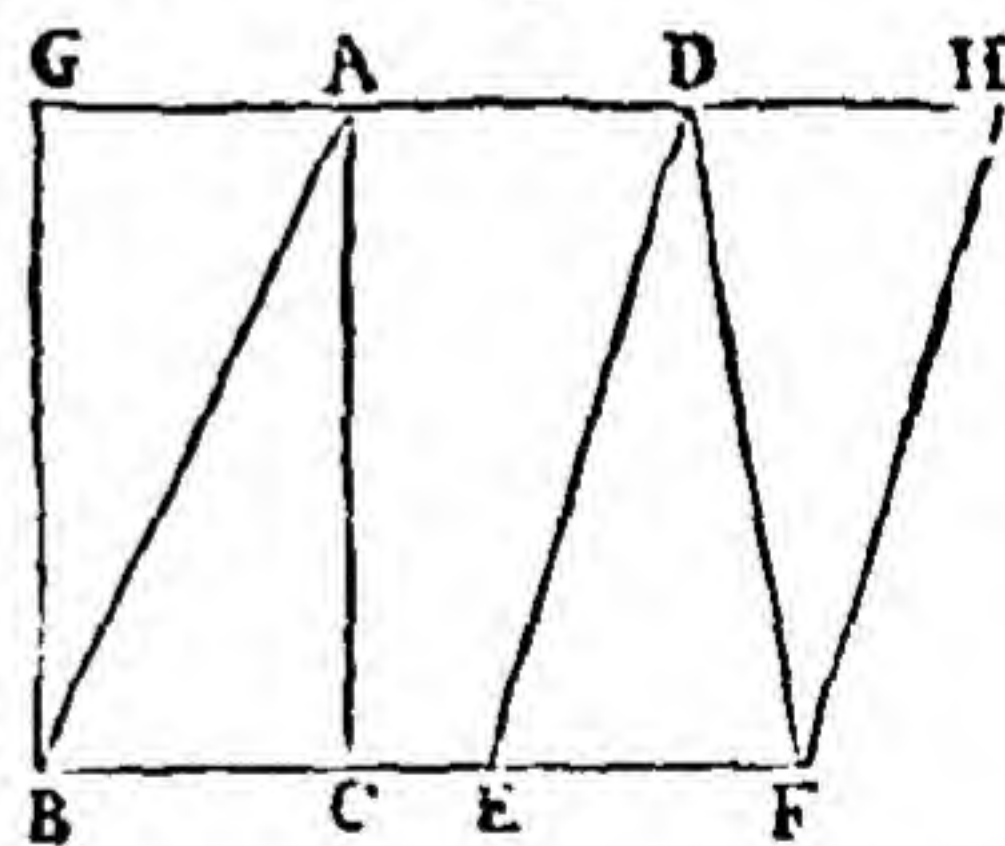


Therefore Triangles on the same base, and in the same parallels, are equal to one another. Which was to be demonstrated.

PROPOSITION XXXVIII.

T *Triangles on equal bases, and in the same parallels are equal to one another.*

Let the Triangles be ABC , DEF , on equal bases BC , EF , and in the same parallels BF , AD . I say, that the Triangle ABC is equal to the Triangle DEF . For let AD be produced both ways to the points G , H , and by B let be drawn BG parallel to CA , and by F , FH parallel to DE , therefore each of these $GBCA$, $DEFH$, is a Parallelogram. And $GBCA$ is equal to $DEFH$; for they are on equal bases BC , EF , and in the same parallels BF , GH . And the Triangle ABC is the half of the Parallelogram $GBCA$; for the Diameter AB cuts the same into halves. And the Triangle DEF is the half of the Parallelogram $DEFH$; for the Diameter DF cut the same into halves; but the halves of equals are equal to one another: wherefore the Triangle ABC is equal to the Triangle DEF .



Therefore Triangles on equal bases, and in the same parallels, are equal to one another. Which was to be demonstrated.

Corollary.

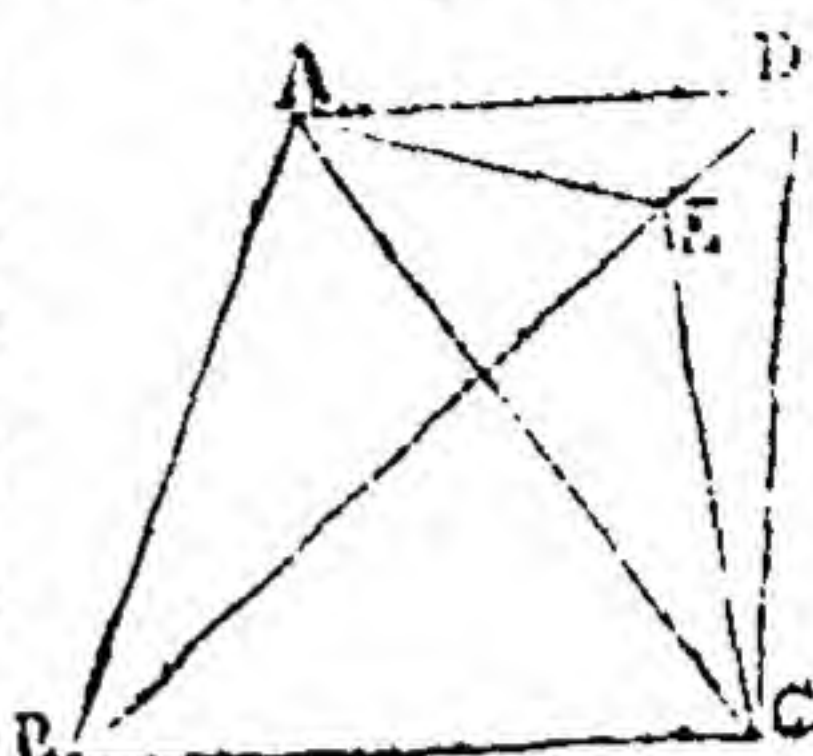
Corollary.

From hence 'tis manifest, that Triangles on unequal bases, and in the same parallels are unequal to one another: on the greater base the greater Triangle, on the less the less.

PROPOSITION XXXIX.

Equal Triangles on the same base, and the same way seated, are in the same parallels.

Let the equal Triangles be ABC , DBC , on the same base BC , and the same way seated. I say, that they are in the same parallels. For let be joyn'd AD ; I say, that AD is parallel to BC . For if not, by the point A let be drawn AE parallel to BC , and let be joyn'd EC : therefore the Triangle ABC is equal to the Triangle EBC ; for they are on the same base BC , and in the same parallels BC , AE . But ABC is equal to DBC : therefore also DBC is equal to EBC , the greater to the less: which is impossible. Therefore AE is not parallel to BC . In like manner may we prove that no other is besides AD : wherefore AD is parallel to BC .

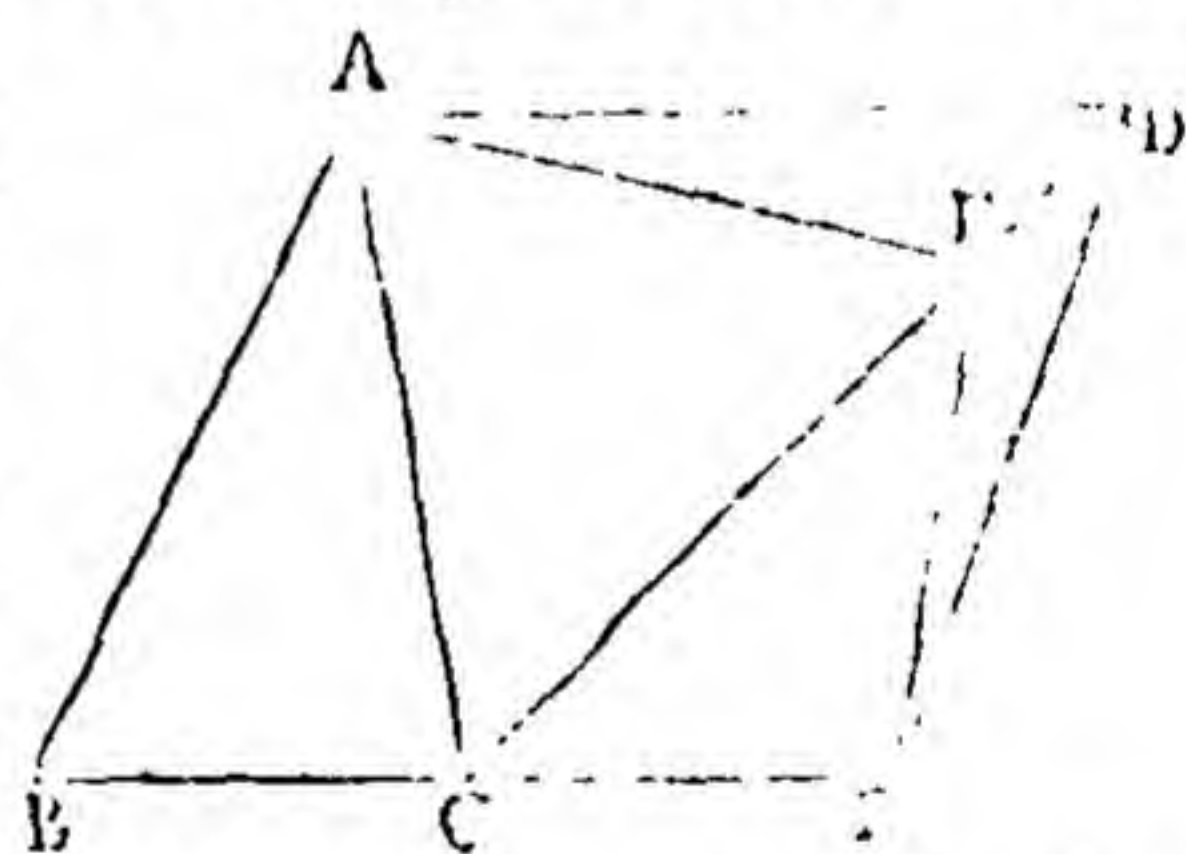


Therefore equal Triangles on the same base, and the same way seated, are in the same parallels. Which was to be demonstrated.

PROPOSITION XL.

Equal Triangles on equal bases, and the same way seated, are in the same parallels.

Let the Triangles be ABC , DCE , on equal bases BC , CE , and the same way seated. I say, that they are in the same parallels. For let be joyn'd AD ; I say, that AD is parallel to BE . For if not, by the point A let be drawn AF parallel to BE , and let be joyn'd FE : therefore the Triangle ABC is equal to the Triangle FCE ; for they are on equal bases BC , CE , and in the same parallels BE , AF . But the Triangle ABC is equal to the Triangle DCE , therefore also the Triangle DCE , is equal to the Triangle FCE , the greater to the less: which is impossible. Therefore AF is not parallel to BE . In like manner we may prove that no other is besides AD : wherefore AD is parallel to BE .



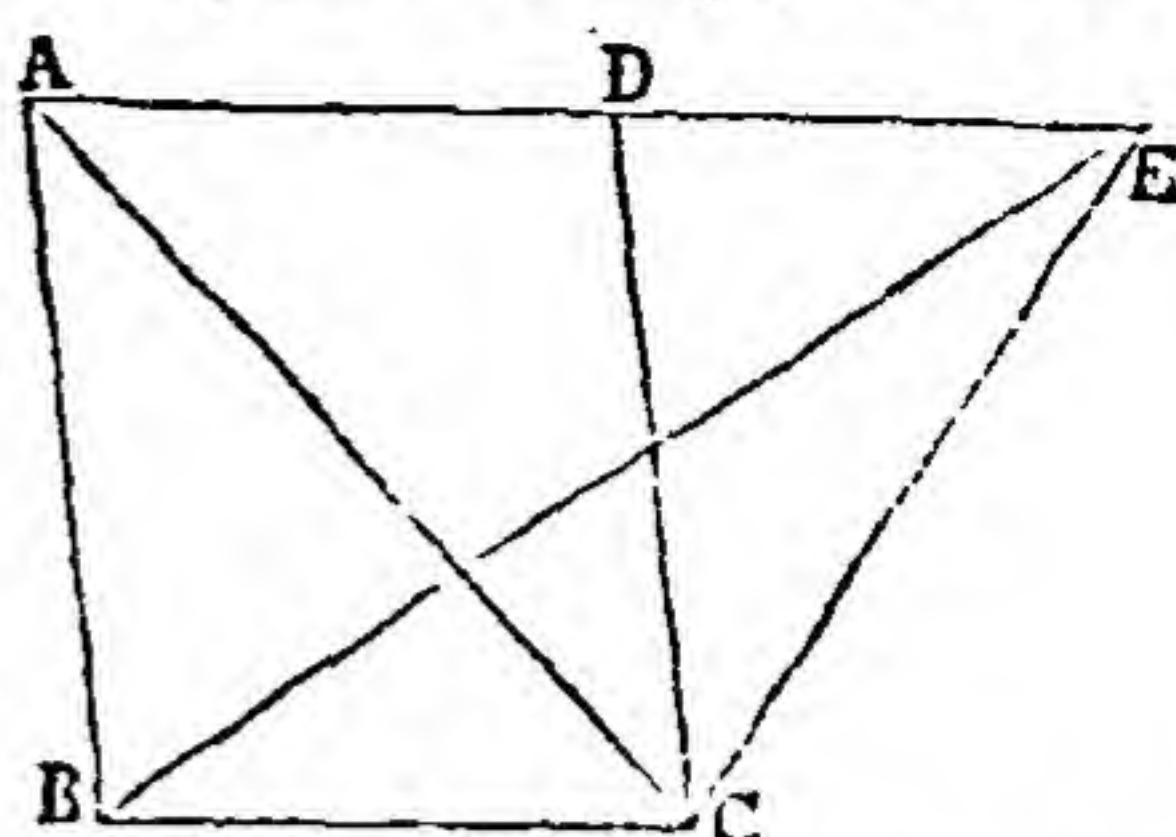
Therefore equal Triangles on equal bases, and the same way seated, are in the same parallels. Which was to be demonstrated.

PROPOSITION XLI.

If a Parallelogram shall have the same base with a Triangle, and be in the same parallels, the Parallelogram shall be double of the Triangle.

For let the Parallelogram $ABCD$ have the same base with the Triangle

Triangle EBC , and be in the same parallels BC, AE . I say, that the Parallelogram $ABCD$, is double of the Triangle EBC . For let be joyn'd AC . Now the Triangle ABC is equal to the Triangle EBC ; for they are on the same base BC , and in the same parallels BC, AE . But the Parallelogram $ABCD$, is double of the Triangle ABC ; for the Diameter AC cuts the same into halves. So that the Parallelogram $ABCD$, is also double of the Triangle EBC .



If therefore a Parallelogram have the same base with a Triangle, and be in the same parallels, the Parallelogram shall be double of the Triangle. Which was to be demonstrated.

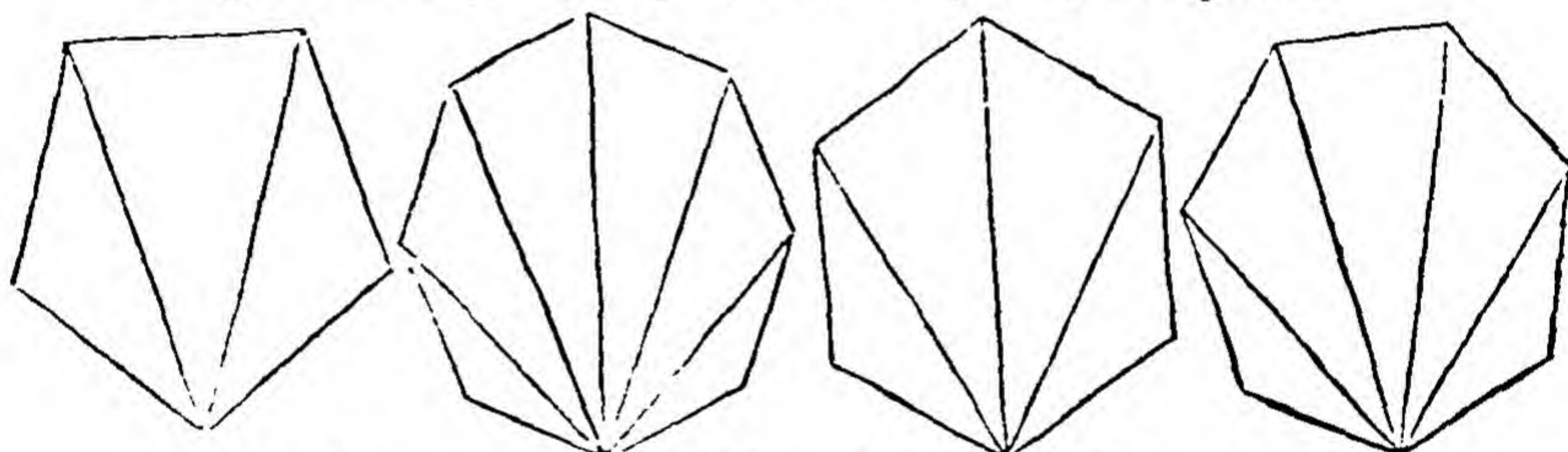
ANNOTATIONS.

This Proposition compleats the Doctrine for Mensuration of plain Surfaces: the Foundation whereof we have fully laid down at Prop. 35th. And whereas it was there shewn, that the Rule, by which all Parallelograms on equal bases, and in the same parallels are to be measured, was to multiply the base into the perpendicular: Now here 'tis further demonstrated, that the Parallelogram on the same base with the Triangle, and in the same parallels is the double of the Triangle: wherefore half of the Parallelogram is equal to the Triangle. And therefore

To find the Area of a Triangle;

Let a perpendicular from any angle of a Triangle to the base, be multiply'd into half the base, it shall give the Area of the Triangle.

And forasmuch as a Triangle is the most simple of all rectilineal Figures, therefore all rectilineal Spaces may be resolved into Triangles, and from the particular Triangles added together, be justly measured by this 41st. Proposition.



Now to reduce any Multilateral Figure into the fewest Triangles, note, that every Multilateral Figure may be divided into so many Triangles, less by two, as is the number of its Sides. As a Pentagon into 3. Triangles, less by 2, that is, into three Triangles. An Hexagon into four. An Heptagon into five. An Octagon into six, &c. As these Figures make apparent.

Corollaries.

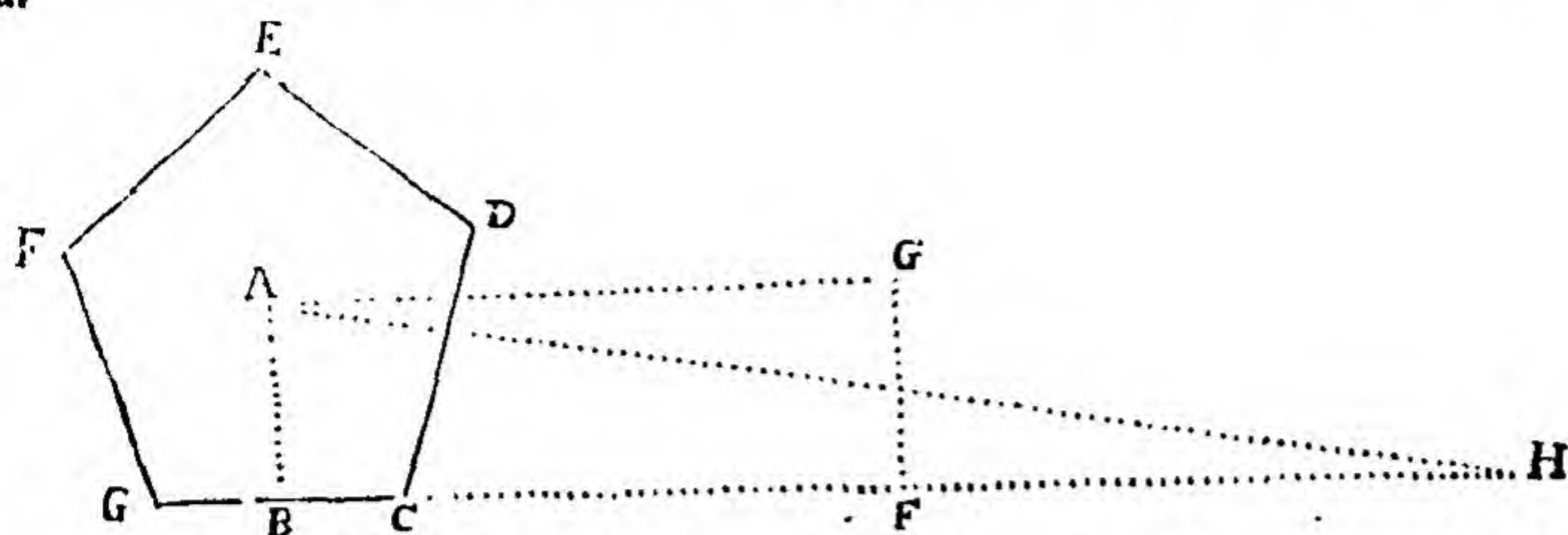
1. For the Mensuration of any Multilateral Figure.

Now to find the Area of any Multilateral Figure, let the Figure be divided after the most convenient manner into the fewest Triangles, and each Triangle be measured by its base and perpendicular, according to the foregoing Rule in multiplying the perpendicular into half of the base; then shall these Triangles added together give the Area of the Multilateral Figure. For further instructions in these kind of Matters, recourse is to be had to the Writers of Practical Geometry. Here we have only touched slightly on the Uses of these Elementary Propositions.

2. For

2. For the Mensuration of a Regular Multilateral Figure.

From hence 'tis manifest, that in a Regular Multilateral Figure a perpendicular from the Center to any of the Sides multiply'd into half the Perimeter gives the Area.

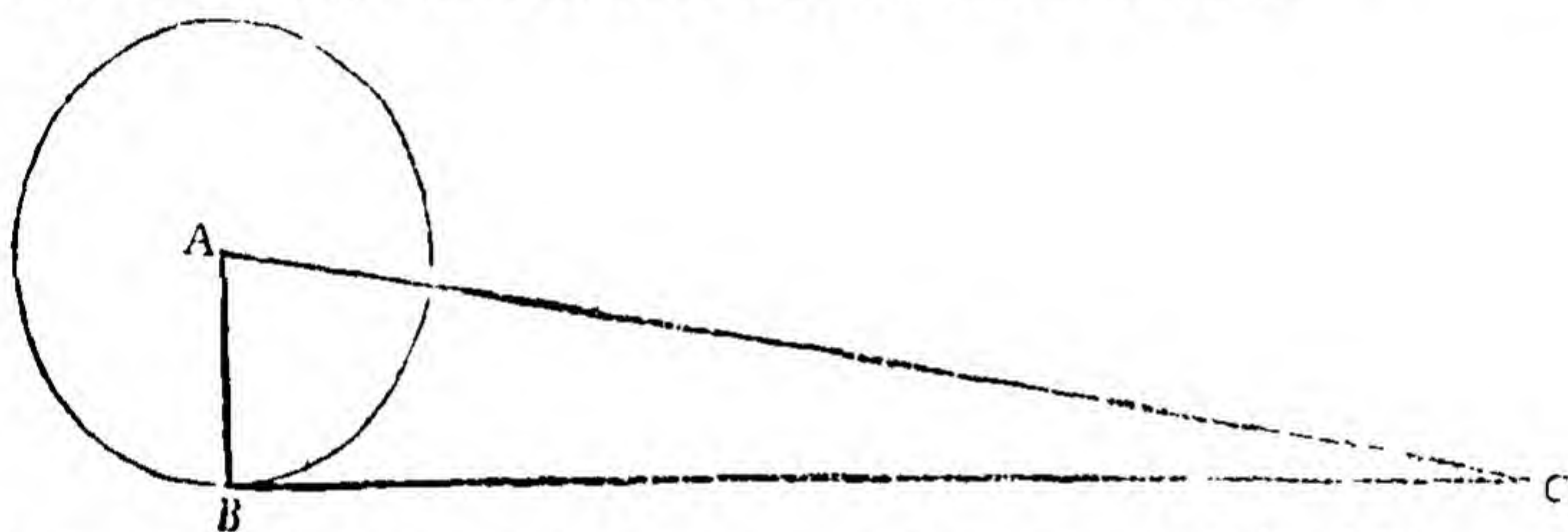


As AB multiply'd into BCDE, half the Perimeter and equal to BF, is equal to the Area of the Figure, that is, to the Parallelogram ABFG. For that of each Triangle the perpendicular AB multiply'd into half the base gives the Area: All which Triangles together are equal to the whole Multilateral Figure.

Therefore also the Right angl'd Triangle ABH, having the Side BH equal to the whole Perimeter BCDEFG, is equal to the same Multilateral Figure.

3. For the Mensuration of a Circle.

And as there is found this equality of Areas between a Regular Polygon and such a Right angl'd Triangle; so *Archimedes* hath demonstrated the same between a Circle and a Right angl'd Triangle, one of whose Sides about the Right angle is equal to the Radius, and the other to the Perimeter of the Circle.



Now the Physical reason of this agreement between a Rectilineal Figure and a Circle seems to be, for that a Circle is, as it were, a *Regular Polygon consisting of infinite equal sides*. So that a Triangle, one of whose sides about the Right angle, is equal to the Radius, and the other equal to the Perimeter, is in Area equal to the Circle, like as it is in Regular Polygons. As of the Triangle ABC, if the side BC be supposed equal to the Perimeter of the Circle, then shall the Triangle ABC be equal to the Circle, as *Archimedes* hath demonstrated. But how Geometrically to exhibit a strait line equal to the Perimeter of a Circle, and to demonstrate the same (as in this Instance BC) to be a line equal to the Perimeter, *Hic labor, hoc opus*.

Archimedes therefore makes a further attempt toward this Matter, and in his wonderful Book of *Spiral Lines*, demonstrates in Prop. 18. that if to the term of a Spiral line described by the first Revolution of the *Genetrix*, a Tangent be drawn: and from the Original, or Central point of the same be likewise drawn a strait line at Right angles to the *Genetrix*, and produced till it meets with the Tangent, then shall this strait line be equal to the Perimeter of the Circle, whose Radius is the line that describes the *Helix*.

But how to draw the Tangent is a work left unfinished. And till a Tangent to an Helix be Geometrically demonstrated, a strait line equal to the Perimeter of a Circle remains unknown.

Archimedes having in these methods proceeded Geometrically toward the investigation of a strait line equal to the circumference of a Circle, without a plenary

satisfaction, endeavours next to come to a nearness of equality, so far as it might be easily practicable, and sufficient for common use.

He begins with the Trisection of a Right angle, or a Quadrant of the circumference of a Circle (commonly signified by the number of 90 Degrees) so that each Segment is a third part of the Quadrant, or 30 Degrees of 90, and therefore a 12^{th} part of the whole circumference, that is, of 360 Degrees.

First, Now he bisects this angle, and by consequence the Arch, which maketh each Segment a 24^{th} part of the circumference. *Secondly*, This bisected, makes each Segment a 48^{th} part of the circumference. *Thirdly*, This again bisected, makes each Segment a 96^{th} part of the circumference: In which Segment *Archimedes* rests.

Thus from the Bisections of a 12^{th} part of the circumference thrice repeated, he takes a regular Polygon of 96 sides circumscribed about a Circle. Then he demonstrates the Perimeter of this Polygon to be to the Diameter of the Circle as 22. to 7. almost, that is, to be triple of the Diameter, and moreover the overplus above the triple to be almost $\frac{1}{7}$ part, or $\frac{1}{7}$ parts of the Diameter: which is the same thing; whether the Diameter be divided into 7. or 70. equal parts. And because the circumference of the contained Circle is less than the Perimeter of the circumscribed Polygon, therefore the overplus of the circumference above the triple, is much less than $\frac{1}{7}$ or $\frac{1}{7}$ of the Diameter.

For of two unequal magnitudes the lesser hath a less proportion to a third magnitude, than the greater hath to the same. As a Groat hath a lesser proportion to a Penny, than a Shilling hath to a Penny.

Again, he takes a regular Polygon of 96 sides inscribed in a Circle, then he demonstrates the Perimeter of this Polygon to be also triple of the Diameter of the Circle, and the overplus above the triple to be greater than $\frac{1}{7}$ parts of the Diameter.

And because the circumference of the Circle is greater than the Perimeter of the contained Polygon, therefore the overplus above the triple is much greater than $\frac{1}{7}$ parts of the Diameter.

So that *Archimedes* uses, first a circumscribed Polygon of 96 sides, whose Perimeter is greater than the circumference of the Circle; and then an inscribed Polygon of 96 sides, whose Perimeter is less than the circumference: and from their proportions to the Diameter he proves the same a *Fortiori*, that the quantity of the Circumference is triple of the Diameter, and somewhat less than $\frac{1}{7}$ parts, yet somewhat greater than $\frac{1}{7}$ parts of the Diameter, being first divided into 70, and again into 71. equal parts. Limits easily comprehended, and exposed in the least and fewest numbers.

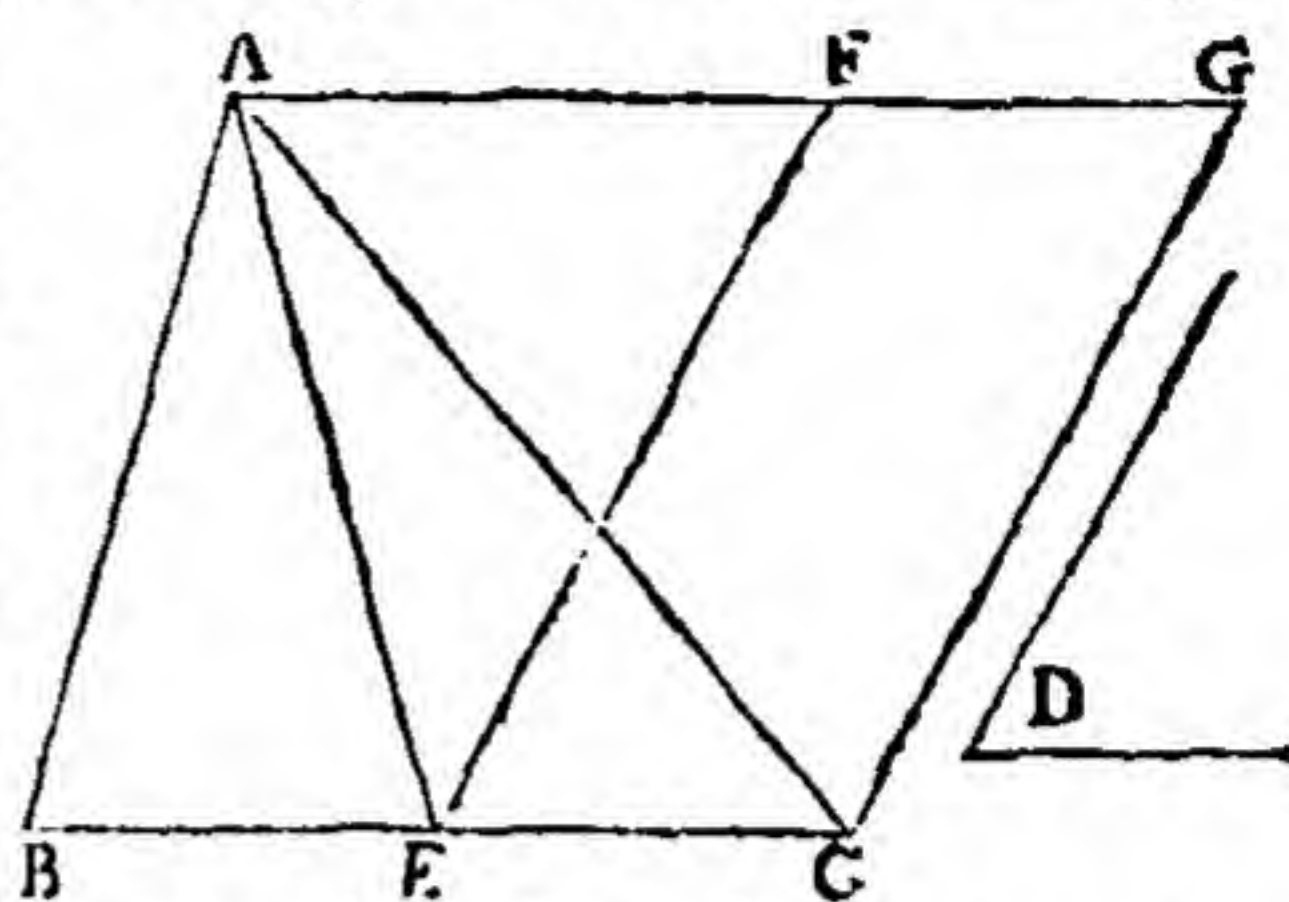
The Moderns indeed have brought this within closer bounds; but in great numbers more nice than necessary. The numbers of *Adrianus Metius* are for usefulness next to those of *Archimedes*. He states the circumference unto the Diameter, as 355 to 113, that is triple, and moreover 16 parts almost of the Diameter divided into 113 equal parts, and thus to be noted $\frac{355}{113}$, or $3\frac{16}{113}$ almost. Whereas $\frac{1}{7}$ multiply'd into 16, makes only $\frac{16}{7}$: which being greater than $\frac{1}{7}$ is not so near to the just and precise Truth, as these numbers of *Adrianus Metius*. But *Archimedes*, who in his Book entitled *Psammites*, or *Arenarius*, does by most artificial Calculations, beginning from a round small Poppy-seed give us this vast number, which reduced into our Decimal form of Notation, is 10000000,00000000,00000000,00000000,00000000,00000000,00000000,00000000,00000000. Which he demonstrates to exceed the number of the Sand of the Seas, if the whole World within the Spherical Concave of the Stars, (which he takes to be as large as the ancient System of *Aristarchus* was, and after two thousand years revived most ingeniously by *Copernicus*) consisted only of such a Mass of Sand; He, I say, could have come to any nearer and nearer terms at pleasure, if he had thought it necessary or convenient. But when after all attempts and labour whatsoever, he knew the matter must end in a bare Approximation, like a great and prudent Master of his Art, rests within the readiest and most useful limits. And all the endeavours of our late Geometricians reach no further, than proceeding in Fractions of greater and greater numbers to bring the overplus above the triple to be lesser and lesser than $\frac{1}{7}$ or $\frac{1}{7}$, yet still to be greater than

than $\frac{1}{11}$. Which *Archimedes* hath demonstrated to be the standing limit on the other side, unto which endless approaches may be nearer and nearer made to very little purpose.

PROPOSITION XLII.

Unto a given Triangle to constitute an equal Parallelogram in an angle equal to a given Strait-lin'd angle.

Let the given Triangle be ABC , and the given strait-lin'd angle D . It is required to constitute a Parallelogram equal to the Triangle ABC , in an angle equal to the strait-lin'd angle D . Let BC be cut into halves in E , and let be joyn'd AE , then to the strait line EC , and to a point in the same E , let be constituted the angle CEF , equal to the angle D [by Prop. 23.]: And by A , let be drawn AG parallel to EC , and by C , CG parallel to EF . Therefore $FECG$ is a Parallelogram.



Now forasmuch as BE is equal to EC , therefore the Triangle ABE is equal to the Triangle AEC . For they are on equal bases BE, EC , and in the same parallels BC, AG . Therefore the Triangle ABC is double of the Triangle AEC . But also the Parallelogram $FECG$ is double of the Triangle AEC ; for it hath the same base and is in the same parallels. Therefore the Parallelogram $FECG$ is equal to the Triangle ABC , and hath the angle CEF equal to the given angle D .

Wherefore to the given Triangle ABC , there is constituted an equal Parallelogram $FECG$ in the angle CEF , which is equal to the angle D . Which was to be done.

ANNOTATIONS.

This Problem concerns the Transformation of Figures one into another; and begins with transmuting a Triangle (the most simple of strait lin'd Figures) into an equal Parallelogram. And by consequence there is imply'd the like transmutation of all Rectilineal spaces into equal Parallelograms; for that every Multilateral Figure may for this end be divided into Triangles. As *Euclide* hath done in the following 45th. Proposition.

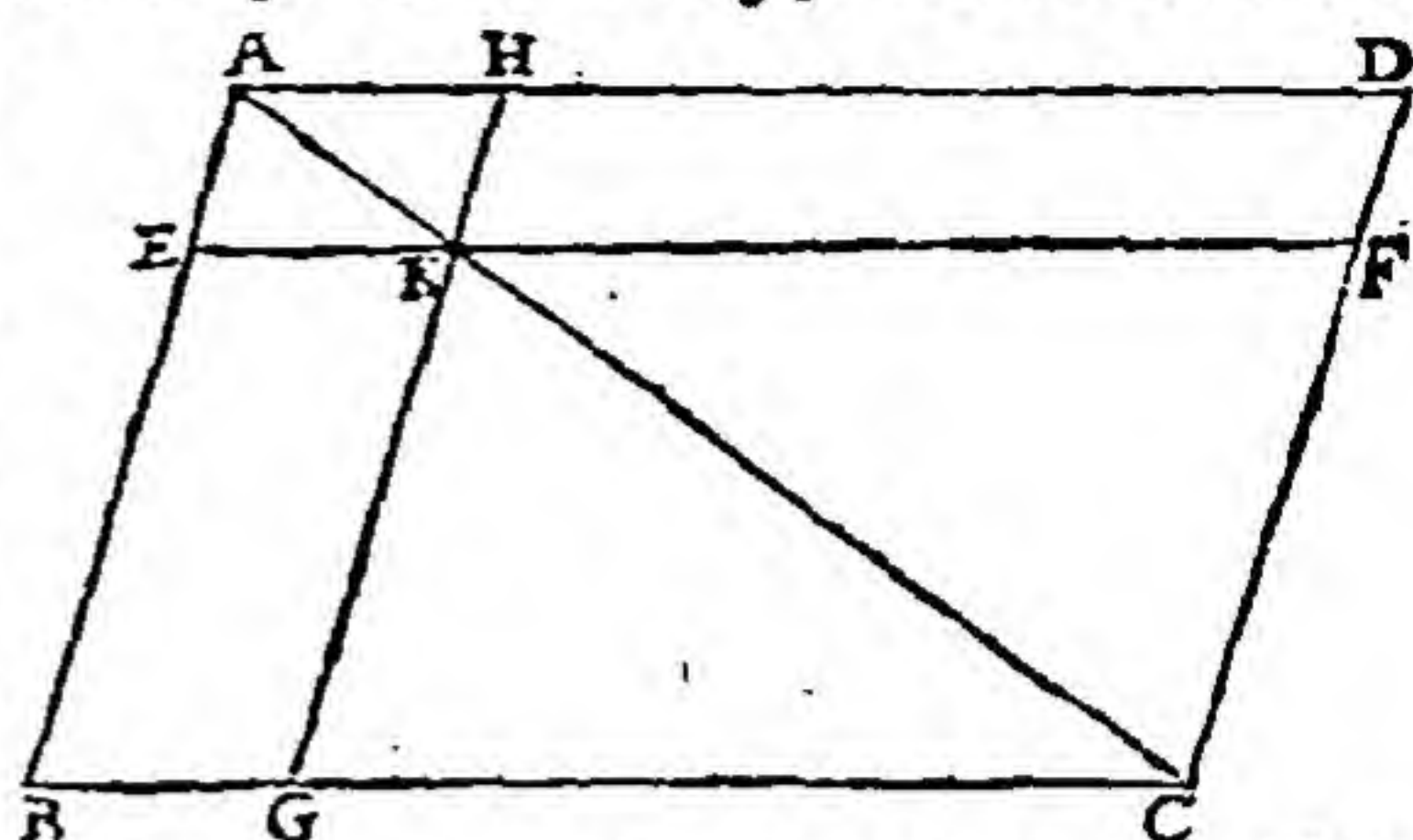
And forasmuch as all Parallelogram spaces are by the Diameter divided into two Triangles, therefore their four angles are equal to four Right [by Prop. 32.]. And because the opposite angles are equal, [Prop. 34.] therefore if one angle of a Parallelogram be given, all the four are given, and determined. For if the given angle be Right, the other three are also right angles. If the given angle be Obtuse, the opposite is also Obtuse, and equal to it, and the other two Acute angles are likewise equal to one another, and together with the two Obtuse do make, or compleat four Right angles. And the like again on the contrary, if the given angle be Acute: so that in one angle given, a Parallelogram is ever to be understood as determined in all its four angles.

PROPOSITION XLIII.

Of every Parallelogram space the complements of the Parallelograms about the Diameter, are equal to one another.

Let the Parallelogram space be $ABCD$, and the Diameter of the
R 2 same

same be AC , and about AC let the Parallelograms be EH , FG , and what are called the complements be BK , KD . I say, that the complement BK is equal to the complement KD . Forasmuch as $ABCD$ is a Parallelogram, and the Diameter thereof AC : therefore the Triangle ABC is equal to the Triangle ADC . Again, because $EKHA$ is a Parallelogram, and the Diameter thereof AK : therefore the Triangle AEK is equal to the Triangle AHK . By the same reason also the Triangle KGC is equal to the Triangle KFC . Now because the Triangle AEK is equal to the Triangle AHK , and the Triangle KGC is equal to the Triangle KFC , therefore the Triangle AEK with the Triangle KGC , is equal to the Triangle AHK , with the Triangle KFC . But the whole Triangle ABE is equal to the whole Triangle ADC : wherefore the remaining complement BK , is equal to the remaining complement KD .



Therefore of every Parallelogram space the complements of the Parallelograms about the Diameter, are equal to one another. Which was to be demonstrated.

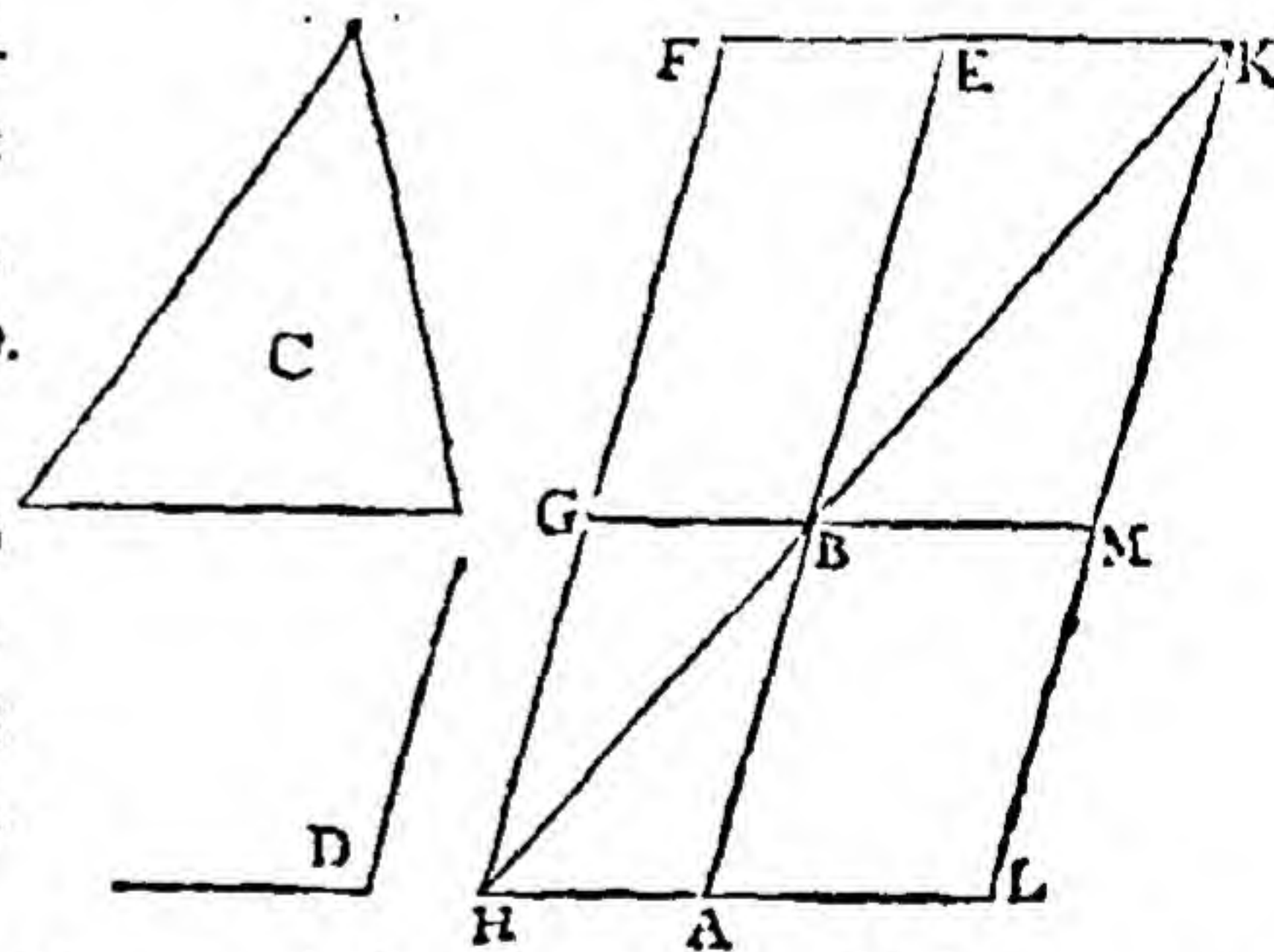
PROPOSITION XLIV.

UNto a given strait line to apply a Parallelogram equal to a given Triangle, in a given strait lin'd angle.

Let the given strait line be AB , and the given Triangle c , and the given strait lin'd angle D . It is required unto the given strait line AB , to apply a Parallelogram equal to the given Triangle c , in an angle equal to D .

Let be constituted the Parallelogram $BEFG$ equal to the Triangle c in the angle EBG , which is equal to D . [by Prop. 42.]

And let BE be put directly to AB , and FG be produced to H . Then by A to either of the lines BG , EF , let AH be drawn parallel, and let be joyn'd HB .



Now forasmuch as the strait line HF falls on the parallels AH , EF , therefore the angles AHF , HFE , are equal to two Right [Prop. 29.]; wherefore BHG , GFE , are less than two Right. But lines infinitely produced from angles less than two Right shall meet: therefore HB , FE , being produced shall meet. Let them be produced, and meet in K : then by the point K to either of the lines EA , FH , let KL be drawn parallel, and let HA , GB , be produced to the points L , M .

There-

Therefore $HLKF$ is a Parallelogram, and HK the Diameter thereof; and the Parallelograms *about* HK are AG , ME , and the Parallelograms called *complements* are LB , BF : therefore LB is equal to BF [Prop. 43.]. But BF is equal to the Triangle c ; therefore also LB is equal to c . And because the angle GBE , is equal to ABM , and also GBE is equal to the angle D ; therefore ABM is equal to the angle D .

Therefore unto the given strait line AB , is apply'd the Parallelogram LB , equal to the given Triangle c , in the angle ABM , which is equal to the given angle D . Which was to be done.

ANNOTATIONS.

It is required unto the given line AB to apply a Parallelogram equal to the given Triangle C .] That is, to constitute a Parallelogram, one of whose Sides shall be the given line AB , and the Parallelogram be also equal to the given Triangle C .

And let BE be put directly to AB] That is, let the Parallelogram $BEFG$ be so constructed, that one of the sides containing the angle EBG , equal to the given angle D , be put directly to AB the given line, unto which a Parallelogram equal to the given Triangle C is required to be apply'd. Now this is to be thus effected.

Produce AB to E , and to the line EB , and to the point B let be constituted the angle EBG , equal to the given angle D [by Prop. 23.]. Then let the Parallelogram $BEFG$, be constituted equal to the given Triangle C [by Prop. 42.]; And let FG the opposite side to EB , be produced indefinitely toward H . And so proceeding onward according to *Euclid's* construction in completing the Diagram, and applying the required Parallelogram LB to the given line AB , as it is at the first posited in any Situation whatsoever given.

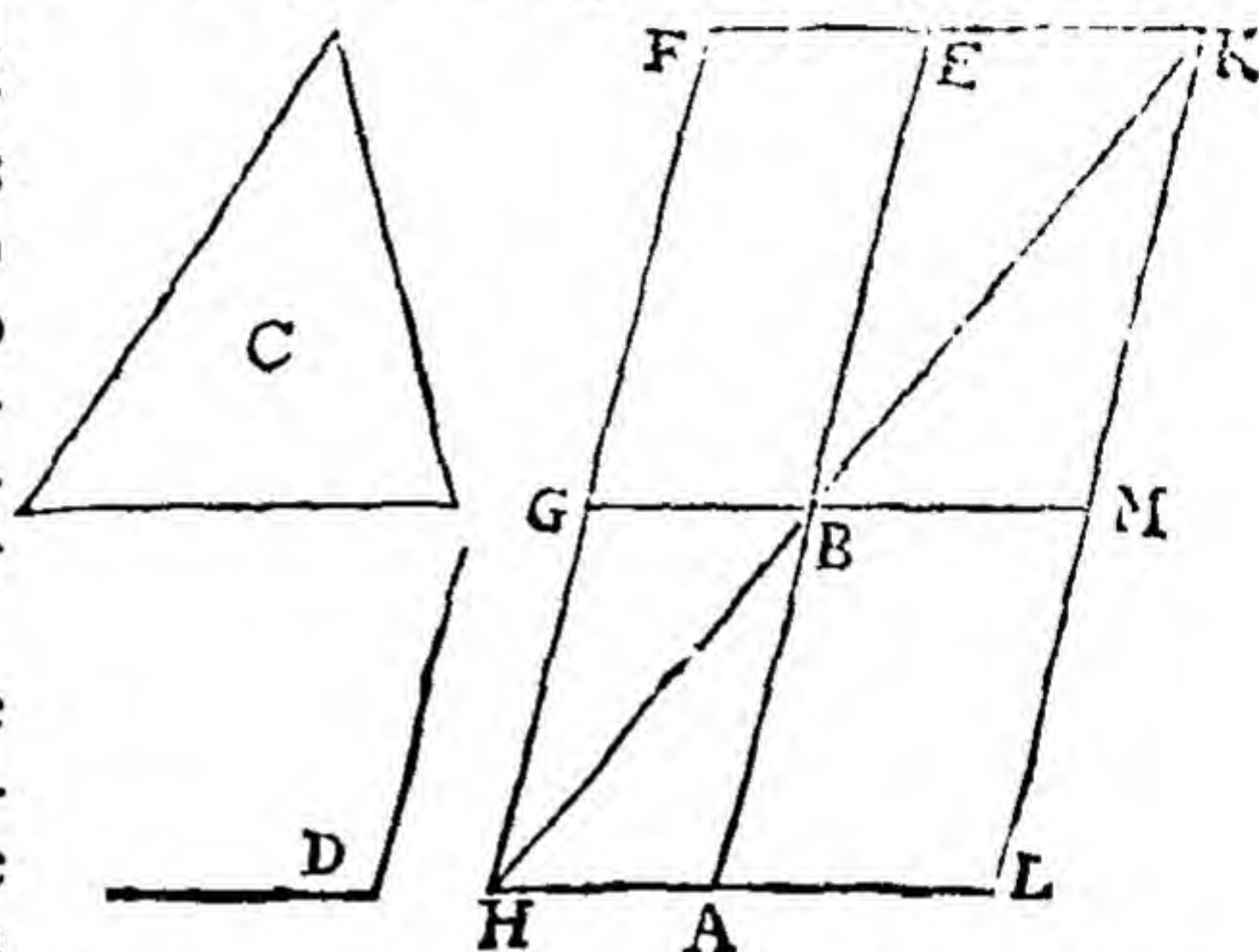
For note, that the Application of that Parallelogram which is required to be equal to the given Triangle, ought to be made *ad Datam Rectam*, that is, to the very line AB in its position, and not to an other line, which shall be put equal to it; as *Clavius* hath in his Exposition of this Proposition without just cause deviated from *Euclide*. But afterward in his Scholion he rightly corrects himself, *ex sententia Euclidis*, and there follows the general Law of Problems, that *The thing is always to be effected according to the Position given*. As in the Use of this Problem it will every where be found necessary, and even in the next following Proposition.

It is farther to be observed, that in the Construction of this Problem there are made four Parallelograms Equiangl'd to one another, and to the whole. *viz.* Two about the Diameter of the whole, and their two Complements. One of which, the Parallelogram LB is first [by Prop. 42.] constituted equal to the given Triangle C : by which means the other Complement BL , equal to the former (and therefore equal to the same given Triangle) is apply'd to the given line AB , according to the full Tenor of this eminent Problem.

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The 42^d. Proposition hath shewed how to constitute a Parallelogram equal to a given Triangle, in an angle equal to a given angle.

Now in this Problem there is moreover required to apply such a Parallelogram also to a given strait line, as well as in a given angle. That is, the given line is to be one side of the apply'd Parallelogram, and an angle of that Parallelogram is to



be equal to the given angle. And always remind, what hath been before noted, that in Parallelograms, if one angle be given, all the four are given, because the opposite angles are equal, and the two inward are equal to two Right, by Prop. 34. and 29. So that in this Problem the Parallelogram is three ways restrained. I. In the given line the Parallelogram is confined to one certain side. II. It is determined in Area, or Magnitude, in that it is to be equal to a given Triangle. III. In the one given angle all the four angles are determined; wherefore of this Parallelogram nothing is left undetermined, but the other side whereby to effect this Problem.

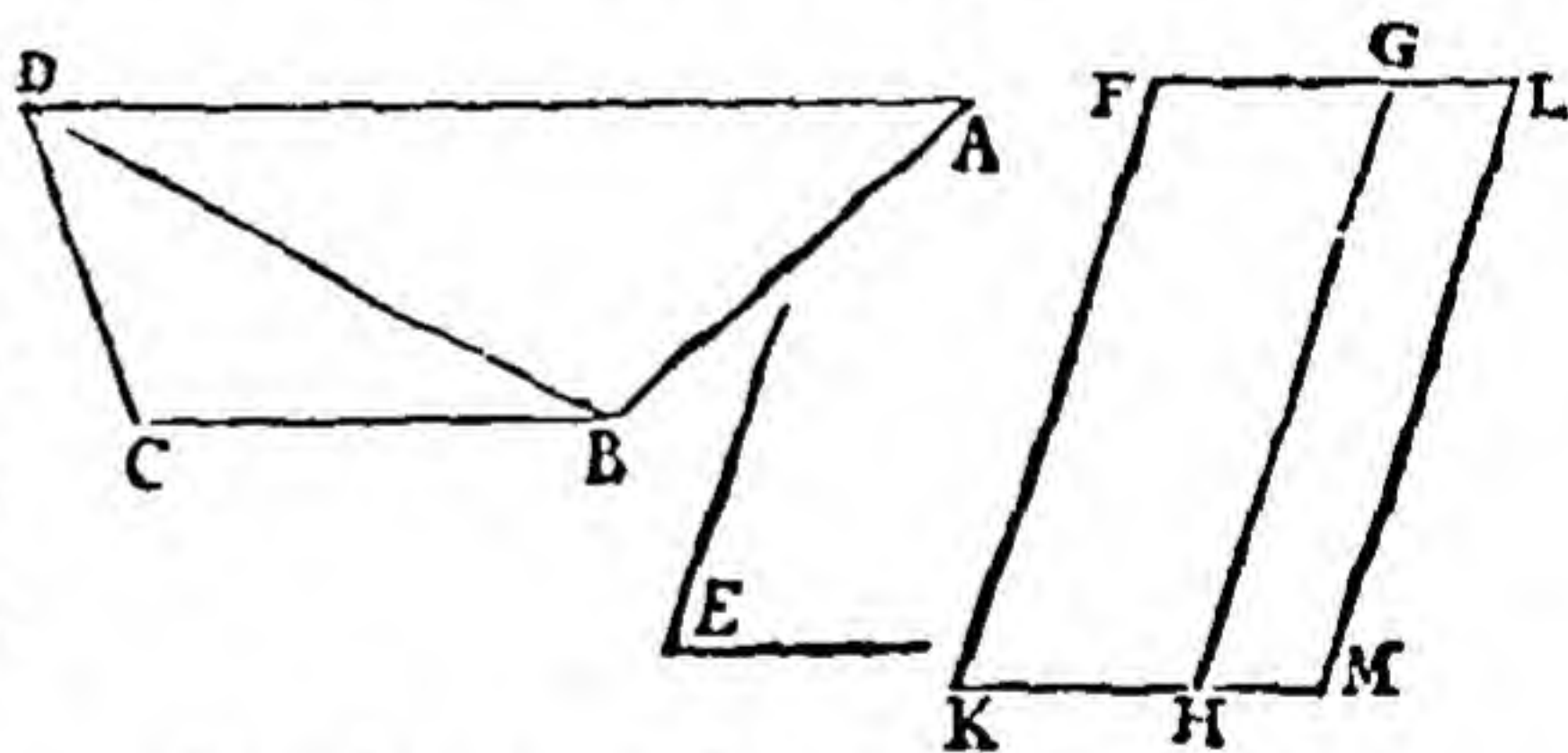
From this *exact* Application, or *Parabolism* of a Parallelogram to a given strait line *precisely*, is one of the Conic Sections named *Parabola*. As *Apollonius* shews in Prop. 11. Lib. I. of his Conic Elements. The strait line to which the Application is made in the Conic Elements, is by *Mydorgius* properly called the *PARAMETER*.

There is likewise in Prop. 28, and 29. El. VI. of *Euclide*, an Application of a given Parallelogram unto a given strait line, and in a given angle: but besides these Restraints, there are added more strict conditions of Defect, and Excess, that is, the Application is to be made either unto a part of the given line in a certain Defect, or to the given line Augmented in a certain Length, or Excess; which Conditions have occasioned the names of *Ellipsis*, and *Hyperbola*, to the other Conic Sections, As Prop. 12, and 13. Lib. I. of *Apollonius* set forth. So that *Euclid's* three fold Application of a Parallelogram to a strait line, has afforded to *Apollonius* names for the three famous Conic Sections. Whereas more anciently, the *Parabola* was called the Section of a Right-angl'd Cone, the *Hyperbola* of an Obtuse-angl'd Cone, the *Ellipsis* of an Acute-angl'd Cone, as we find in *Archimedes*. But for applying these three Sections in general to any one Cone of whatsoever angle, and accordingly moulding the old Conic Elements, *Apollonius* was in his time, and ever since renowned with the Title of the *great Geometrician*.

PROPOSITION XLV.

TO a given Rectilineal space to constitute an equal Parallelogram in an angle equal to a given strait-lin'd angle.

Let the given Rectilineal space be $ABCD$, and the given strait-lin'd angle E . It is required to constitute a Parallelogram equal to the Rectilineal space $ABCD$, in an angle equal to E . For let be joyn'd DB ; and let be constituted [by Prop. 42.] the Parallelogram FH equal to the Triangle ADB in the angle HKF , which is equal to E .



Then unto the strait line GH , let be apply'd the Parallelogram GM , equal to the Triangle DBC in the angle GHM , which is equal to E , [by Prop. 44.]

Now forasmuch as the angle E is equal to each of the angles FKH , GHM ; therefore FKH is equal to GHM . Let KHG be added in common, therefore the angles FKH , KHG , are equal to the angles KHG , GHM . But the angles FKH , KHG , are equal to two Right [Prop. 29.]; therefore also KHG , GHM , are equal to two Right. Now to the strait line GH , and to a point in the same H , the two strait lines KH , HM , not lying the same way, make the consequent angles equal to two Right; therefore KH is direct to HM [Prop. 14.]. And because the strait

strait line HG falls on the parallels KM , FG , therefore the Alternate angles MHG , HGF , are equal to one another [Prop. 29.]. Let HGL be added in common, therefore MHG , HGL , are equal to HGF , HGL . But the angles MHG , HGL , are equal to two Right: therefore FG is direct to GL [Prop. 14.]. And because KF is equal, and parallel to HG , and likewise HG to ML ; therefore also KF is equal, and parallel to ML [Prop. 30.]. And the strait lines KM , FL , joyn the same, therefore KM , FL , are also equal, and parallel [Prop. 33.]; wherefore $KFLM$ is a Parallelogram. And because the Triangle ADB , is equal to the Parallelogram HF , and BDC to GM , wherefore the whole Rectilineal space $ABCD$, is equal to the whole Parallelogram $KFLM$. Therefore the Parallelogram $KFLM$ is constituted equal to the Rectilineal space $ABCD$ in the angle FKM , which is equal to the given angle E . Which was to be done.

ANNOTATIONS.

And let be constituted the Parallelogram FH equal to the Triangle ADB .] To effect this Problem, the given Rectilineal space is to be divided into Triangles: and first to one of these Triangles there is to be constituted an equal Parallelogram. As here the Parallelogram FH is constituted equal to the Triangle ADB , by Prop. 42. But is not required as *Clavius* proposes to be apply'd to any certain strait line, tho' the other Parallelograms are; and after this manner following.

Then unto the strait line GH , let be apply'd the Parallelogram GM , equal to the Triangle DBC , in the angle GHM , which is equal to E .] Altho' the first Parallelogram FH , equal to the Triangle ADB , was not confined to any given strait line; yet the next Parallelogram GM equal to the Triangle DBC , is of necessity. to be apply'd to the strait line GH , as it lyes in a given position; that by this means there might be constituted from such particular Parallelograms one entire Parallelogram $KFLM$, equal to the given Rectilineal space $ABCD$. And so forward, if the Rectilineal space required a division into more Triangles, Parallelogram is after Parallelogram to be apply'd to such a certain strait line, that makes successively one common Side; and all of them are equal to one another, and the Parallelograms are together equal to the whole Parallelogram space, which was required to be constituted equal to a given Rectilineal space.

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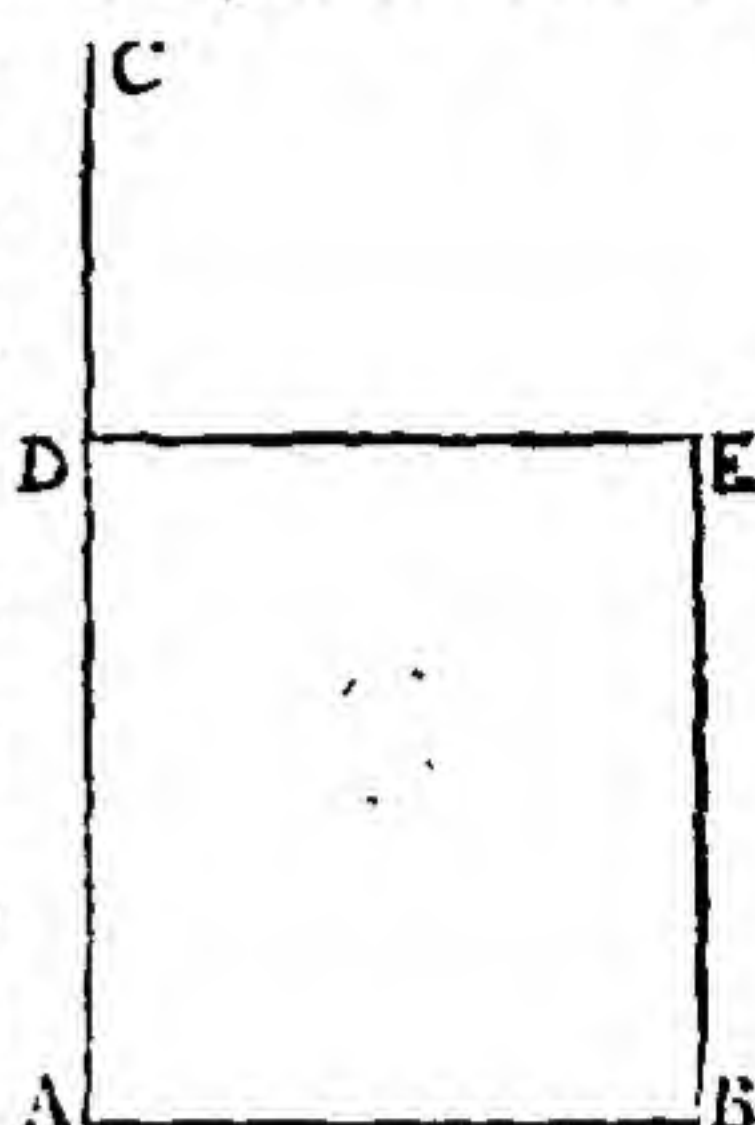
The Application of Rectilineal spaces to a given strait line, if also the given angle be a Right angle, does truly answer to the operation of Division in Arithmetic. For the given plain space is as the Dividend in Numbers, the given strait line as the Divisor, and the other Side of the Parallelogram emergent from this Application is as the Quotient. And again, as the Divisor multiply'd into the Quotient makes up the Dividend, so the given side drawn into the emergent side, gives the Area of the Parallelogram.

PROPOSITION XLVI.

ON a given strait line to describe a square.

Let the given strait line be AB . It is required on the strait line AB to describe a square. To the strait line AB from a given point in
the

the same A ; let be drawn at Right angles the line AC , and to AB let AD be put equal: then by the point D , let DE be drawn parallel to AB , and by the point B let BE be drawn parallel to AD ; therefore $ADEB$ is a Parallelogram: wherefore AB is equal to DE , and AD to BE [Prop. 33.]. But also BA is equal to AD ; therefore the four lines BA, AD, DE, EB , are equal to one another: wherefore the Parallelogram $ADEB$ is Equilateral: I say, it is also Right angl'd. For because on the parallels AB, DE , falls the strait line AD , therefore the angles BAD, ADE , are equal to two Right [Prop. 29.]. But BAD is a Right angle, also ADE is a Right angle, and of Parallelogram spaces the opposite sides and angles are equal to one another [Prop. 34.]; wherefore each of the opposite angles ABE, BED , is a Right angle, therefore $ADEB$ is Right angl'd. But also it hath been prov'd Equilateral: therefore it is a Square; And it is describ'd on the strait line AB . Which was to be done.



ANNOTATIONS.

Euclid, before he makes use of a square, does demonstrate the Being and Construction of such a Figure. And therefore hath here premised this problem in order toward the demonstration of the next following Theorem, which is the first wherein squares are concern'd.

And according to his 29th. Definition he now shews how to describe a Figure of four Equal sides, and four Right Angles on any given strait line.

Thus having demonstrated this Figure, he does hereafter upon occasion justly assume from the nature of a Square, that *Squares describ'd on equal strait lines are equal to one another*. As likewise that *Equal Squares* are describ'd on equal strait lines. Altho' *Commandinus* and *Clavius* after *Proclus*, have thought fit to demonstrate these most natural Conceptions: which were before as evident of themselves, and immediately conjoyn'd in common sense with the definition of a Square.

PROPOSITION XLVII.

IN a Right angled Triangle, the Square of the side subtending the Right angle, is equal to the Squares of the sides containing the Right angle.

Let the Right angl'd Triangle be ABC , having the Right angle BAC . I say, that the square of BC , is equal to the squares of BA, AC . For on BC , let be describ'd the square $BDEC$, and on AB, AC , the squares GB, HC , and by A let AL be drawn parallel to either of the lines BD, CE , and let be joyn'd AD, FC .

Now forasmuch as each of the angles BAC, BAG , is a right angle, and to the strait line BA , and to a point in the same A , the two strait lines AC, AG , not lying the same way, make the consequent angles equal to two Right, therefore CA is direct to AG : by the same reason also AB is direct to AH . And because the angle DBC is equal to the angle FBA , for each is a right angle, let the angle

ABC

ABC be added in common, therefore the whole angle DBA is equal to the whole angle FBC . And because the two lines DB, BA , are equal to the two lines CB, BF , each to each, and the angle DBA is equal to the angle FBC , therefore the base AD is equal to the base FC , and the Triangle ABD is equal to the Triangle FBC .

Now the Parallelogram BL is double of the Triangle ABD , for they have the same base BD , and are in the same parallels BD, AL [Prop. 41.]. Also the square GB is double of the Triangle FBC , for they have the same base FB , and are in the same parallels FB, GC . Now the doubles of equals are equal to one another; Therefore the Parallelogram BL is equal to the square GB .

In like manner AE, BK , being joyn'd, may be proved that the Parallelogram CL , is equal to the square HC ; therefore the whole square $BDEC$ is equal to the two squares GB, HC , and the square $BDEC$, is described on BC , and GB, HC on BA, AC ; wherefore the square of the side BC , is equal to the squares of the sides BA, AC .

Therefore in Right angl'd Triangles, the square of the side subtending the Right-angle, is equal to the squares of the sides containing the Right angle. Which was to be demonstrated.

ANNOTATIONS.

This Proposition among Geometricians most famous, is said to have been found out by *Pythagoras*, and the Invention publickly celebrated with a Sacrifice to the Muses. Yet the hint from whence the discovery of this Truth might first arise, seems to be very obvious.

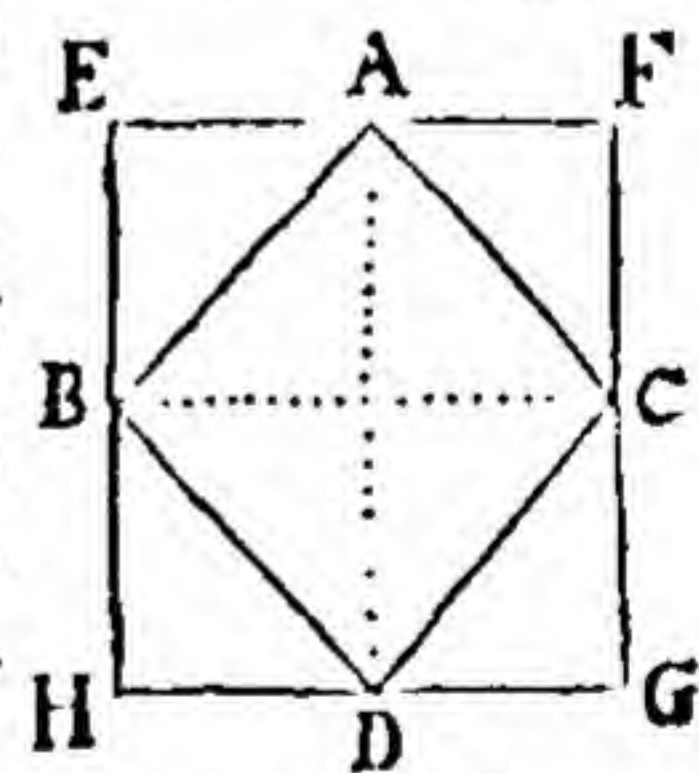
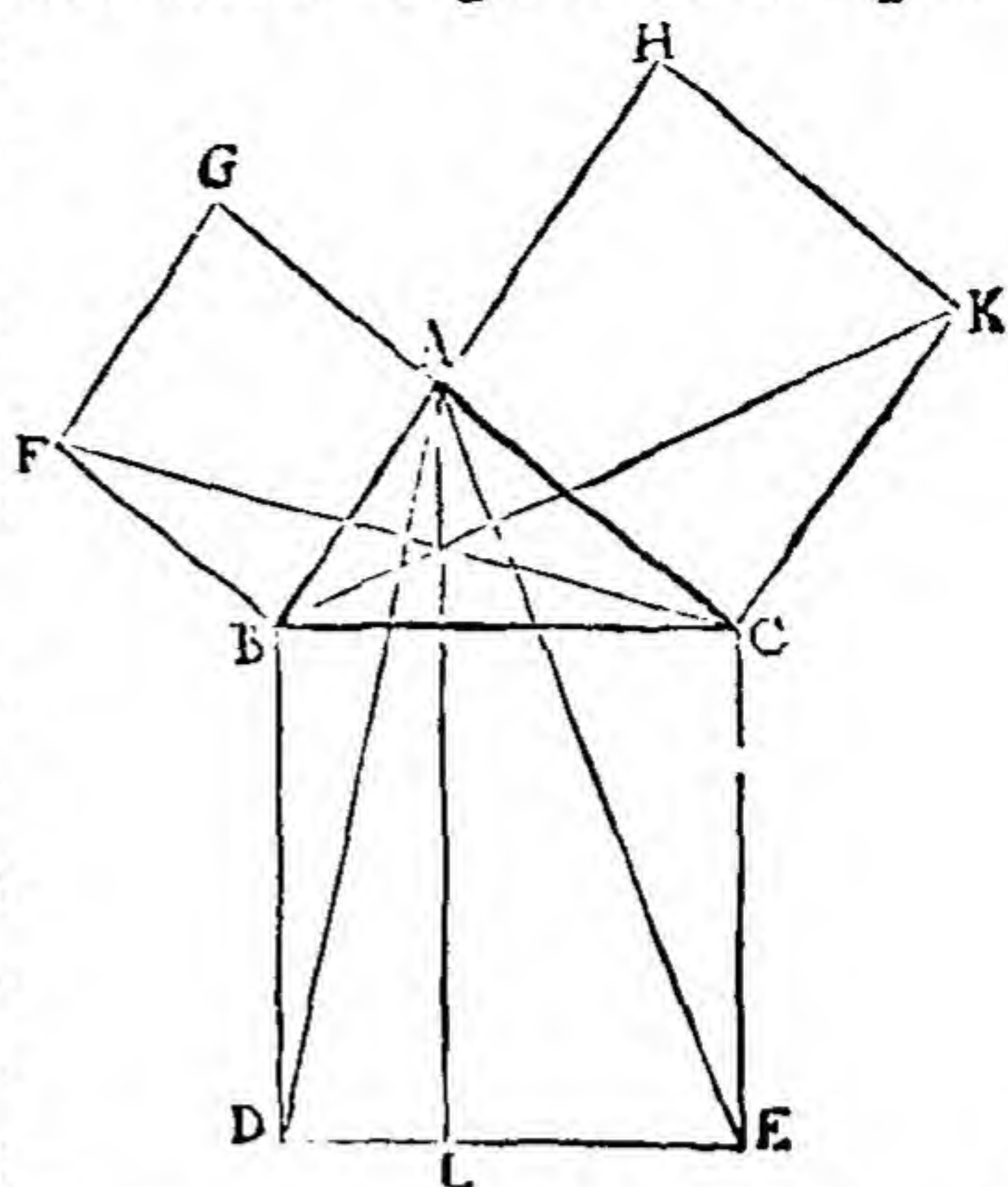
For in this Figure the square $EFGH$, is apparently double of the square $ABDC$; but $EFGH$ is described on EF , which is equal to BC , the side subtending the Right angle BAC of the Equicrural Triangle ABC ; and the square $ABDC$ is described on either of the sides AB, AC , containing the Right angle BAC , of the same Equicrural Triangle ABC . It is therefore hereupon very reasonable to conceive, that the same property might likewise belong to Scalene Right-angl'd Triangles, and give the occasion of a farther enquiry into this matter.

Thus Geometricians often happen to discover a Truth, before they have framed a legitimate demonstration of it: and find out their Propositions one way (which they usually conceal) but prove them in an other. We have an Example of this kind in the Remains of *Archimedes*, who shews, how first he found the Quadrature of a Parabola Mechanically, as he calls it, and afterwards gives a Geometrical demonstration.

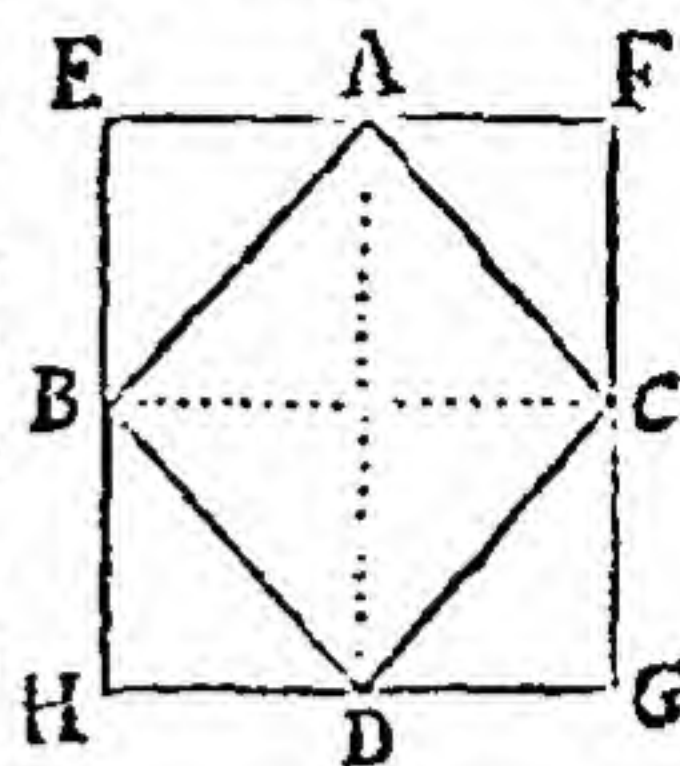
Now *Euclid's* demonstration reaches in general all Right-angled Triangles, Equicrural, and Scalene: and is very easy, natural and immediate, being framed from the two Fundamental Propositions 35th and 41th: by which all plain Surfaces are measured, and proved to be equal, or unequal to one another.

S

But



But forasmuch as no square number added to it self, can make a square number, nor any square number be the double of an other square number: therefore in an Equicrural Right-angl'd Triangle, as ABC , if the length of the sides AB , or AC , containing the Right angle, be expressed by the number of any known measure, as 2 inches, Feet, &c. then the length of the side BC subtending the Right angle is not expressible by a number of the same measure, or by any possible part thereof. For the square of 2 is 4, which added to it self makes 8 the square of BC , whose length is less than 3, because the square of 3 is 9. And on the contrary, if the length of BC be put 4 Inches, Feet, &c. then the length of either side AB or AC is not expressible by any part of the same measure. For the square of 4 is 16, therefore the square of AB or AC , is 8, and the length of the side AB or AC , is less than 3, and expressible by no part or parts of BC . For tho' BC be divided into 4000000 &c. of parts of the same measure; yet the case is still as before, and not one of those parts shall measure AB , which will ever be less than 3000000 &c. of these parts, and greater than 2999999 &c. of the same.

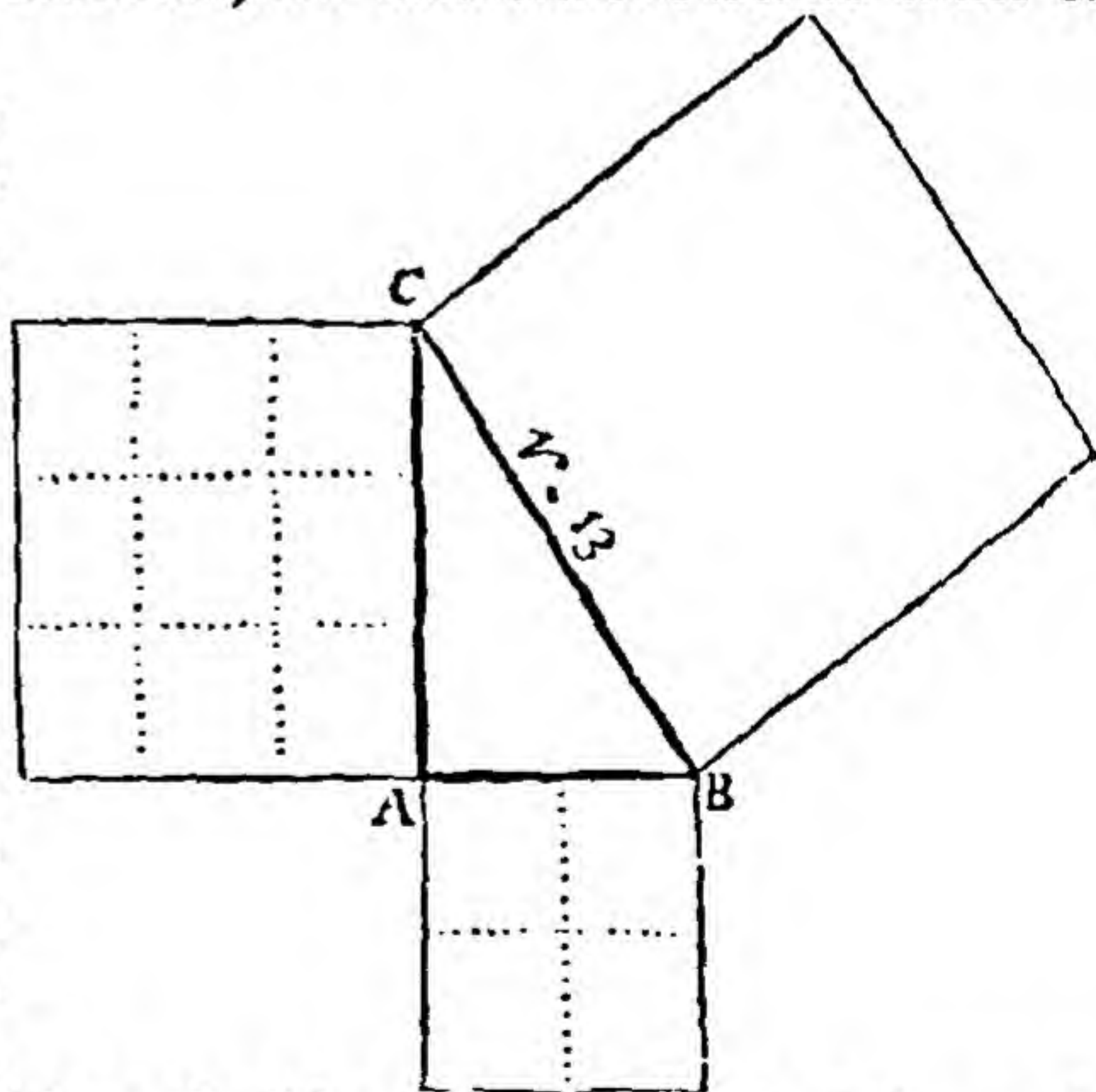


So that in an Equicrural Right-angl'd Triangle, if the side subtending the Right angle be *Rational*, that is, be expressed in quantity by a certain Number, and Measure, then each of the other sides is *Irrational* to that Measure, and cannot be expressed in quantity by any number of parts belonging to the same.

And because every square is divisible by the Diameter into two equal and Equicrural Right-angl'd Triangles, therefore if the side of a square be Rational, the Diameter is Irrational: and if the Diameter be put Rational, the side is Irrational. And the 10th. Element of *Euclide* ends with a demonstration, that the side of a Square, and the Diameter, are to one another incommensurable, that is, whatsoever length any certain times repeated shall make up exactly, and measure either the side, or the Diameter, the same tho' never so small, cannot precisely measure the other, but repeated shall fall under, or over it. As let any length exactly measure AB from the point A to the point B ; the same shall not precisely measure BC , from B to C ; but repeated will either come short of the point C , or pass beyond it. And this is meant by incommensurable Magnitudes. For tho' every finite Magnitude is in it self mensurable, yet all finite Magnitudes are not capable of the same measure so as to have their quantities signified by any one and the same, as appears from this 47th. Proposition.

There are infinite other strait lines of the like nature: some of which are the whole subject of that most subtle 10th. Element. As also here, not only in all Equicrural but in all Scalene Right-angl'd Triangles, where the squares of the sides containing the Right angle added together, make not a square number, there the side subtending the Right angle is irrational, and incommensurable to either of them.

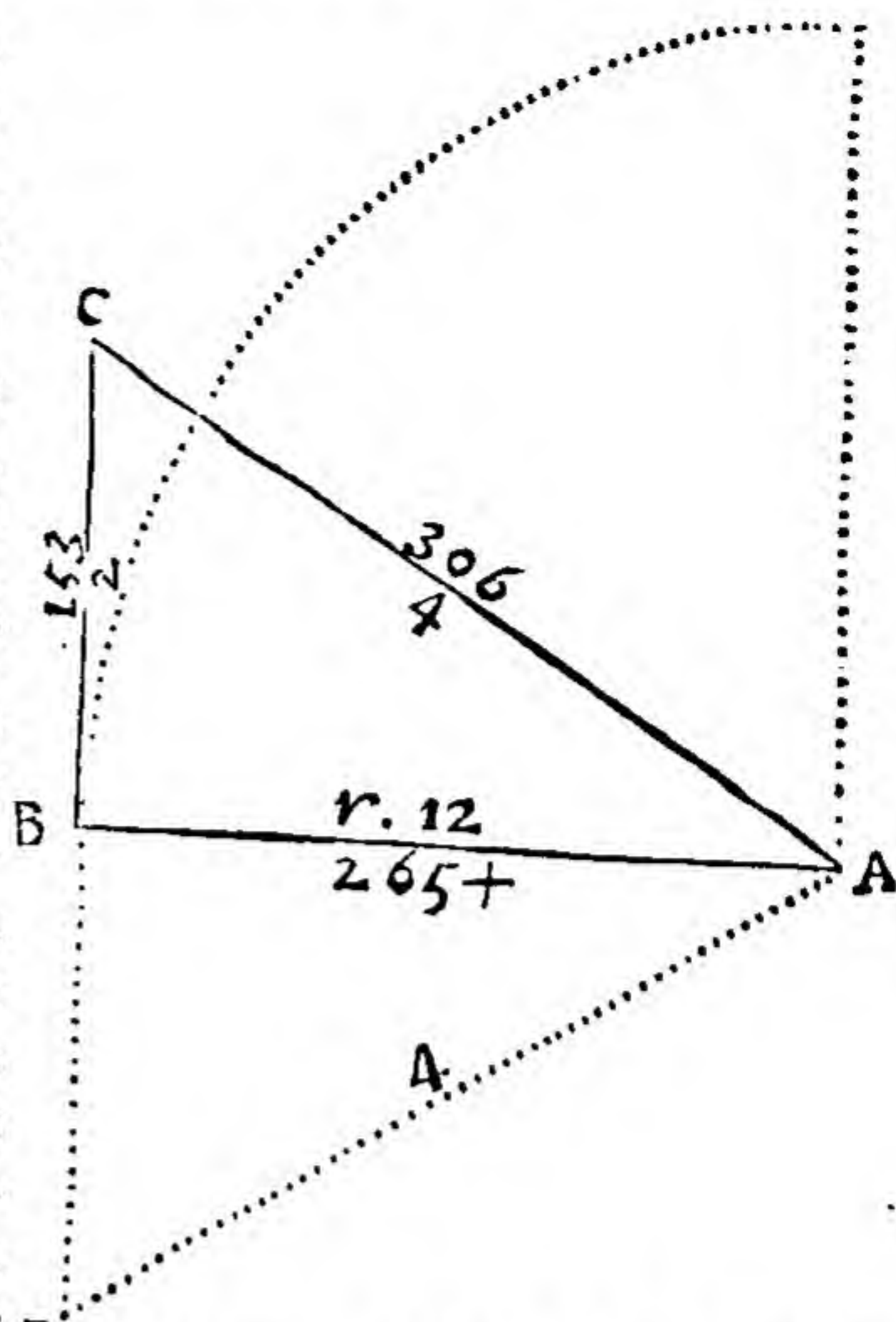
As in the Right-angl'd Triangle ABC , let the sides containing the Right angle be of a certain length, as AB 2 inches, and AC 3 inches; their squares are then 4, and 9, which added together, make the square 13. wherefore the side BC subtending the Right angle is greater than 3, whose square is but 9, and less than 4, whose square is 16, so that the quantity of BC , is only expressible by a surd number between 3 and 4, and is no part, nor any possible fraction, that can be made out of 3, or 4; but is called the Square Root of 13; and therefore irrational it is, and incommensurable to the sides AB and AC . And if an inch



were

were divided into an hundred thousand millions of parts, the case would be still the same, and not one of those particles should exactly measure, and make up the line BC .

Again, let a Right-angl'd Triangle ABC , be the half of an Equilateral Triangle ACD , each of whose sides let be put 4, so that AC is 4, and BC the half of AC is 2. Now the square of AC is 16, the square of BC is 4, and therefore the square of AB is 12, or 16 less 4, whose side AB is a furd between 3 and 4, and called the Square Root of 12. Yet these Squares are commensurable, and the square of AC is quadruple of the square BC , and the square of AB is triple of the same; tho' the side AB be irrational and incommensurable to the other sides AC, BC . This is the *Archimedean* Right-angl'd Triangle exposed in small numbers, such as plainly give the mutual proportions of the sides and squares, on which his Cyclometries are founded; where the furd side AB he makes the Radius of his Circle. The numbers of *Archimedes* are these, AC is put 306, therefore BC the half of AC is 153, and the Radius

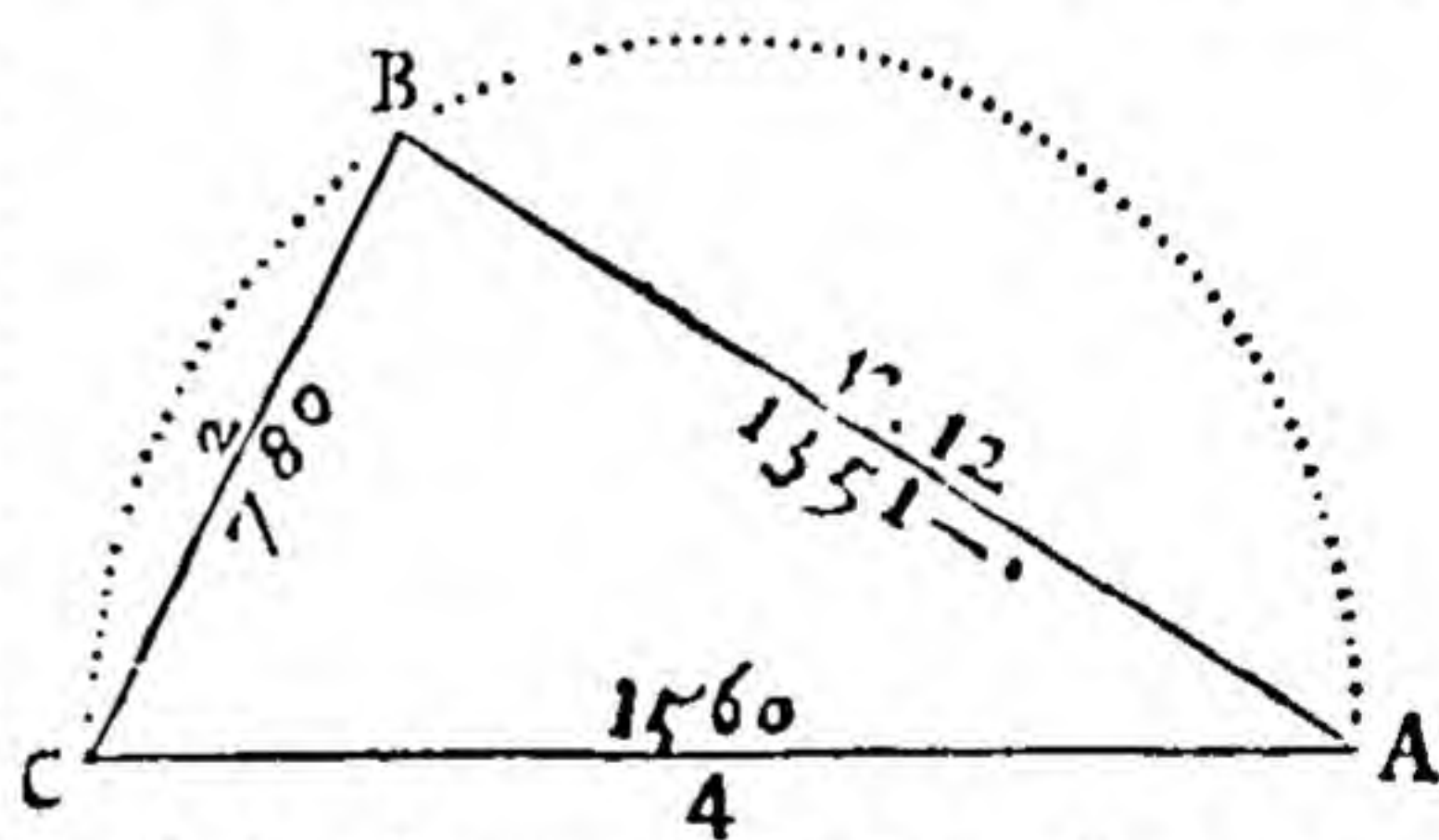


AB is a furd, somewhat greater than 265. For the square of AC 306 is 93636: and the square of BC 153, is 23409; which subtracted from the square 93636, there remains 70227, which is the square of AB : so that AB is somewhat greater than 265, whose exact square is 70225, and only less than 70227, the square of AB by two units: and therefore AB is a furd, somewhat insensibly greater than 265, the Root of the square number 70225.

Now from the sides, and squares of a Right-angl'd Triangle, which is half of an Equilateral, *Archimedes* demonstrates by repeated Bisections of angles, and arches, and just Calculations thereupon, that a Polygon of 96 sides circumscribed about a Circle, and by consequence that the Circumference of the Circle, is triple, and moreover somewhat less than $\frac{1}{3}$ or $\frac{10}{13}$ of the Diameter.

And further, upon this occasion to declare fully the use *Archimedes* makes of this 47th. Prop. in the mensuration of a

Circle, let there be again such an other Right-angl'd Triangle ABC , being the half of an Equilateral. And let AC subtending the Right angle ABC , be the Diameter of a Circle, and supposed to consist of 1560 equal parts: therefore BC the half of AC is 780. And the side AB is a furd, somewhat less than 1351.



For the square of AC 1560 is 2433600: and the square of BC 780 is 608400: which subtracted from the square 2433600, there remains 1825200, which is the square of AB : so that AB is somewhat less than 1351, whose exact square is 1825201, and only exceeds 1825200 the square of AB by a single unite; and therefore AB is a furd, somewhat insensibly less than 1351 the Root of the square number 1825201.

Now from the Sides, and Squares of this Right-angl'd Triangle, *Archimedes* demonstrates, that a Polygon of 96 sides inscribed in a Circle, and by consequence that the Circumference of the Circle, is triple, and moreover somewhat greater than $\frac{1}{3}$ parts of the Diameter.

Thus far upon this 47th. Prop. we have explained the Grounds of *Archimedes* his Cyclometries taken from an Equilateral Triangle, where the angles are always known, and the Sides may be put of any quantity at pleasure, and out of which is

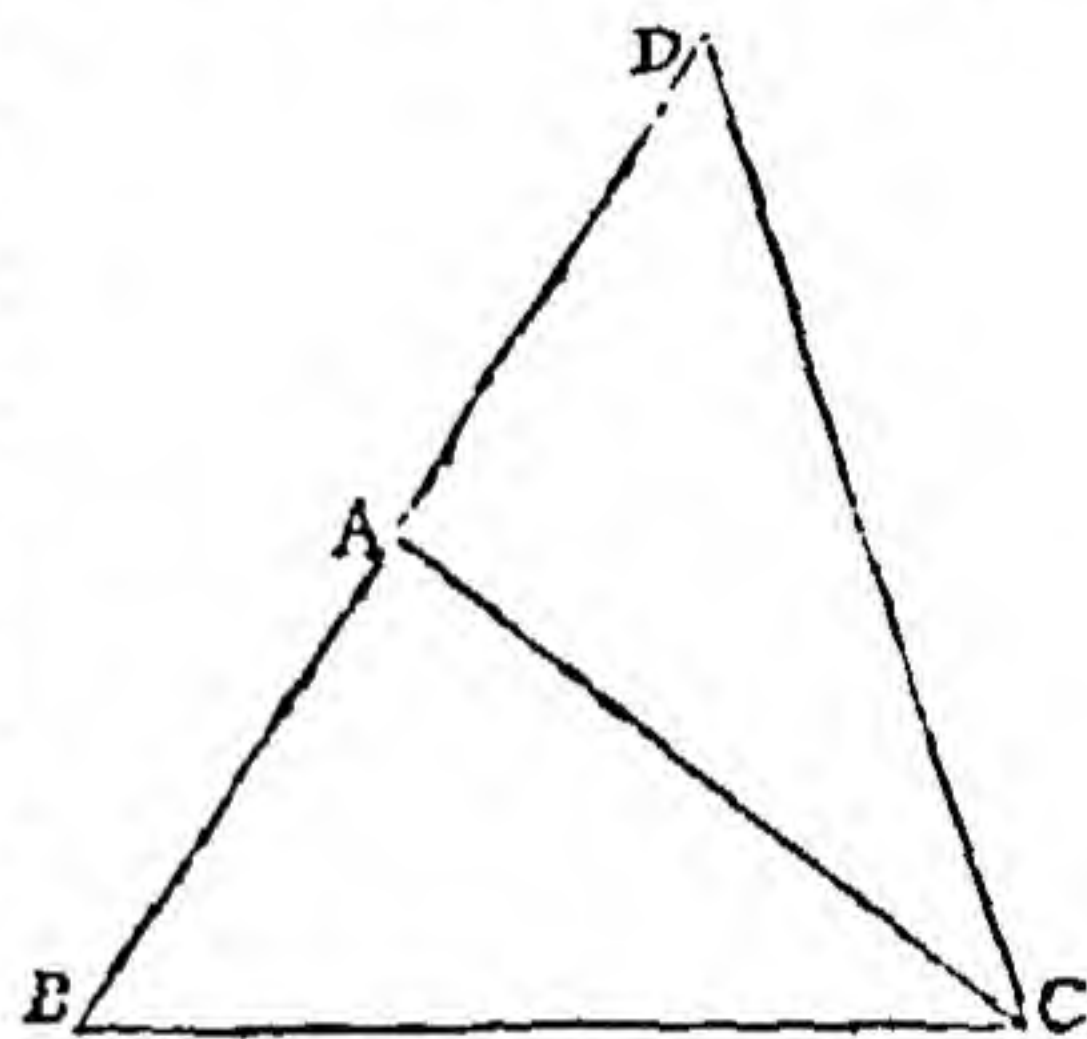
made such a Right-angl'd Triangle of known angles and sides, that whoever is but moderately exercised in the use of numbers, may now with ease go forward through the rest of *Archimedes* his demonstrations, which would otherwise be somewhat obscure and difficult to younger Students. For whose sake also we have from this Proposition given in brief some easy, and useful Notions of incommensurable magnitudes. And farther tis to be noted, that between incommensurable magnitudes it is at the first arbitrary and changeable at pleasure, which of them be put Rational; as whether the Side, or the Diameter of a Square. But these are of themselves in nature incommensurable to one another, which is the apparent cause of their mutual Irrationality. But the cause of incommensurability is more subtil, and lyes in the infinite divisibility of magnitude, which hath not in nature a *Minimum* to be a common measure of magnitudes, as an Unite is of Numbers.

This incommensurability between magnitudes of the same kind (whereof we shall have occasion to speak more at large in the Fifth Element) is one of the many inexplicable Mysteries in Geometry. *Habet enim Geometria miracula sua.*

PROPOSITION XLVIII.

I*f the square of one side of a Triangle be equal to the squares of the two remaining sides of the Triangle, the angle contained by the two remaining sides of the Triangle, is a Right angle.*

For of the Triangle ABC , let the square of BC be equal to the squares of BA, AC . I say, that the angle BAC is a Right angle. For from the point A to the line AC , let be drawn at Right angles the line AD ; and to BA let AD be put equal, and DC be joyn'd. Now forasmuch as DA is equal to AB , therefore also the square of DA is equal to the square of AB . Let the square of AC be added in common; therefore the squares of DA, AC , are equal to the squares of BA, AC . But the square of DC is equal to the squares of DA, AC [Prop. 47.]; for the angle DAC is a Right angle; and the square of BC is put equal to the squares of BA, AC ; therefore the square of DC is equal to the square of BC . So that also the side DC is equal to the side BC . And because DA is equal to AB , and AC common; therefore the two lines DA, AC , are equal to the two lines BA, AC , and the base DC is equal to the base BC ; therefore the angle DAC is equal to the angle BAC ; but the angle DAC is a Right angle, therefore also BAC is a Right angle.



If therefore the square of one side of a Triangle be equal to the squares of the two remaining sides of the Triangle, the angle contained by the two remaining sides of the Triangle is a Right angle. Which was to be demonstrated.

ANNOTATIONS.

This Proposition is the Converse of the precedent. And forasmuch as here is demonstrated, that of a Triangle if the square of one side be equal to the squares of the other two, then shall that be a Right angled Triangle: wherefore to form a Right angled Triangle of Rational, and Commensurable sides, we are to find three strait lines,

lines, whose quantities may be expressed in numbers of one, and the same measure common to all the three sides; so that also of those numbers the square of one shall be equal to the squares of the other two, that is, to find two square numbers, which added together shall make a square number according to this Problem.

A Problem.

From two given numbers to derive three numbers, where the square of one shall be equal to the squares of the other two.

The Solution.

Of the two given numbers the SUM of the squares; the DIFFERENCE of the squares, and TWICE THE RECTANGLE make a Right angl'd Triangle of Rational, and Commensurable sides.

In *Species* the Rule is thus set forth. Let the given numbers be A, and E: then shall $Aq + Eq$. $Aq - Eq$. $2A$ into E make a Right angl'd Triangle of Rational, and Commensurable sides.

As to begin with 1 and 2. The square of 1 is 1. The square of 2 is 4. The sum of the squares, $4 + 1$ is 5. The difference of the squares $4 - 1$ is 3. The Rectangle is 1 multiply'd into 2, which makes 2: and 2 twice taken is 4. So that the sides of this Right angl'd Triangle are 3, 4, 5. Where the square of 5 is 25, and equal to the two squares 16, and 9, whose sides are 4, and 3. This is the first Triangle of this sort.

Again, the second is this. Let be given the numbers 2, and 3. Now the square of 2 is 4. The square of 3 is 9. The sum of the squares $9 + 4$ is 13. The difference of the squares $9 - 4$ is 5. The Rectangle of 2 multiply'd into 3 makes 6, which twice taken is 12. So that the sides of this Right angl'd Triangle are 5, 12, 13. Where the square of 13 is 169, and equal to the two squares 144, and 25, whose sides are 12, and 5.

The third of these kind of Triangles, made from 3 and 4, is 7, 24, 25. The fourth from 4 and 5, is 9, 40, 41. The fifth from 5 and 6 is 11, 60, 61. and so forth infinitely. Where note, that in such kind of Triangles the two greater sides only differ by an unite; and the sum of these is always equal to the square of the least side.

THE SECOND ELEMENT.

DEFINITIONS.

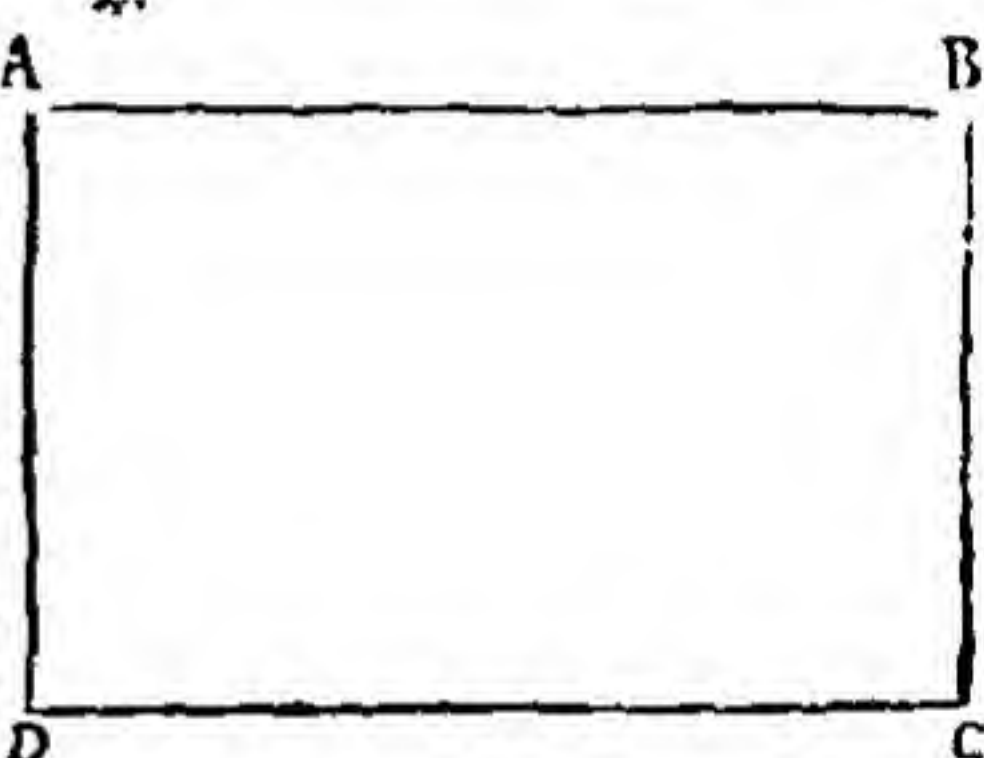
DEFINITION I.

Every Right angl'd Parallelogram is said to be contained by any two of the strait lines, which contain a Right angle.

Altho' a Right angl'd Parallelogram be comprehended, or encompassed by four strait lines, yet because by Prop. 34. El. I. the opposite sides of Parallelograms are equal to one another, therefore any of the two sides, which contain a Right angle, are said to contain the whole Parallelogram.

For AB being equal to DC, and AD to BC, therefore AB, and BC, or BC and CD, or CD, and DA, or DA, and AB, are indifferently said to contain the whole Right angl'd Parallelogram ABCD.

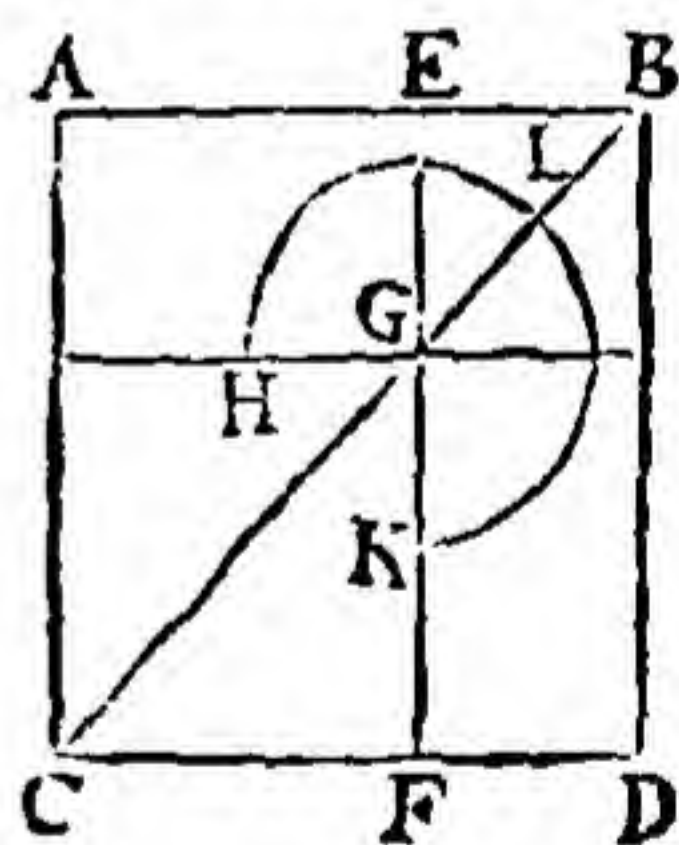
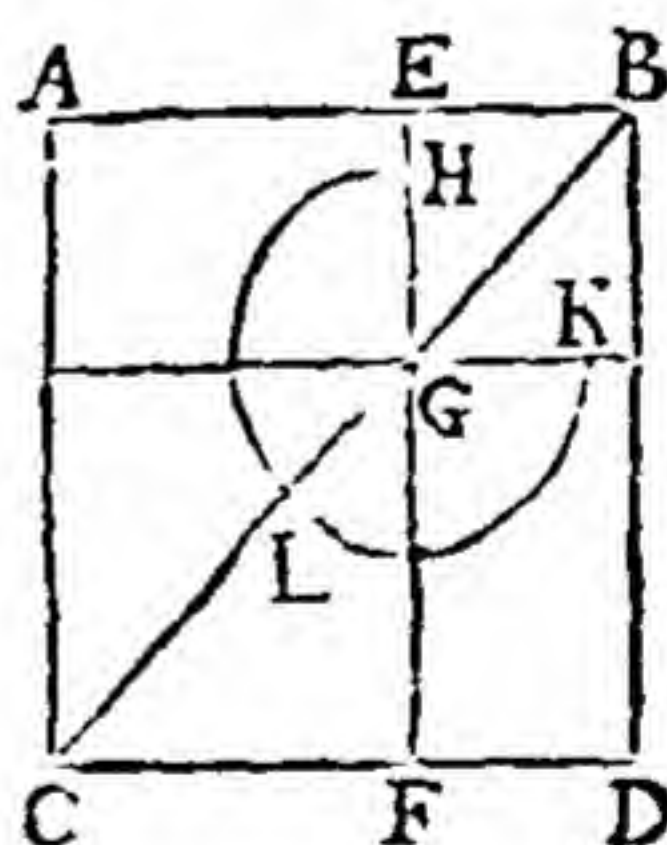
Observe also, that a Parallelogram space is for brevity sake, often noted by two opposite letters, as the Parallelogram AC, or DB; by either of which is signified the Parallelogram ABCD.



DEFINITION II.

In every Parallelogram space, any one of the Parallelograms about the Diameter, together with the two Complements, shall be called a Gnomon.

What are Parallelogram spaces about the Diameter, and what are Complements, is before declared in Prop. 43. El. I. Now here, the two Complements with either of the Diagonal spaces, taken together, are for brevity sake, in one word, as a term of Art, called a GNOMON; as in the Parallelogram space AD, the two Complements AG, GD, with any one of the Parallelograms about the Diameter, either CG, or BG, are called the GNOMON HLK.



Because this Second Element treats of the powers of strait lines, and of the Section of strait lines into several Segments, which are in power variously compared to one another, it will be requisite to add to *Euclide* these two following Definitions.

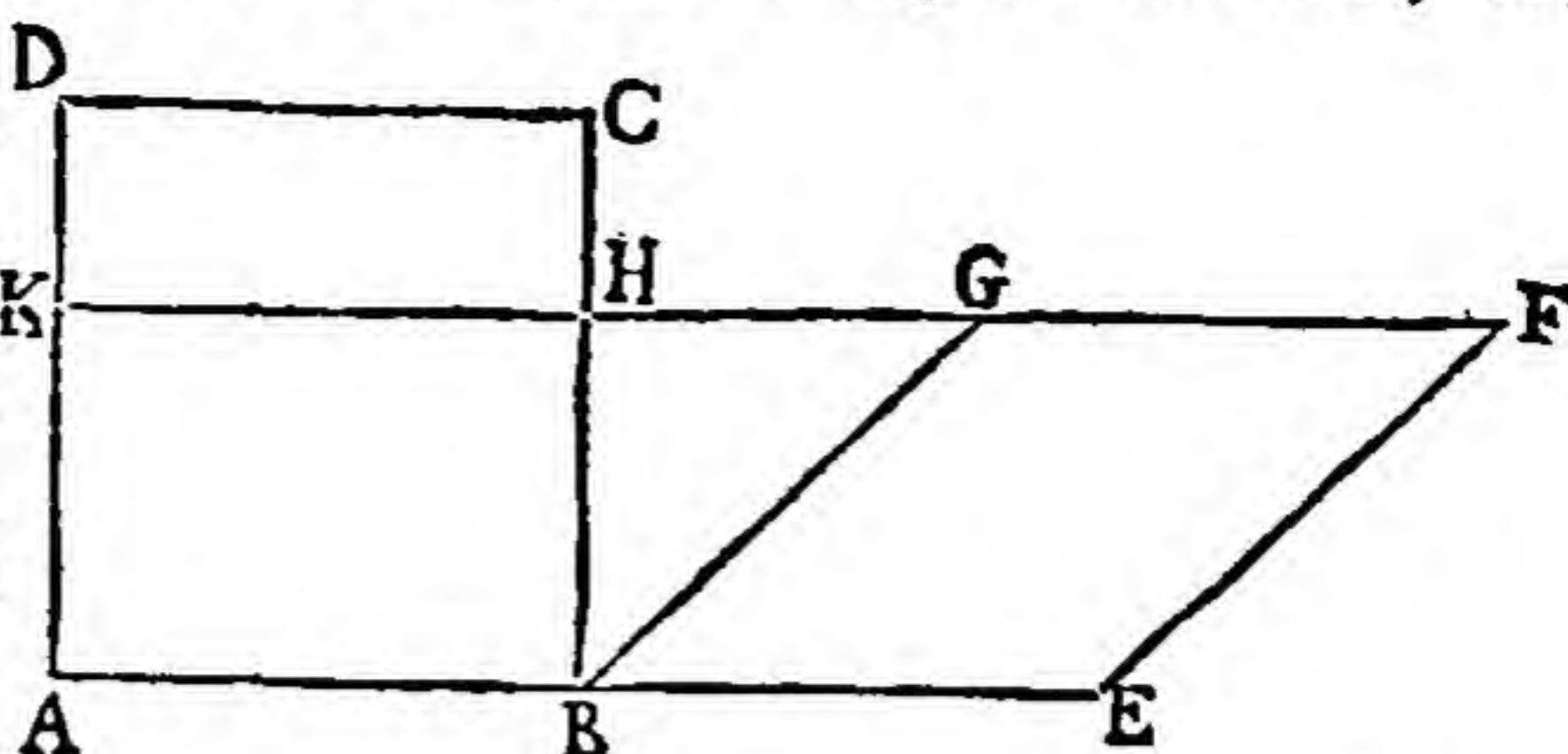
DEFINITION III.

The power of a strait line is the square of the same line.

As on the line AB, let there be described the square ABCD, then the square ABCD is said to be the power of the line AB.

Geome-

Geometricians have with good reason used the term *Power*, in this sense; because the square is the greatest Parallelogram space that can be described on one and the same line. For produce the line AB , and put BE equal to AB ; and on BE describe the Rhombus $BEFG$, 'tis manifest, that the Rhombus $BEFG$ is less than the square $ABCD$. For producing the line FG , to K , the Rhombus $BEFG$ is equal to the Rectangle $ABHK$ [by Prop. 36. El. I.] which is but a part of the square $ABCD$.

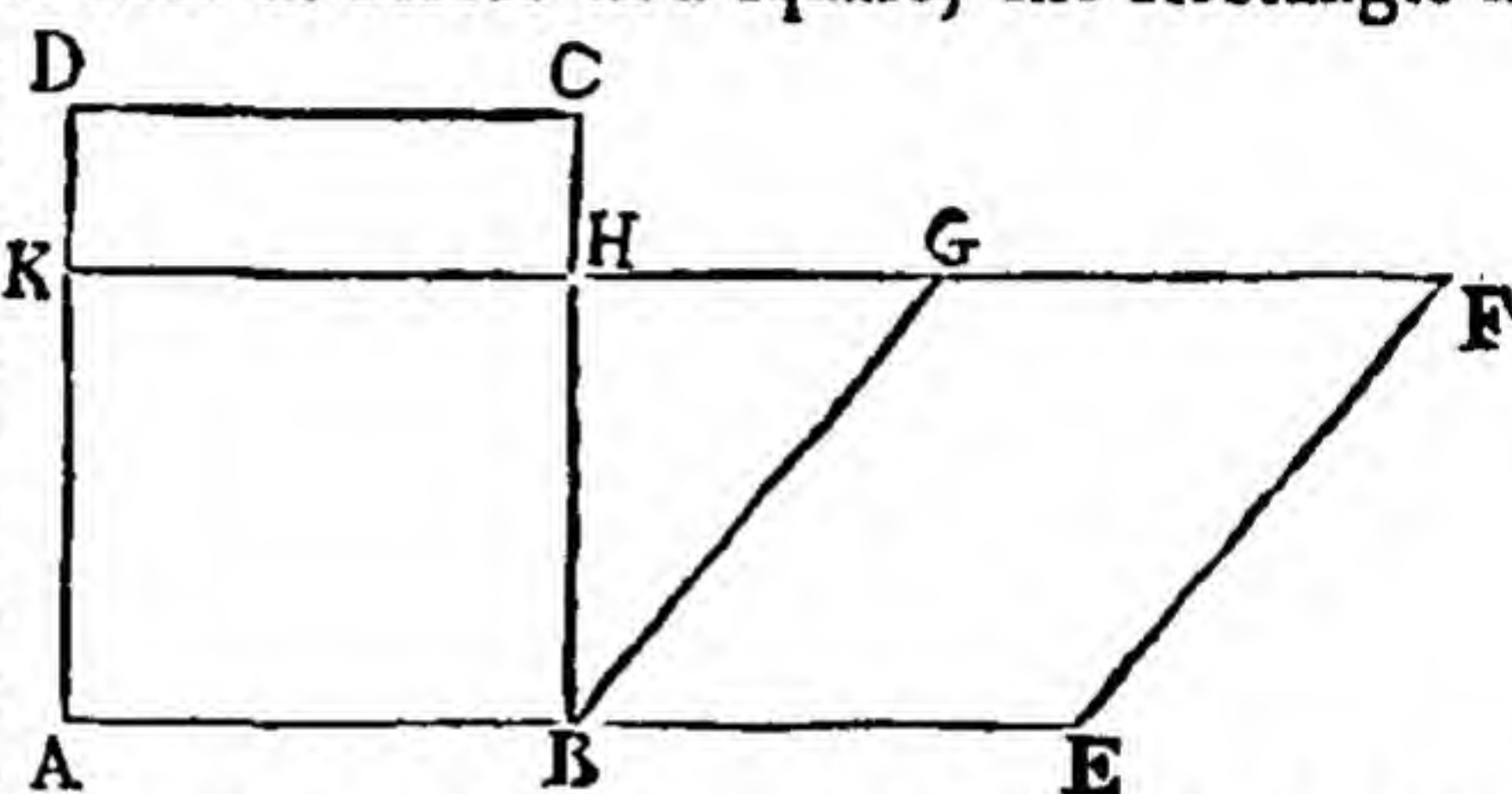


DEFINITION IV.

The power of two strait lines is the Rectangle contained by those lines.

As the Rectangle $ABCD$, contained by the lines AB , BC , is said to be the power of the lines AB , BC . Because here as before in a square, the Rectangle is the greatest Parallelogram space that

can be comprehended by any two and the same lines. For produce the line AB , and put BE equal to AB , and from the point E draw EF equal to BC , and complete the Parallelogram Figure; then by the same Prop. 36. El. I. the Rhomboides $BEFG$ is equal to the Rectangle $ABHK$, which is but a part of the Rectangle $ABCD$: yet the Rectangle $ABCD$, and the Rhomboides $BEFG$, are comprehended by the same, that is, by equal lines, for that BE is equal to AB , and EF to BC . Wherefore only the Rectangle $ABCD$ is said to be the power of the lines AB , BC .



PROPOSITION I.

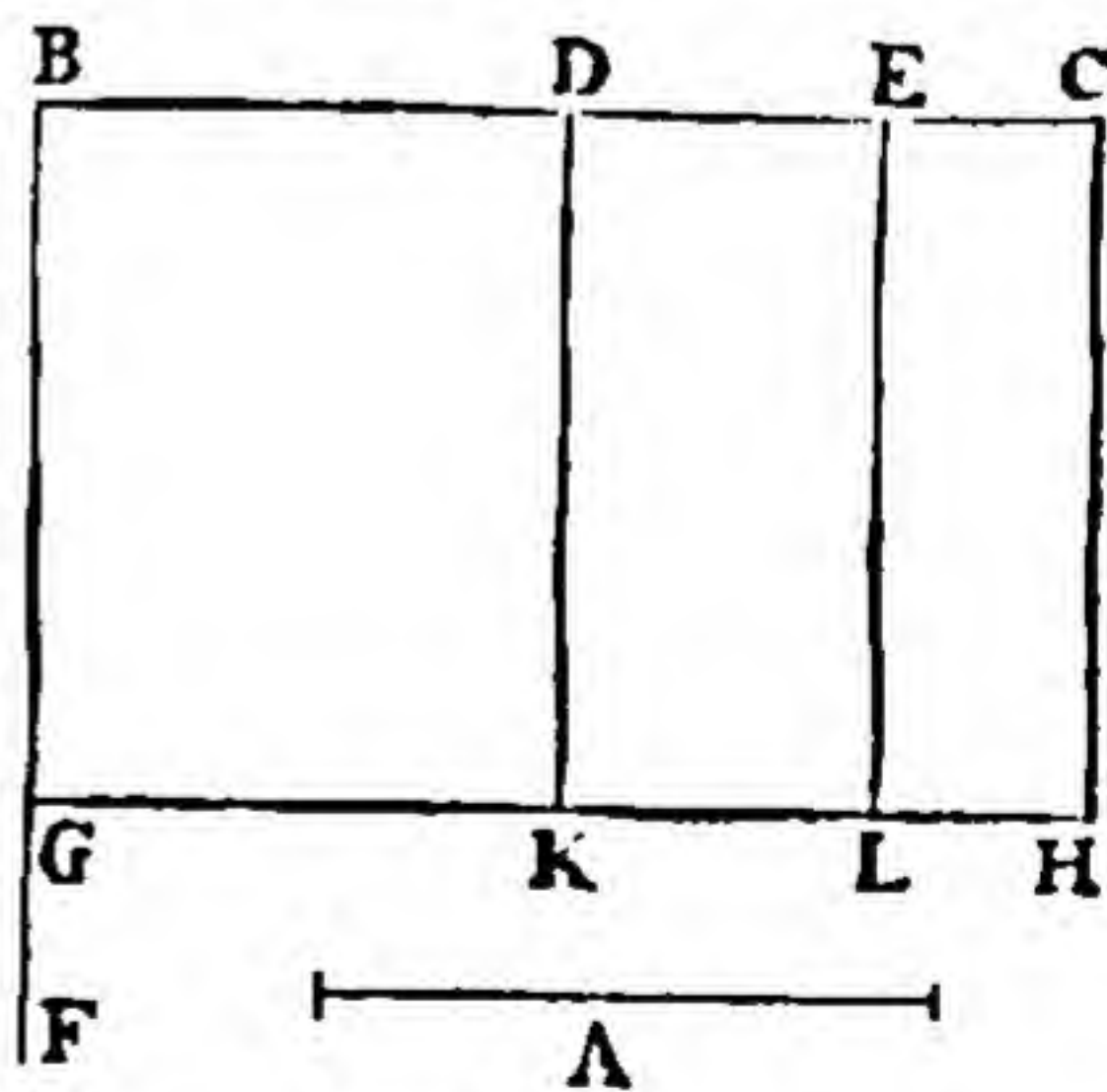
If there be two strait lines, and one of them be cut into how many Segments soever, the Rectangle contained by the two strait lines, is equal to the Rectangles contained by the undivided line, and the several Segments of the other line.

Let there be two strait lines A , BC , and let BC be cut at adventure in the points D , E . I say, that the Rectangle contained by the lines A , BC , is equal to the Rectangles contained by A , BD , and by A , DE , and also by A , EC .

For from the point B , let there be drawn BF , at right angles to BC ; and let BG be put equal to A [Prop. 3. I.]: Then by the point G let GH be drawn parallel to BC . Again, by the points D , E , C , let there be drawn DK , EL , CH , parallels to BG .

Now the Rectangle BH is equal to the Rectangles BK , DL , EH ; and BH is contained by A , BC , for it is contained by GB , BC , but GB is equal to A .

Again,



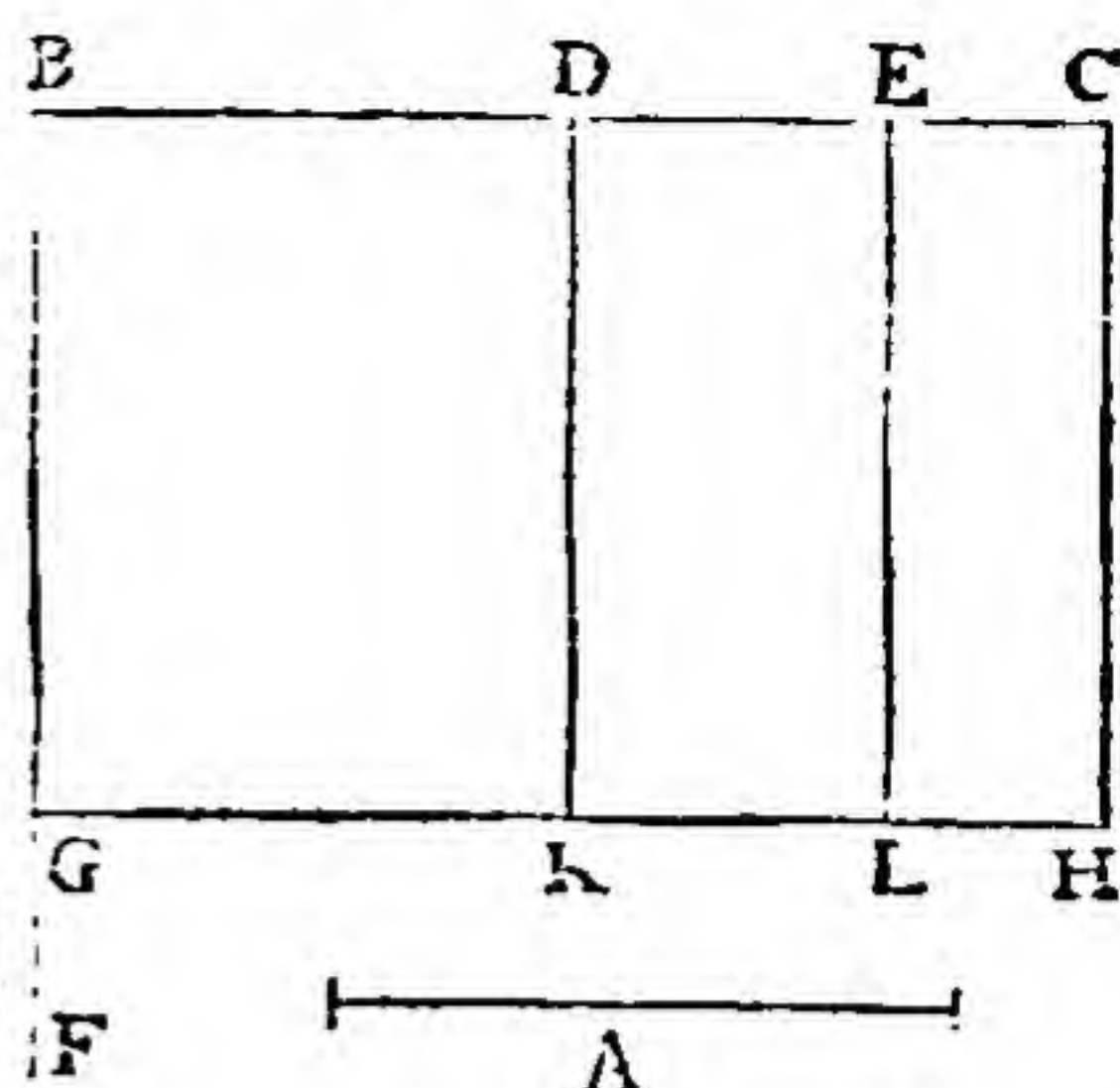
Again, the Rectangle BK is contained by A, BD , for it is contained by GB, BD , but GB is equal to A .

Also the Rectangle DL is contained by A, DE , for DK , that is BG , [Prop. 34. El. I.] is equal to A .

And in like manner the Rectangle EH is contained by A, EC .

Wherefore the Rectangle contained by A, BC , is equal to the Rectangles contained by A, BD , and by A, DE , and also by A, EC .

Therefore if there be two strait lines, and one of them be cut into how many Segments soever, the Rectangle contained by the two strait lines, is equal to the Rectangles contained by the undivided line. and the several Segments of the other line. Which was to be demonstated.



PROPOSITION II.

If a strait line be cut at adventure, the Rectangles contained by the whole line, and each of the Segments are equal to the square of the whole line.

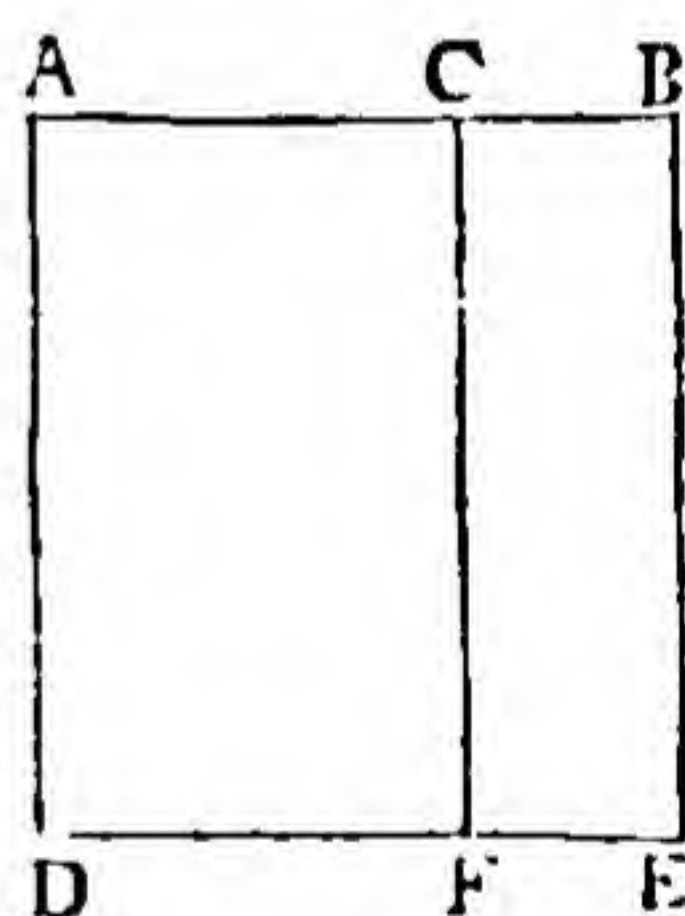
Let the strait line AB be cut at adventure in the point c .

I say, that the Rectangle contained by AB, AC , together with the Rectangle contained by AB, BC , is equal to the square of AB .

For on the line AB , let there be described the square $ADEB$; and by c , draw CF parallel to either AD , or BE .

Now AE is equal to AF, CE ; but AE is the square of AB , and AF is the Rectangle contained by AB, AC ; for it is contained by DA, AC ; but DA is equal to AB .

Also CE is contained by AB, BC , for BE is equal to AB ; wherefore the Rectangle contained by AB, AC , together with the Rectangle contained by AB, BC , is equal to the square of AB .



Therefore if a strait line be cut at adventure, the Rectangles contained by the whole line, and each of the Segments are equal to the square of the whole line. Which was to be demonstrated.

PROPOSITION III.

If a strait line be cut at adventure, the Rectangle contained by the whole, and one of the Segments is equal to the Rectangle contained by the Segments, and the square of the foresaid Segment.

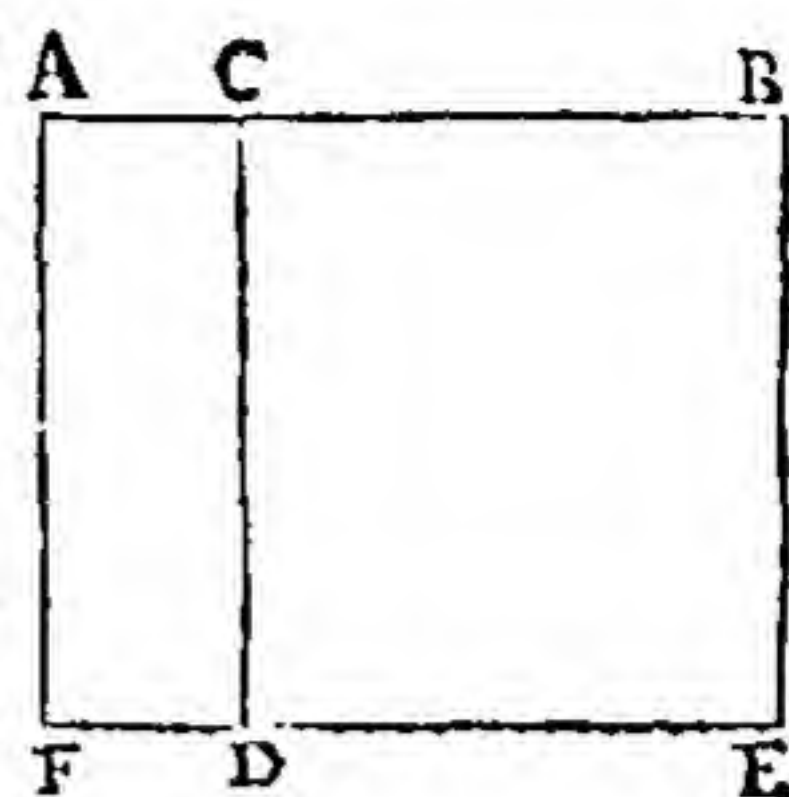
Let the strait line AB be cut at adventure in the point c .

I say, that the Rectangle contained by AB, BC , is equal to the Rectangle contained by AC, CB , together with the square of CB .

For on the line CB let there be described the square $CDEB$, and produce

produce ED to F : Then by the point A , draw [by Prop. 31. El. I.] AF , parallel to either CD , or BE .

Now the Rectangle AE is equal to the Rectangle AD , and the square CE ; and AE is the Rectangle contained by AB, BC ; for it is contained by AB, BE , but BE is equal to BC .



Again, AD is contained by AC, CB , for DC is equal to CB ; also DB is the square of CB ; wherefore the Rectangle contained by AB, BC , is equal to the Rectangle contained by AC, CB , together with the square of CB .

Therefore if a strait line be cut at adventure, the Rectangle contained by the whole and one of the Segments, is equal to the Rectangle contained by the Segments, and the square of the foresaid Segment. Which was to be demonstrated.

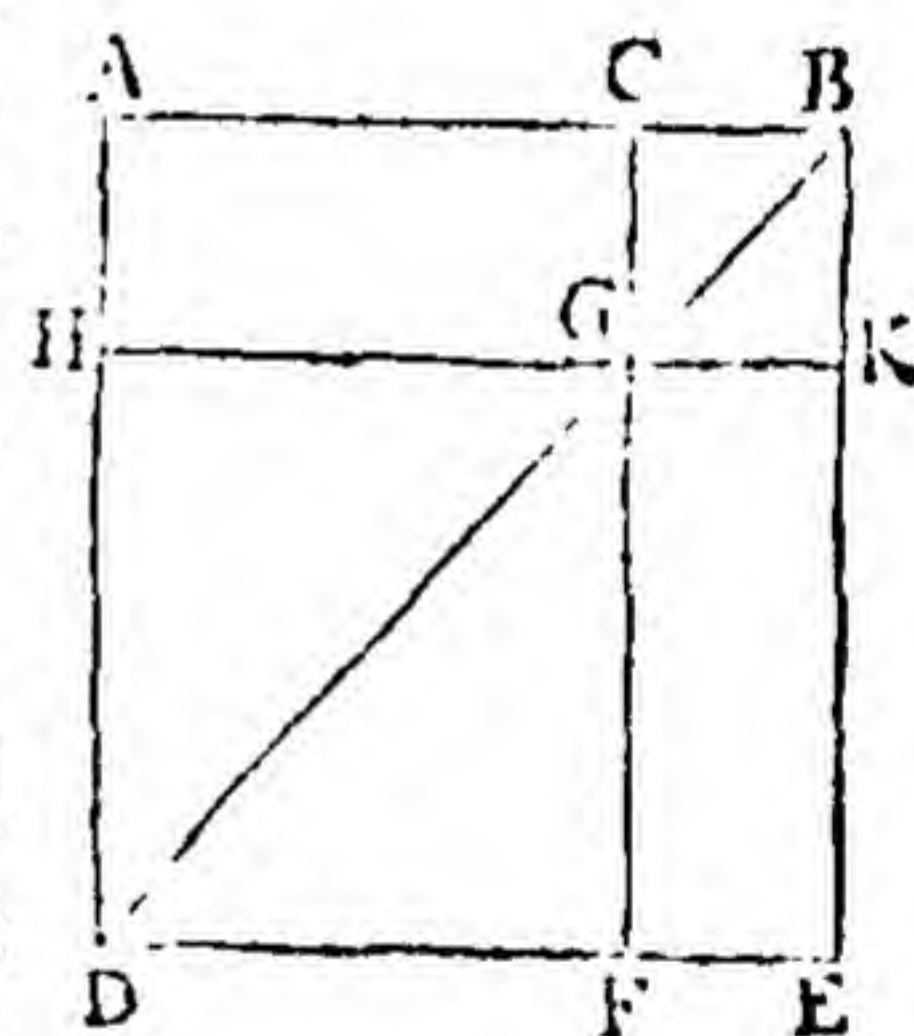
PROPOSITION IV.

If a strait line be cut at adventure, the square of the whole is equal to the squares of the Segments, and to the Rectangle twice contained by the Segments.

Let the strait line AB be cut at adventure in the point C .

I say, that the square of AB is equal to the squares of AC, CB , and to the Rectangle twice contained by AC, CB .

For, on the line AB let there be described the square $ADEB$, and draw the line BD . Then by the point C , let there be drawn parallel to either AD , or BE , the line CGF : and by the point G , draw HK , parallel to AB , or DE .



Now because CF is parallel to AD , and there falls on them the line BD , therefore the outward angle BGC is equal to the inward and opposite BDA ; but the angle BDA is equal to the angle DBA , because the side AB is equal to the side AD [by Prop. 5. El. I.]; wherefore also the angle CGB is equal to the angle CBG , and therefore the side CB , is equal to the side CG [by Prop. 6. El. I.]; but CB is equal to GK , and CG to BK [by Prop. 34. El. I.]; therefore also GK is equal to KB , and the Parallelogram $CGKB$ is Equilateral.

I say also, that it is Rectangular. For because CG is parallel to BK , and there falls on them the line CB , therefore the inward angles KBC, GCB , are equal to two right angles [Prop. 29. El. I.]. But the angle KBC is a right angle [by Construction]; wherefore the angle GCB is a right angle, and also the opposite angles CGK, GKB , are right angles [Prop. 34. El. I.]; wherefore the Parallelogram $CGKB$ is Rectangular. And it hath been proved to be Equilateral; therefore $CGKB$ is a square, and it is described on the line CB .

By the same reason also the Parallelogram HF is a square, and it

is described on the line HG , that is on AC ; therefore HF , CK , are the squares of AC , CB , the Segments of the line AB .

Moreover, because the Complement AG is equal to the Complement GE [by Prop. 43. El. I.], and AG is contained by the lines AC , CB , for that GC is equal to CB ; therefore also GE is equal to the Rectangle contained by AC , CB ; wherefore AG , GE , are equal to the Rectangle twice contained by the lines AC , CB : But also HF , CK , are the squares of AC , CB .

Therefore the four Parallelograms HF , CK , AG , GE , are equal to the squares of AC , CB , and to the Rectangle twice contained by AC , CB . But HF , CK , AG , GE , are the whole $ADEB$, which is the square of AB ; wherefore the square of AB is equal to the squares of AC , CB , and to the Rectangle twice contained by AC , CB .

Therefore, if a strait line be cut at adventure, the square of the whole is equal to the squares of the Segments, and to the Rectangle twice contained by the Segments. Which was to be demonstrated.

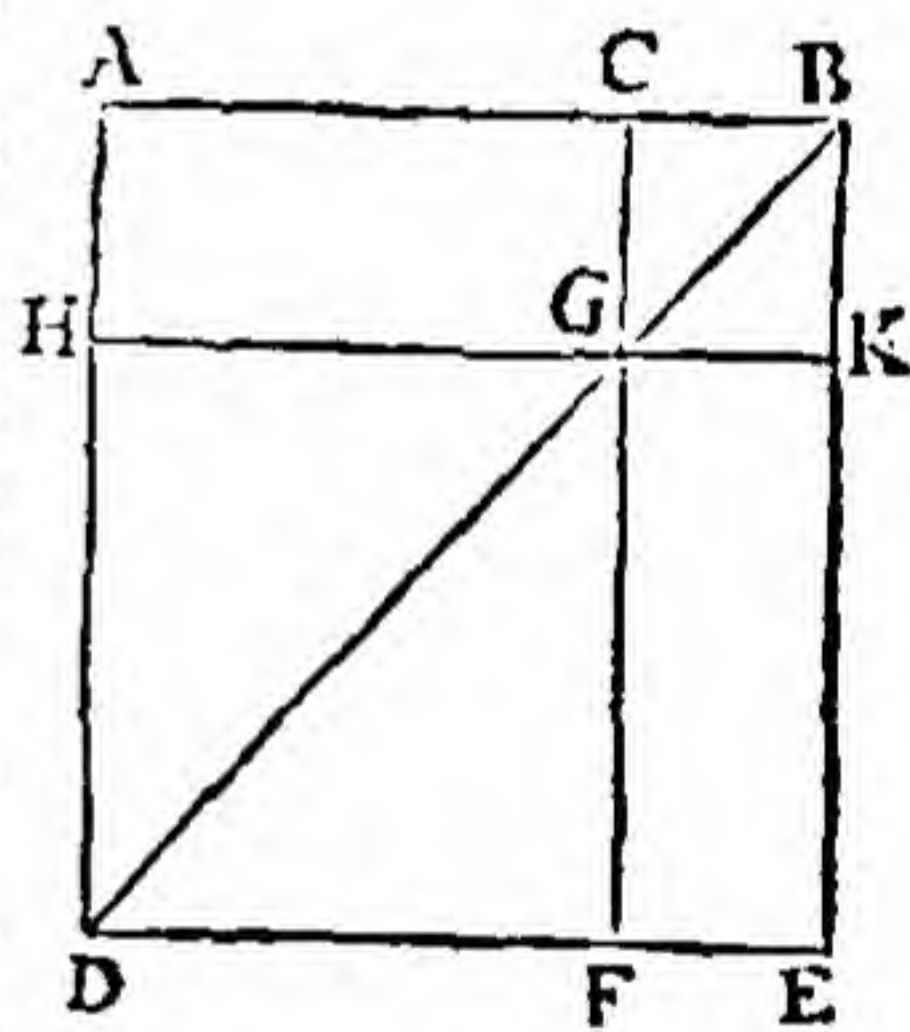
Otherwise.

I say, that the square of AB is equal to the squares of AC , CB , and to the Rectangle twice contained by AC , CB . For in the same Figure, because AB is equal to AD , therefore the angle ABD , is equal to the angle ADB [Prop. 5. El. I.]. And forasmuch as the three angles of every Triangle are equal to two right angles [Prop 32. El. I.]; therefore of the Triangle ABD , the three angles ABD , ADB , BAD , are equal to two right. But the angle BAD is a right angle; therefore the remaining angles ABD , ADB , are equal to one right angle, and they are also equal to one another; wherefore each of the angles ABD , ADB , is the half of a right angle. But the angle BCG is a right angle [Prop. 29. El. I.]; for it is equal to the inward and opposite angle at A ; wherefore the remaining angle CGB is the half of a right angle; and therefore the angle CGB is equal to the angle CBG , and the side CB is equal to the side CG [Prop. 6. El. I.]; but CB is equal to GK , and CG to BK [Prop. 34. El. I.]; wherefore the Parallelogram CK is Equilateral.

And because the angles BCG , CBK , are right angles, therefore CK is also a square. And it is described on the line CB .

Likewise by the same reason, the Parallelogram HF is a square, and equal to the square of AC ; wherefore HF , CK , are squares, and equal to the squares of AC , CB .

Moreover, because the Rectangle AG is equal to the Rectangle GE [Prop. 43. El. I.], and AG is contained by the lines AC , CB , (for that CG is equal to CB) therefore also GE is equal to the Rectangle contained by AC , CB ; wherefore AG , GE , are equal to the Rectangle twice



twice contained by AC, CB ; but also HF, CK , are equal to the squares of AC, CB .

Therefore HF, CK, AG, GE , are equal to the squares of AC, CB , and to the Rectangle twice contained by AC, CB . But HF, CK, AG, GE , are the whole AE , which is the square of AB ; therefore the square of AB is equal to the squares of AC, CB , and to the Rectangle twice contained by AC, CB . Which was to be demonstrated.

Corollaries.

1. From hence 'tis manifest, that in Squares, the Parallelograms, which are about the Diameter, are also Squares.

2. And if the side of a Square be cut into halves, then the Complements are also Squares. And the Square of the whole line is quadruple to the Square of the half.

On the fourth Proposition.

This is a most remarkable Proposition, and of excellent and various uses: The Analysis or Resolution of a Square, which, by Arithmeticians, is called the Extraction of the Square-root, wholly depends upon it. But because all things are resolved into those parts whereof they are at first constituted, it will be requisite, to begin with the Genesis or Construction of every compound Square.

The better to explain this matter, we shall apply to the Scheme of this fourth Proposition, by the addition of E , making it $A + E$: Then square it, that is, draw $A + E$ into $A + E$, thus,

$$\begin{array}{r} A + E \\ A + E \\ \hline Aq + AE \\ \quad + AE + Eq \\ \hline Aq + 2AE + Eq \end{array}$$

This Genesis of a Square exactly answers to *Euclid's* Proposition, and back again, by Analysis, shews how the most simple division of a Square must necessarily fall into these kinds of parts. But indeed *Euclid* begins with the square it self, and then resolves it, by dividing it into any two parts, because in magnitude, there is no Square singly the First, as Unity is in Numbers, which answers to all Powers. And therefore dividing a line, on which a Square is described, into any two parts, he shews of what parts the most simple that may be a Square can consist. And accordingly the practice of extracting the Square root answers to it. As for example.

Every Square is encreased by the addition of twice the root for the Complements, and 1 for the Diagonal Square, thus $2\sqrt{+1}$. If there be put Aq , it is encreased $2A + 1$: So that $Aq + 2A + 1$, is the next Square.

PROPOSITION V.

IF a strait line be cut into two equal Segments, and two unequal, the Rectangle contained by the unequal Segments of the whole, together with the Square of the line between the Sections, is equal to the Square of half the line.

Let the strait line AB be cut into two equal Segments, at the point c , and into two unequal, at the point D .

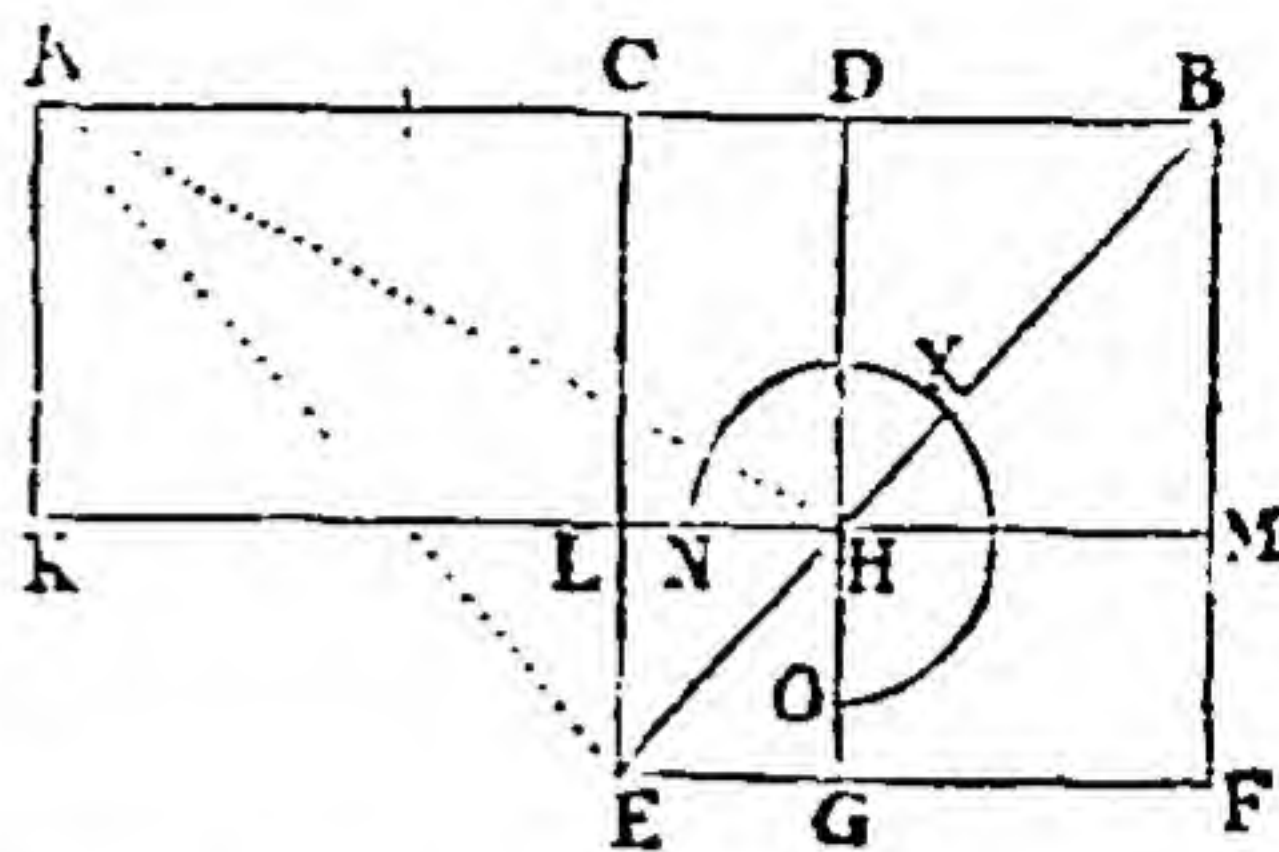
I say, that the Rectangle contained by AD, DB , together with the square of CD , is equal to the square of CB .

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For on the line CB , let there be described the square $CEFB$, and draw the line BE . Then by the point D , let there be drawn parallel to either CE , or BF , the line DHG , and by the point H , draw KM , parallel to CB , or EF . And again, by the point A , draw AK , parallel to CL , or BM .

Now forasmuch as the Complement CH is equal to the Complement HF [Prop. 43. El. I.], let DM be added in common to both; therefore the whole CM is equal to the whole DF . But CM is equal to AL , for that AC is [by Supposition] equal to CB : wherefore also AL is equal to DF . Add in common CH , therefore the whole AH is equal to DF , DL .



But AH is the Rectangle contained by AD , DB , for that DH is equal to DB [by Coroll. 1. Prop. 4. El. II.], and DF , DL , is the Gnomon NXO ; therefore the Gnomon NXO is equal to the Rectangle contained by AD , DB .

Again, add in common LG , which is equal to the square of CD ; therefore the Gnomon NXO , and LG , are equal to the Rectangle contained by AD , DB , and to the square of CD .

But the Gnomon NXO , and LG , are the whole square $CEFB$, which is described on the line CB ; therefore the Rectangle contained by AD , DB , together with the square of CD is equal to the square of CB .

If therefore a strait line be cut into two equal Segments, and two unequal, the Rectangle contained by the unequal Segments of the whole, together with the square of the line between the Sections, is equal to the square of half the line. Which was to be demonstrated.

PROPOSITION VI.

IF a strait line be cut into two equal Segments, and to it be added another strait line directly, the Rectangle contained by the whole with the adjunct (as one line) and by the adjunct, together with the square of the half, is equal to the square described on the half and the adjunct, as one line.

Let the strait line AB be cut into two equal Segments at the point C , and to AB , let BD be added directly. I say, that the Rectangle contained by AD , DB , together with the square of CB is equal to the square of CD .

For on the line CD , let there be described the square $CEFD$, and draw the line DE . Then by the point B , let there be drawn parallel to either CE or DF , the line BHG . And by the point H , draw KLM , parallel to AD , or EF : And also by the point A draw AK , parallel to CL , or DM .

Now

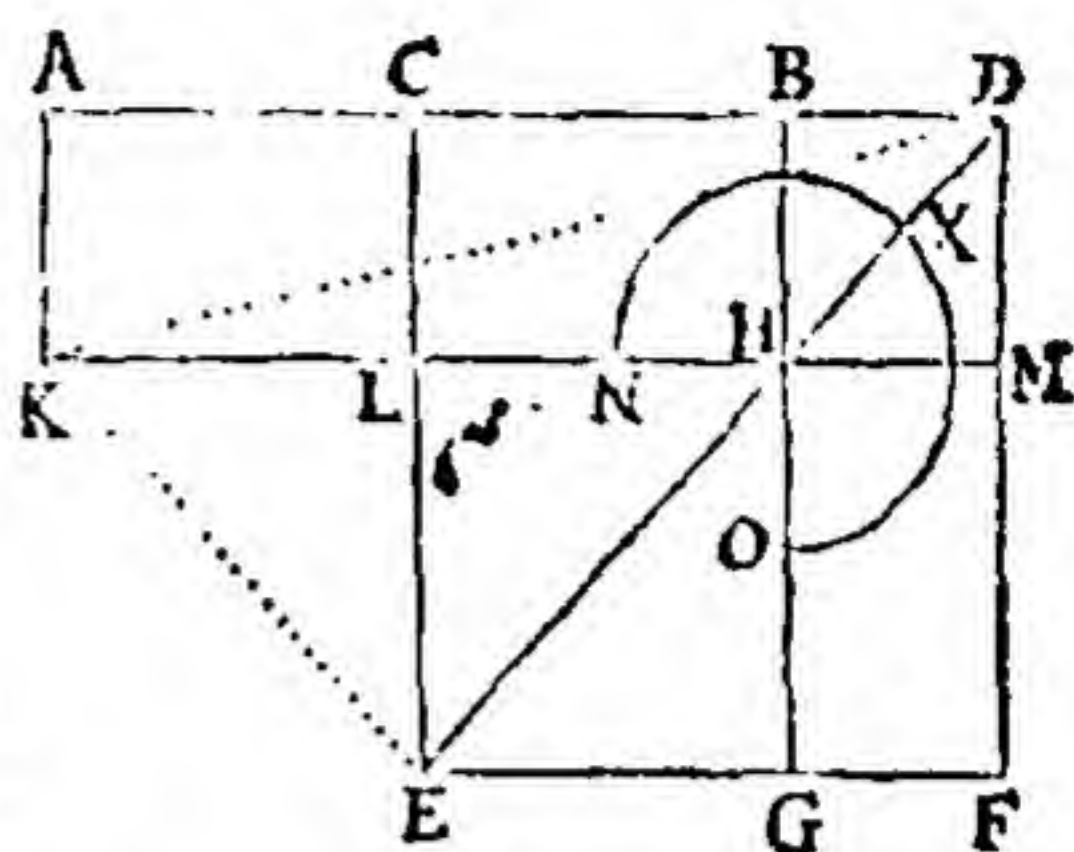
Now forasmuch as AC is equal to CB [by Supposition], therefore the Rectangle AL is equal to the Rectangle CH [by Prop. 36. El. I.]. But CH is equal to HF [Prop. 43. El. I.]; wherefore AL is equal to HF . Let CM be added in common; therefore the whole AM , is equal to the Gnomon NXO .

But AM is the Rectangle contained by AD, DB , for that DM is equal to DB [by Corol. I. Prop. 4. El. II.]; wherefore the Gnomon NXO is equal to the Rectangle contained by AD, DB .

Again, add in common LG , which is equal to the square of CB ; therefore the Rectangle contained by AD, DB , together with the square of CB , is equal to the Gnomon NXO , and to the square LG .

But the Gnomon NXO , and LG are the whole square $CEFD$, which is described on the line CD ; therefore the Rectangle contained by AD, DB , together with the square of CB , is equal to the square of CD .

If therefore a strait line be cut into two equal Segments, and to it be added another strait line directly, the Rectangle contained by the whole with the adjunct (as one line) and by the adjunct, together with the square of the half, is equal to the square described on the half and the adjunct, as one line. Which was to be demonstrated.



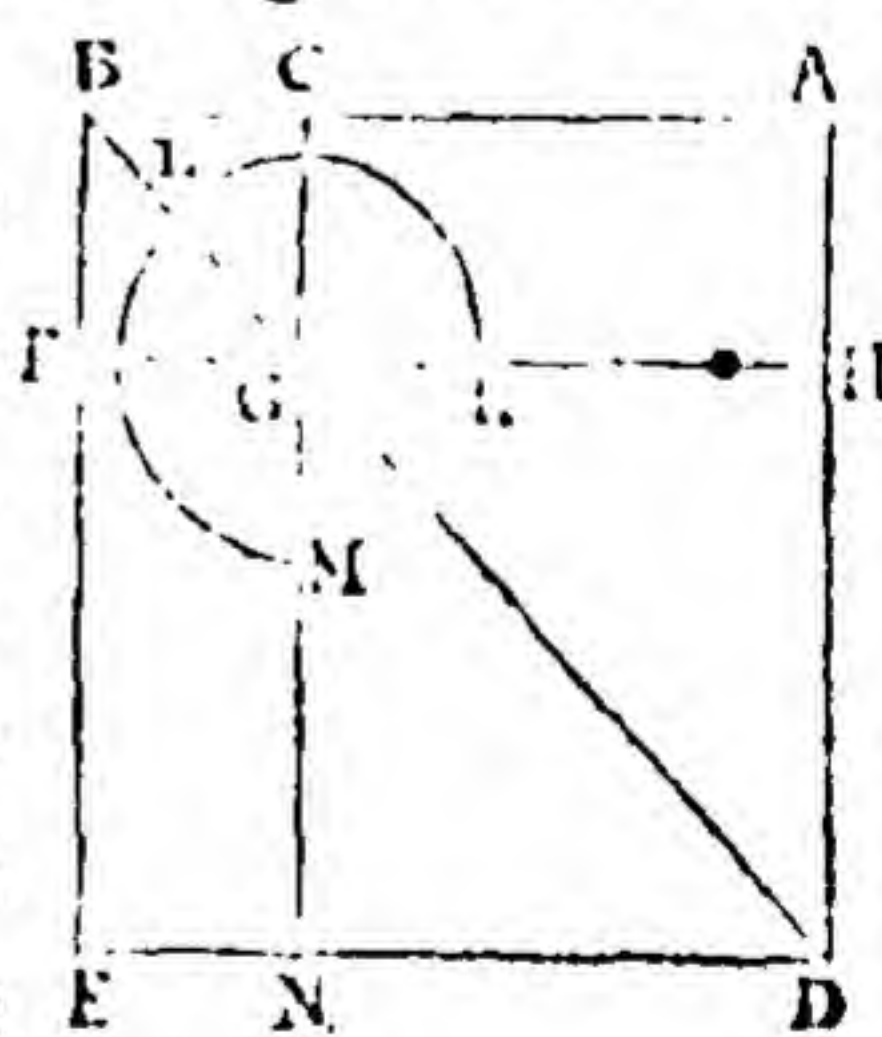
PROPOSITION VII.

If a strait line be cut at adventure, the square of the whole, and the square of one of the Segments, both these squares together are equal to the Rectangle twice contained by the whole, and the said Segment, and also to the square of the remaining Segment.

Let the strait line AB be cut at adventure in the point C . I say, that the squares of AB, BC , are equal to the Rectangle twice contained by AB, BC , and to the square of AC .

For on the line AB , let there be described the square $ADEB$, and let the Figure be constructed.

Now forasmuch as AG is equal to GE [Prop. 43. El. I.], add in common CF : therefore the whole AF is equal to the whole CE : wherefore AF, CE , are double to AF . But AF, CE , are the Gnomon KLM , and the square CF ; therefore the Gnomon KLM , and the square CF , are double to AF .



But the Rectangle twice contained by AB, BC , is double to AF , for BF is equal to BC ; therefore the Gnomon KLM , and the square CF are equal to the Rectangle twice contained by AB, BC .

Again, let there be added in common HN , which is the square of AC ; therefore the Gnomon KLM , and the squares CF, HN , are equal

to the Rectangle twice contained by AB, BC , and to the square of AC .

But the Gnomon KLM , and the squares CF, HN , are the whole $ADEB$, and CF , which are the squares of AB, BC ; therefore the squares of AB, BC , are equal to the Rectangle twice contained by AB, BC , together with the square of AC .

If therefore a strait line be cut at adventure, the square of the whole, and the square of one of the Segments, both these squares together are equal to the Rectangle twice contained by the whole, and the said Segment; and also to the square of the remaining Segment. Which was to be demonstrated.

PROPOSITION VIII.

If a strait line be cut at adventure, the Rectangle four times contained by the whole, and one of the Segments, together with the square of the other Segment, is equal to the square described on the whole, and the said Segment, as one line.

Let the strait line AB be cut at adventure in the point c . I say, that the Rectangle four times contained by AB, BC , together with the square of AC , is equal to the square described on AB, BC , as one line.

For let the line AB be produced directly to D , and let BD be put equal to CB ; then on AD let there be described the square $AEDF$, and let the double Figure be constructed.

Now forasmuch as CB is equal to BD , but also CB is equal to GK , and BD to KN [Prop. 34. El. I.], therefore GK is equal to KN . By the same reason also PR is equal to RO .

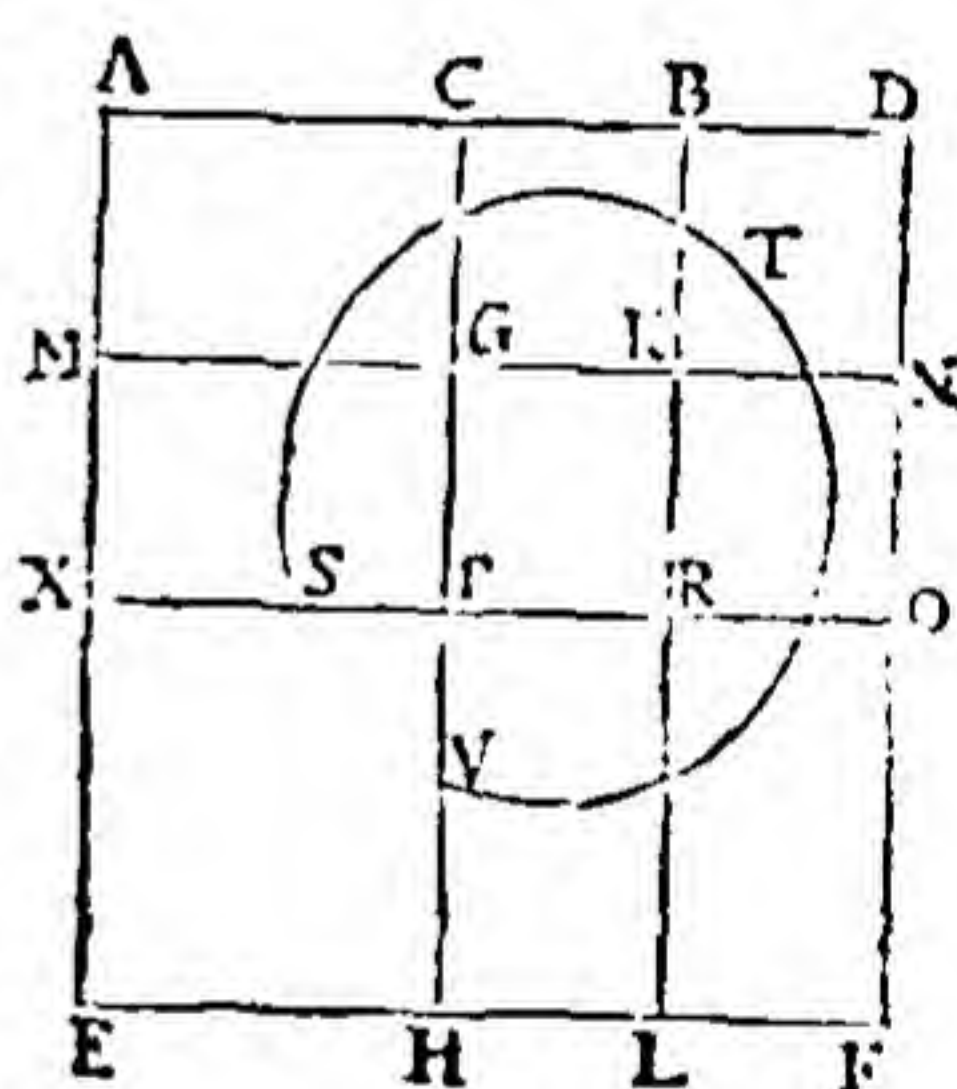
And because CB is equal to BD , and GK to KN , therefore the Rectangle CK is equal to the Rectangle KD ; and GR to RN [Prop. 36. El. I.]. But CK is equal to RN (because they are the Complements of the Parallelogram CO) wherefore also KD is equal to GR : therefore the four Rectangles CK, KD, GR, RN , are equal to one another; and these four therefore are quadruple to CK .

Again, because CB is equal to BD , but also BD is equal to BK [by Coroll. 1. Prop. 4. El. II.] that is, to CG , [by Prop. 36. El. I.]; and also because CB is equal to GK , that is, to GP [by Coroll. 1. Prop. 4. El. II.]; therefore CG is equal to GP .

And because CG is equal to GP , and PR to RO , therefore the Rectangle AG is equal to the Rectangle MP , and PL to RF [by Prop. 36. El. I.].

But MP is equal to PL (because they are the Complements of the Parallelogram ML); wherefore also AG is equal to RF .

There-



Therefore the four Rectangles AG, MP, PL, RF , are equal to one another; and these four therefore are quadruple to AG .

But the four Rectangles CK, KD, GR, RN , have been proved quadruple to CK .

Wherefore the eight Rectangles which contain the Gnomon STV , are quadruple to the Rectangle AK .

And because AK is contained by AB, BC , for BK is equal to BD , that is to BC , therefore the Rectangle four times contained by AB, BC , is quadruple to AK .

But the Gnomon STV hath been proved quadruple to AK ; therefore the Rectangle four times contained by AB, BC , is equal to the Gnomon STV .

Let there be added in common XH , which is equal to the square of AC ; wherefore the Rectangle four times contained by AB, BC , together with the square of AC , is equal to the Gnomon STV , and XH .

But the Gnomon STV , and XH are the whole square $AEDF$, which is described on the line AD ; therefore the Rectangle four times contained by the lines AB, BC , together with the square of AC , is equal to the square of AD , that is, to the square described on AB, BC , as one line.

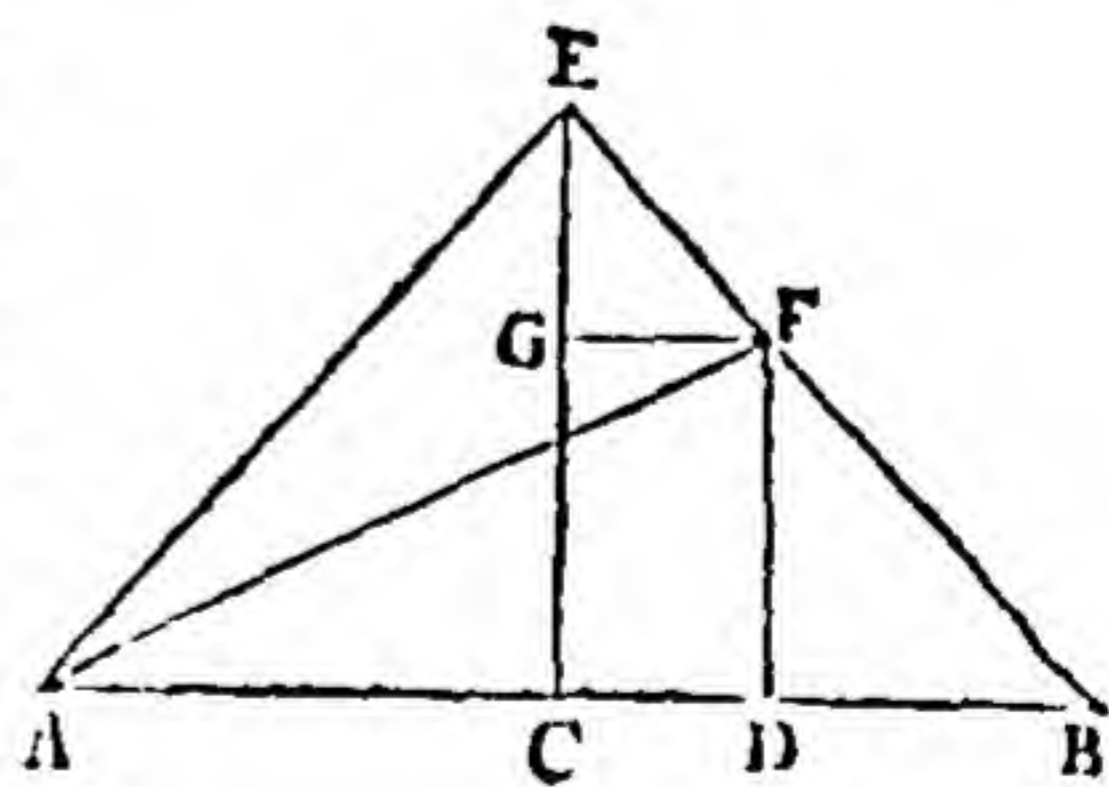
If therefore a strait line be cut at adventure, the Rectangle four times contained by the whole, and one of the Segments, together with the square of the other Segment, is equal to the square described on the whole and the said Segment, as one line. Which was to be demonstrated.

PROPOSITION IX.

If a strait line be cut into two equal Segments, and two unequal, the squares of the unequal Segments of the whole line are double to the square of the half, and to the square of the line between the Sections.

Let the strait line AB be cut into two equal Segments at the point C , and into two unequal, at the point D . I say, that the squares of AD, DB , are double to the squares of AC, CD .

For from the point C to the line AB , let there be drawn at right angles CE , and let it be put equal to either AC , or CB ; then draw AE, EB . And by the point D , let there be drawn, parallel to CE , the line DF , and also by the point F , draw FG parallel to AB . Then let there be drawn AF .



Now forasmuch as AC is equal to CE , therefore the angle EAC is equal to the angle AEC . And because the angle at C is a right angle, therefore the remaining angles AEC, EAC , are equal to one right angle [Prop. 32. El. I.], and they are equal to one another: wherefore

fore each of the angles AEC , EAC , is half of a right angle.

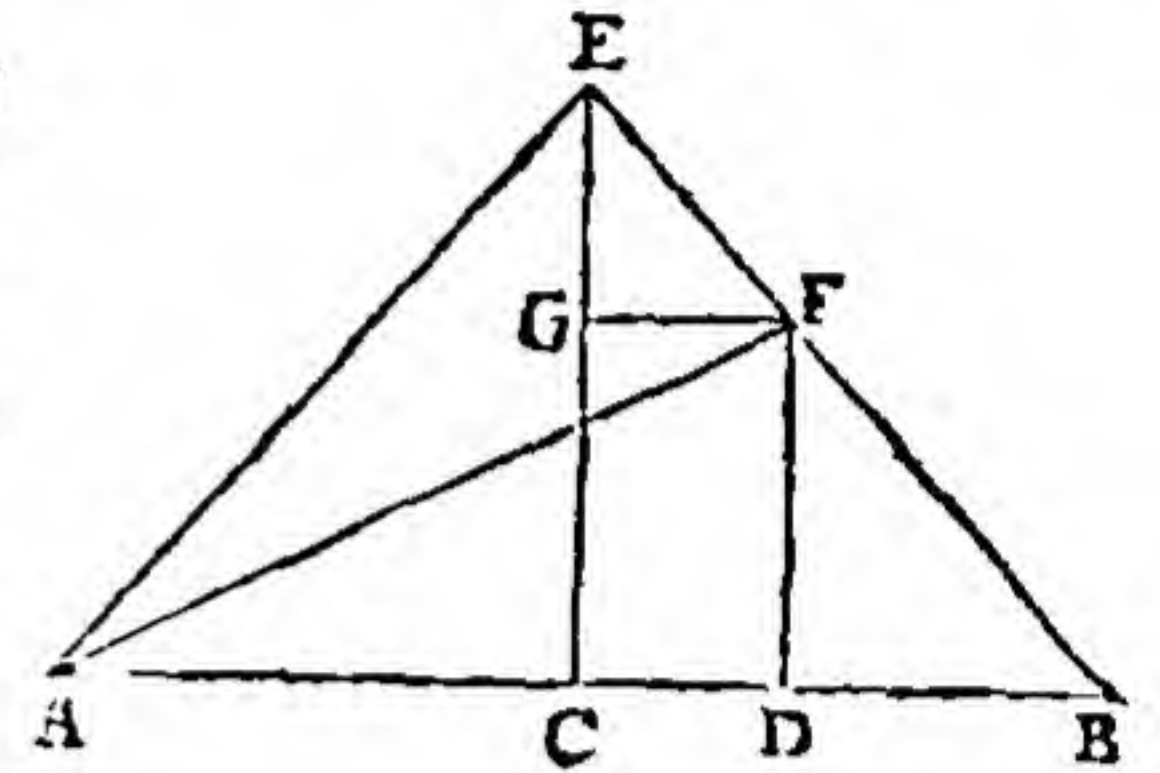
By the same reason also each of the angles BEC , EBC , is half of a right angle.

Therefore the whole angle AEB is a right angle.

And forasmuch as GEF is half of a right angle, but EGF is a right angle [Prop. 29. El. I.], (because it is equal to the inward and opposite ECB) therefore the remaining angle FGC is also half of a right angle.

Wherefore the angle FGC is equal to GEF , and therefore the side EG , is equal to the side FG . [Prop. 6. El. I.].

Again, forasmuch as the angle at B is half of a right angle, and that FDB is a right angle (because it is equal to the inward and opposite ECB) therefore the remaining angle BFD , is also half of a right angle.



Wherefore the angle at B is equal to the angle DFB , so that also the side DF is equal to the side DB [Prop. 6. El. I.].

Now because AC is equal to CE , and so the square of AC is equal to the square of CE , therefore the squares of AC , CE , are double to the square of AC . But the square of AE is [Prop. 47. El. I.] equal to the squares of AC , CE [for ACE is a right angle by Construction]; wherefore the square of AE is double to the square of AC .

Again, forasmuch as EG is equal to GF ; and so the square of EG is equal to the square of GF ; therefore the squares of EG , GF , are double to the square of GF . But the square of EF is equal to the squares of EG , GF ; therefore the square of EF is double to the square of GF . But the line GF is equal to CD [Prop. 34. El. I.]; wherefore the square of EF is double to the square of CD .

But the square of AE is double to the square of AC .

Therefore the squares of AE , EF , are double to the squares of AC , CD .

But the square of AF is equal to the squares of AE , EF , [for AEF is a right angle]; therefore the square of AF is double to the squares of AC , CD .

But again, to the square of AF are equal the squares of AD , DF , (for the angle at D is a right angle) therefore the squares of AD , DF , are double to the squares of AC , CD .

But DF is equal to DB , wherefore the squares of AD , DB , are double to the squares of AC , CD .

If therefore a strait line be cut into two equal Segments, and two unequal, the squares of the unequal Segments of the whole line, are double to the square of the half, and to the square of the line between the Sections. Which was to be demonstrated.

PROPO.

PROPOSITION X.

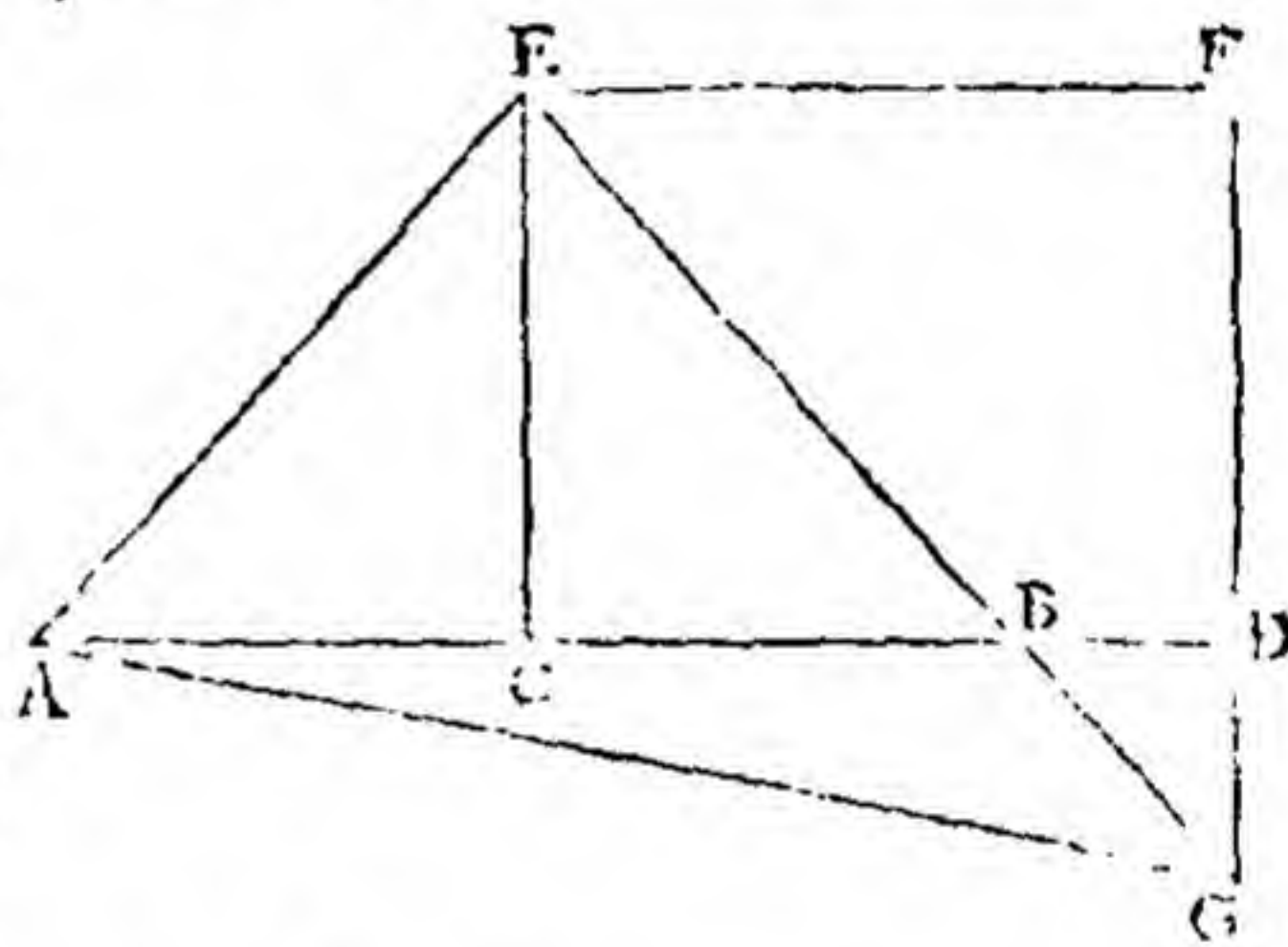
If a straight line be cut into two equal Segments, and to it be added another straight line directly, the square of the whole, with the adjunct (as one line) and the square of the adjunct, these both together are double to the square of the half, and to the square described on the half, and the adjunct, as one line.

Let the straight line AB be cut into two equal Segments, at the point C , and to AB let BD be added directly. I say, that the squares of AD , DB , are double to the squares of AC , CD .

For from the point C to the line AB , let there be drawn at right angles CE , and let it be put equal to either AC , or CB ; then draw AE , EB . And by the point E let there be drawn, parallel to CD , the line EF , and also by the point D , draw DF parallel to CE . Now because there falls on the parallels EC , FD , the straight line EF ; therefore the inward angles CEF , EFD , are equal to two right angles; wherefore the angles FEB , EFD , are less than two right angles: but lines infinitely produced from angles less than two right do meet together; wherefore EB , FD , produced towards B , D shall meet. Let them be produced, and meet at the point G . Then draw AG .

Now forasmuch as AC is equal to CE , therefore the angle AEC is equal to the angle EAC . And the angle at C is a right angle; wherefore each of the angles EAC , AEC , is half of a right angle.

By the same reason also each of the angles BEC , EBC , is half of a right angle.



Therefore the whole angle AEB is a right angle.

And forasmuch as EBC is half of a right angle, therefore DBG is also half of a right angle [Prop. 15. I.]; but BDG is a right angle, because it is equal to the alternate angle DCE [Prop. 29. I.]; therefore the remaining angle DGB is half of a right angle; wherefore DGB is equal to DBG . So that also the side BD is equal to the side DG [Prop. 6. I.].

Again, forasmuch as EGF is half of a right angle, and that the angle at F is a right angle, because it is equal to the opposite angle at C [Prop. 34. El. I.]; therefore the remaining angle FEG is half of a right angle. Wherefore EGF is equal to GEF ; so that also the side GF is equal to the side EF .

Now because EC is equal to CA ; and so the square of EC , to the square of CA ; therefore the squares of EC , CA , are double to the square of CA . But the square of EA is equal to the squares of EC , CA [Prop. 47. I.].

Therefore the square of EA is double to the square of AC .

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Again,

contained by CF, FA , together with the square of AE is equal to the square of EF [Prop. 6, El. II.].

But EF is equal to EB ; therefore the Rectangle contained by CF, FA , together with the square of EA , is equal to the square of EB .

But to the square of EB are equal the squares of BA, AE ; for the angle at A is a right angle.

Therefore the Rectangle contained by CF, FA , together with the square of AE , is equal to the squares of BA, AE .

Let the common square of AE be taken away. Therefore the remaining Rectangle contained by CF, FA , is equal to the square of AB ; and the Rectangle contained by CF, FA , is the Rectangle FK (for AF is equal to FG) and also the square of AB is the square AD ; therefore FK is equal to AD .

Let the common Rectangle AK be taken away. Therefore the remainder FH is equal to the remainder HD .

But HD is the Rectangle contained by AB, BH (for AB is equal to BD by Construction), and FH is the square of AH .

Therefore the Rectangle contained by AB, BH , is equal to the square of AH .

Wherefore the straight line given AB , is cut in the point H , so that the Rectangle contained by the whole AB , and by the Segment BH , is equal to the square of the other Segment AH . Which was to be done.

PROPOSITION XII.

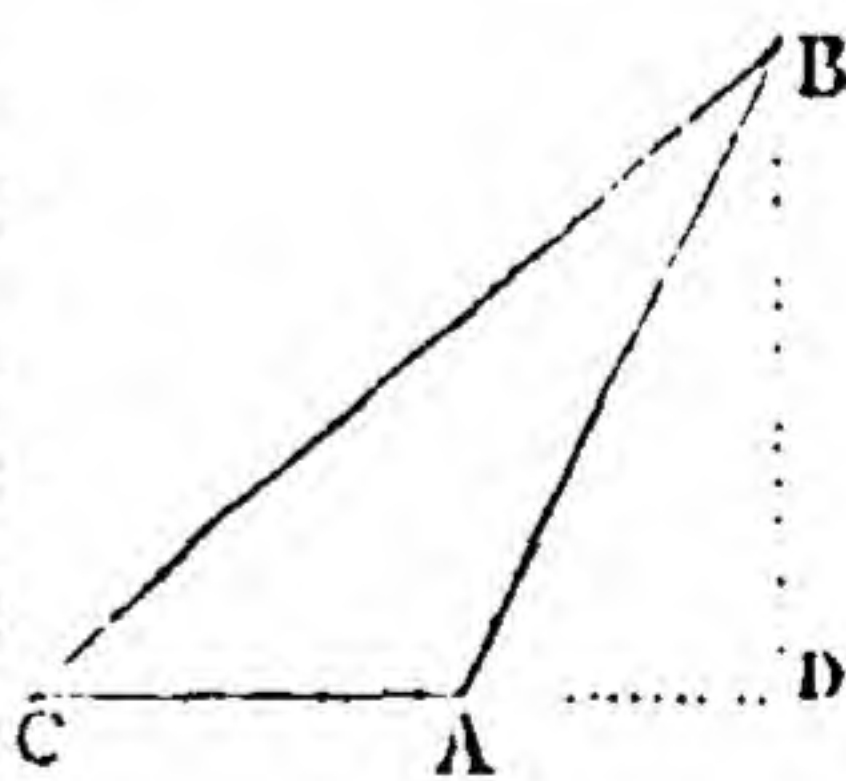
IN Obtuse angl'd Triangles, the square of the side subtending the Obtuse angle, is greater than the squares of the sides containing the Obtuse angle, by the Rectangle twice contained by one of the sides about the Obtuse angle, on which being produced a perpendicular falls; and by the line intercepted without, between the perpendicular and the Obtuse angle.

Let ABC be an Obtuse angl'd Triangle, having the Obtuse angle BAC ; and from the point B on the side CA produced, let there be drawn the perpendicular BD .

I say, that the square of BC is greater than the squares of BA, AC , by the Rectangle twice contained by CA, AD .

For because CD is cut at adventure in the point A , therefore the square of CD is equal to the squares of CA, AD , and to the Rectangle twice contained by CA, AD [Prop. 4. El. II.]. Add in common the square of DB ; therefore the squares of CD, DB , are equal to the squares of CA, AD, DB , and to the Rectangle twice contained by CA, AD .

But to the squares of CD, DB , is equal the square of CB (for the angle at D is a right angle), and to the squares of AD, DB , is equal the square of AB .



Therefore the square of CB is equal to the squares of CA, AB , and to the Rectangle twice contained by CA, AD .

So that the square of CB is greater than the squares of CA, AB , by the Rectangle twice contained by CA, AD .

Therefore in Obtuse angl'd Triangles, the square of the side subtending the Obtuse angle, is greater than the squares of the sides containing the Obtuse angle, by the Rectangle twice contained by one of the sides about the Obtuse angle, on which being produced a perpendicular falls; and by the line intercepted without, between the perpendicular and the Obtuse angle. Which was to be demonstrated.

PROPOSITION XIII.

IN *Acute angl'd Triangles the square of the side subtending the Acute angle, is less than the squares of the sides containing the Acute angle, by the Rectangle twice contained by one of the sides about the Acute angle, on which a perpendicular falls; and by the line intercepted within, between the perpendicular and the Acute angle.*

Let ABC be an Acute angl'd Triangle, having an Acute angle at B ; and from the point A , on the side BC , let there be drawn the perpendicular AD .

I say, that the square of AC is less than the squares of CB, BA , by the Rectangle twice contained by CB, BD .

For because CB is cut at adventure in the point D , therefore the squares of CB, BD , are equal to the Rectangle twice contained by CB, BD , and to the square of DC [Prop. 7. El. II.]. Add in common the square of AD .

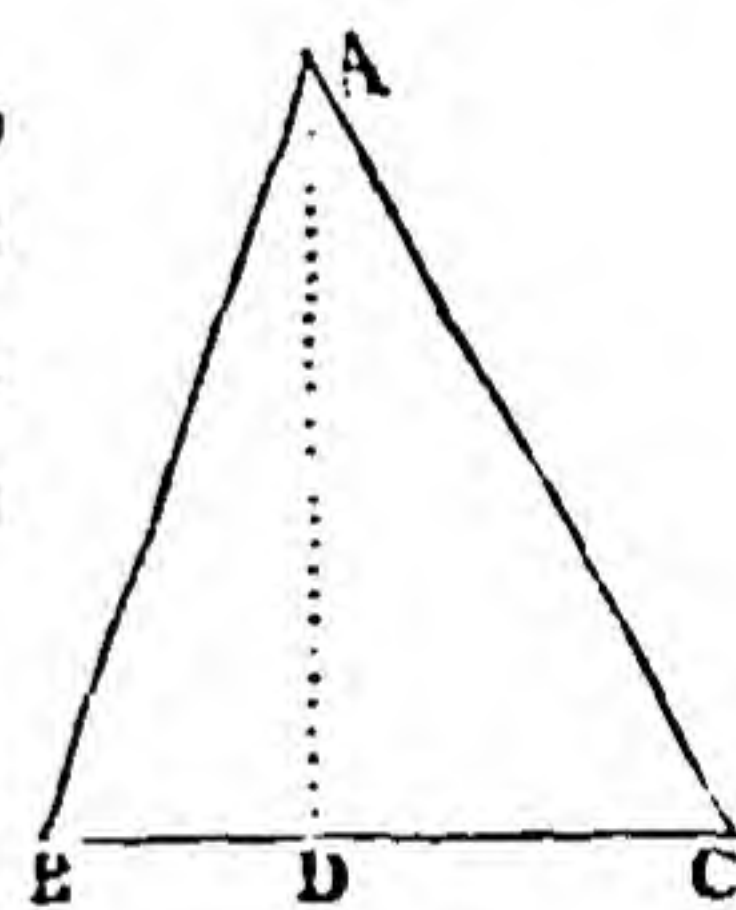
Therefore the squares of CB, BD, DA , are equal to the Rectangle twice contained by CB, BD , and to the squares of DA, DC .

But to the squares of BD, DA , is equal the square of AB (for the angle at D is a right angle), and to the squares of DA, DC , is equal the square of AC .

Therefore the squares of CB, BA , are equal to the square of AC , and to the Rectangle twice contained by CB, BD .

So that the single square of AC is less than the squares of CB, BA , by the Rectangle twice contained by CB, BD .

Therefore in Acute angl'd Triangles the square of the side subtending the Acute angle, is less than the squares of the sides containing the Acute angle, by the Rectangle twice contained by one of the sides about the Acute angle, on which a perpendicular falls; and by the line intercepted within, between the perpendicular and the Acute angle. Which was to be demonstrated.



PROPO.

PROPOSITION XIV.

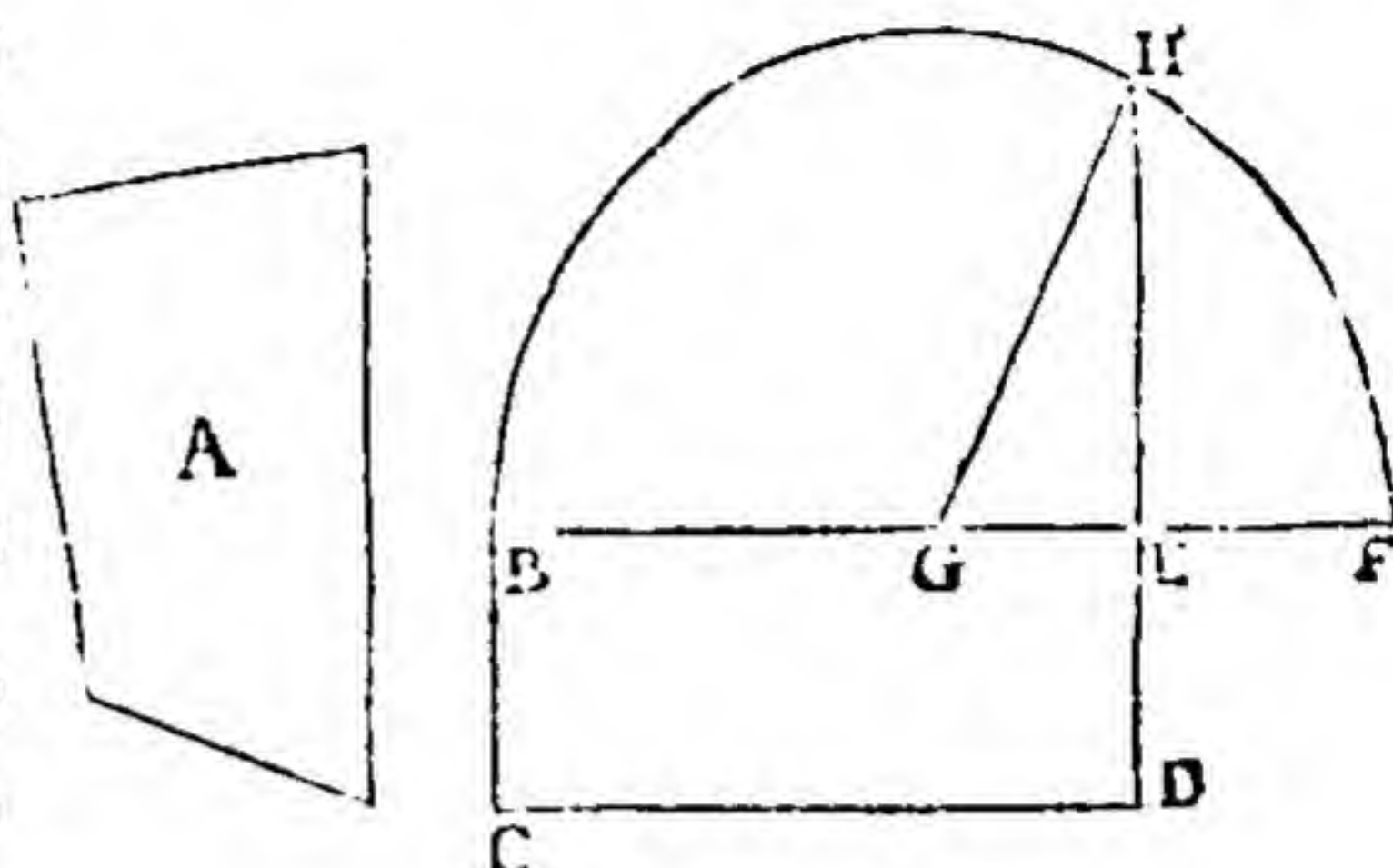
TO constitute a square equal to a right-lin'd Figure given.

Let the right-lin'd Figure be A . It is required to constitute a square equal to A .

By Prop. 45. El. I. let the Rectangled Parallelogram BD be constituted equal to the right-lin'd Figure A .

If therefore the line BE be equal to the line ED , then what was required is now done, for there is constituted the square BD , equal to the right-lin'd Figure A .

But if not, then one of the lines BE, ED , is the greater. Let BE be the greater; and let it be produced to F ; and let EF be put equal to ED . Then let BF be cut into halves at the point G ; and to the center G , and the distance one of the lines GB, GF , let there be described the Semi-



circle BHF ; and let DE be produced to H . Then let GH be joyn'd.

Now forasmuch as BF is cut into two equal Segments at G , and into two unequal at E ; therefore the Rectangle contained by BE, EF , together with the square of GE , is equal to the square of GF [Prop 5. El. II.]. But GF is equal to GH , therefore the Rectangle contained by BE, EF , together with the square of GE , is equal to the square of GH . But to the square of GH , are equal the squares of GE, EH , therefore the Rectangle contained by BE, EF , together with the square of GE , is equal to the squares of GE, EH . Let the common square of GE be taken away. Therefore the remaining Rectangle contained by BE, EF , is equal to the square of EH . But the Rectangle contained by BE, EF , is the Parallelogram BD , because EF is equal to ED .

Therefore the Parallelogram BD is equal to the square of EH . But the Parallelogram BD is equal to the right lin'd Figure A ; wherefore the right-lin'd Figure A , is equal to the square of EH .

Therefore there is constituted a square described on EH equal to A , the right-lin'd Figure given. Which was to be done.

THE THIRD ELEMENT.

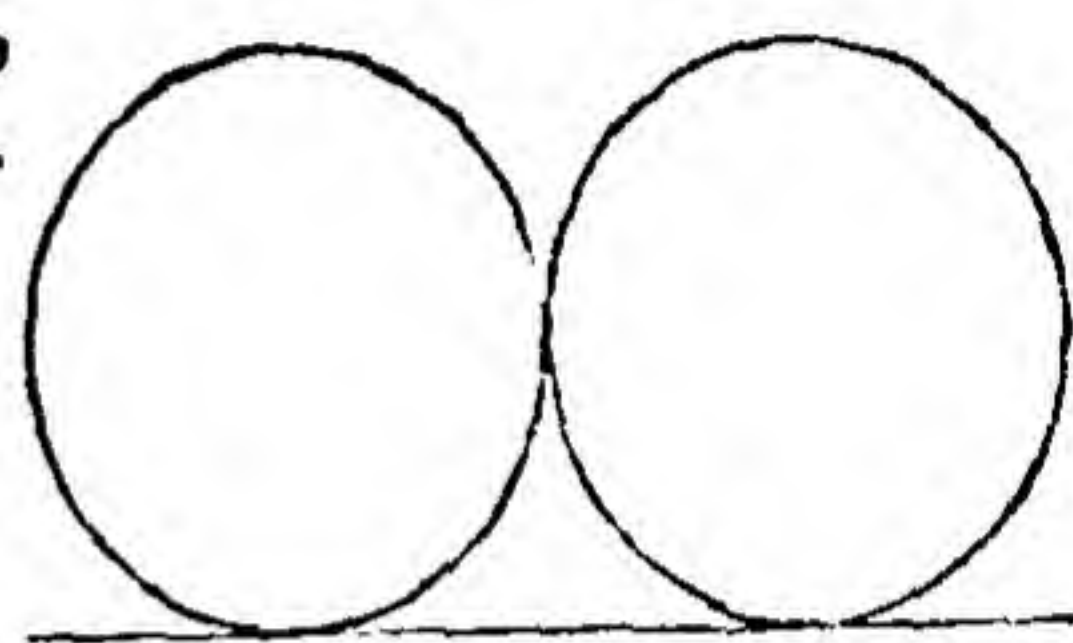
DEFINITIONS.

DEFINITION I.

Equal Circles are such, whose Diameters are equal; or, whose lines from the Center are equal.

DEFINITION II.

A Strait line is said to touch a Circle, which meeting a Circle, and being produced, cuts not the Circle.



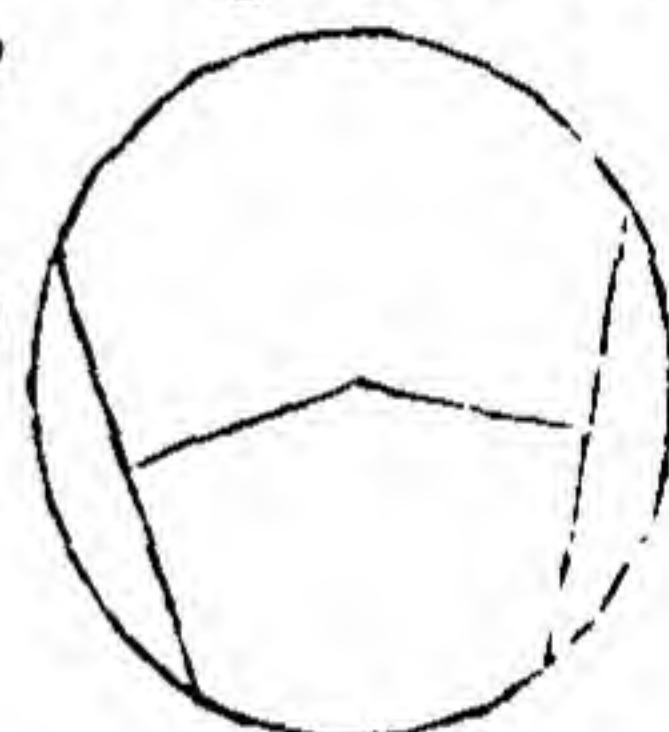
DEFINITION III.

Circles are said to touch one another, which meeting together, do not cut one another.

DEFINITION IV.

In a Circle strait lines are said to be equally distant from the Center, when the perpendiculars drawn from the Center to the same lines are equal.

And that line is said to be more distant; on which the greater perpendicular falls.



DEFINITION V.

A Segment of a Circle is a Figure comprehended by a strait line, and the circumference of the Circle. The strait line is called the base of the Segment.



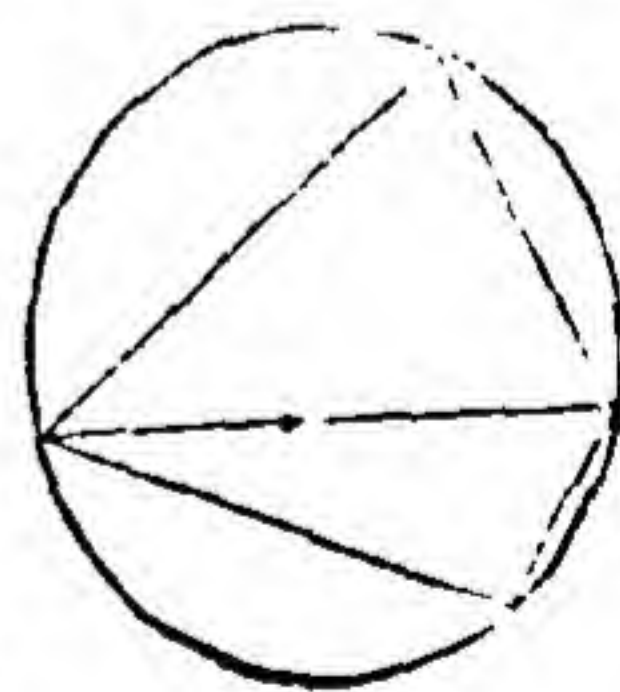
DEFINITION VI.

An angle of a Segment is that which is contained by a strait line, and the circumference of the Circle.



DEFINITION VII.

An angle in a Segment is when in the circumference of a Segment shall be taken a point, and from that point to the ends of the line, which is the base of the Segment, shall be drawn strait lines: it is the angle contained by those same lines.



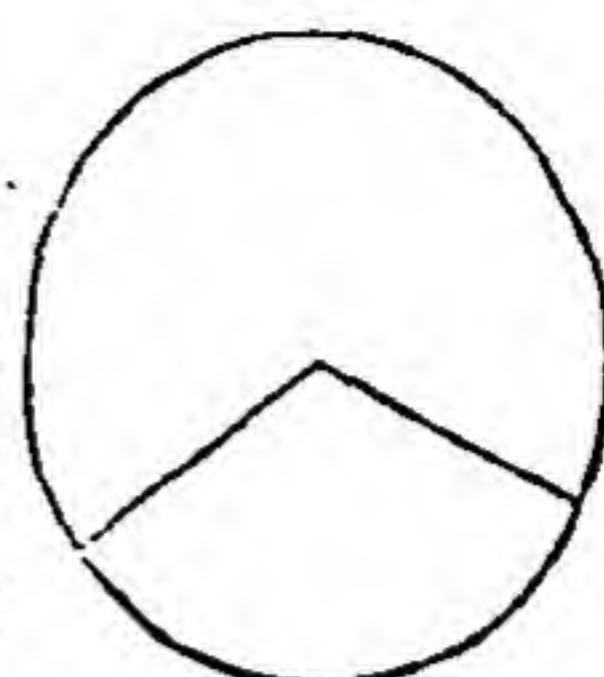
DEFI-

DEFINITION VIII.

AND when the strait lines containining the angle do assume a circumference, the angle is said to insist upon that circumference.

DEFINITION IX.

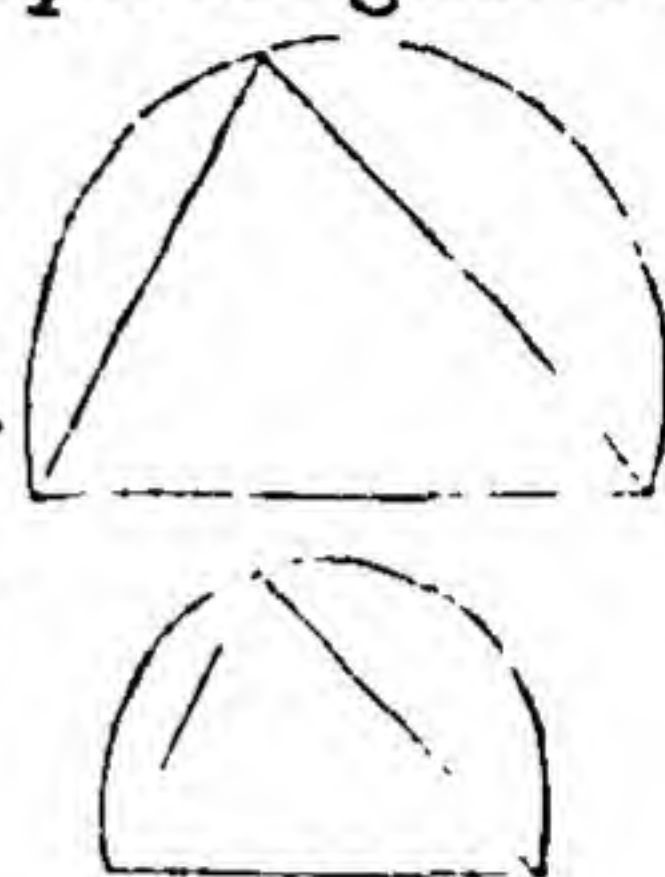
A Sector of a Circle is a Figure, which, when an angle shall be stated at the center of the Circle; is comprehended by the strait lines containing that angle, and by the circumference assumed under the same lines.



DEFINITION X.

LIke Segments of Circles are such, which receive equal angles. Or in which the angles are equal to one another.

That is, if the Segments of two Circles are supposed like, or so proved, then we are to allow the angles in those Segments to be equal. Or contrarily, if the angles be supposed, or proved equal, then we are to allow the Segments to be like to one another.

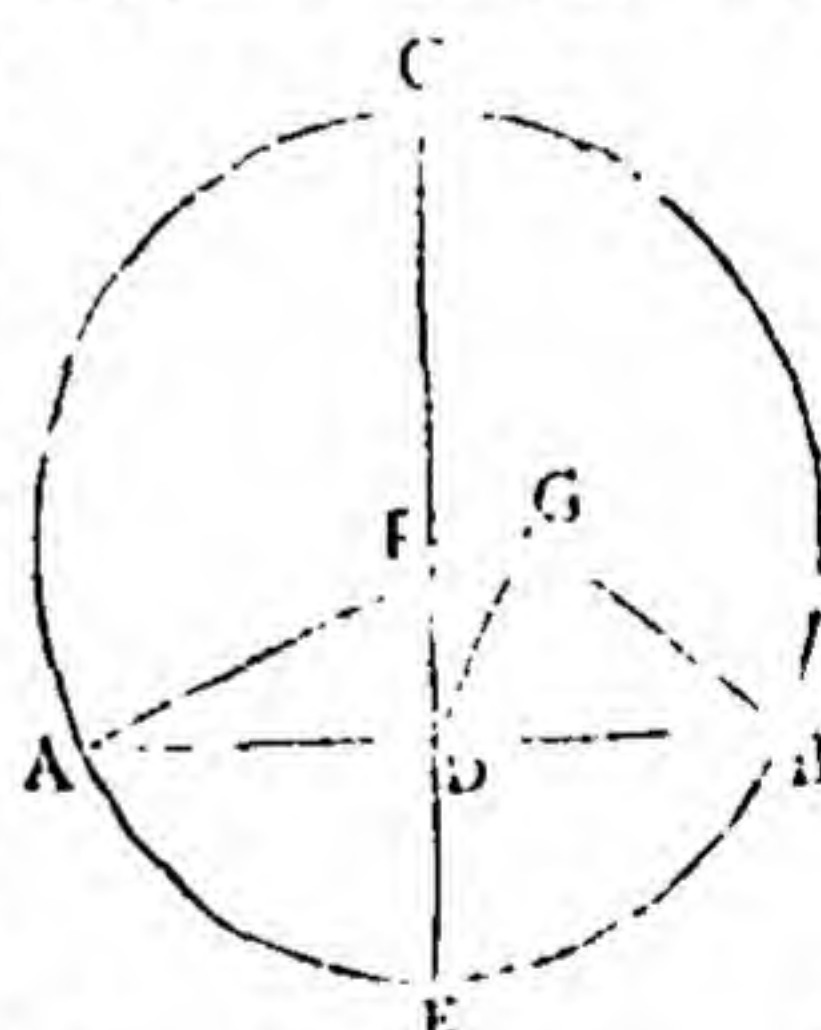


PROPOSITION I.

TO find the center of a given Circle.

Let the given Circle be ABC . It is required of the Circle ABC , to find the center.

Let in the Circle be drawn at adventure any strait line as AB ; and let it be cut into halves in the point D ; and from D draw DC at right angles to AB , and produce it to E . Then let CE be cut into halves in F . I say, the point F is the center of the Circle ABC . For if not: let it, if possible, be the point G ; and let GA, GD, GB , be joyn'd. Now forasmuch as AD is equal to DB and DG is common; therefore there are the two lines AD, DG , equal to the two lines DB, DG , each to each, and the base GA is equal to the base GB , for they are from the center G ; wherefore the angle ADG is equal to the angle GDB [Prop. 8. El. I.]. But when a strait line standing upon a strait line, makes the angles on each side equal to one another, each of the equal angles is a right angle; therefore GDB is a right angle. But also FDB is a right angle [by Construction]; therefore FDB is equal to GDB , the greater to the less, which is impossible. Wherefore G is not the center of the Circle ABC . And in like manner may we prove that no other point can be besides F .



Therefore the point F is the center of the Circle ABC . Which was to be done.

Corol-

Corollary.

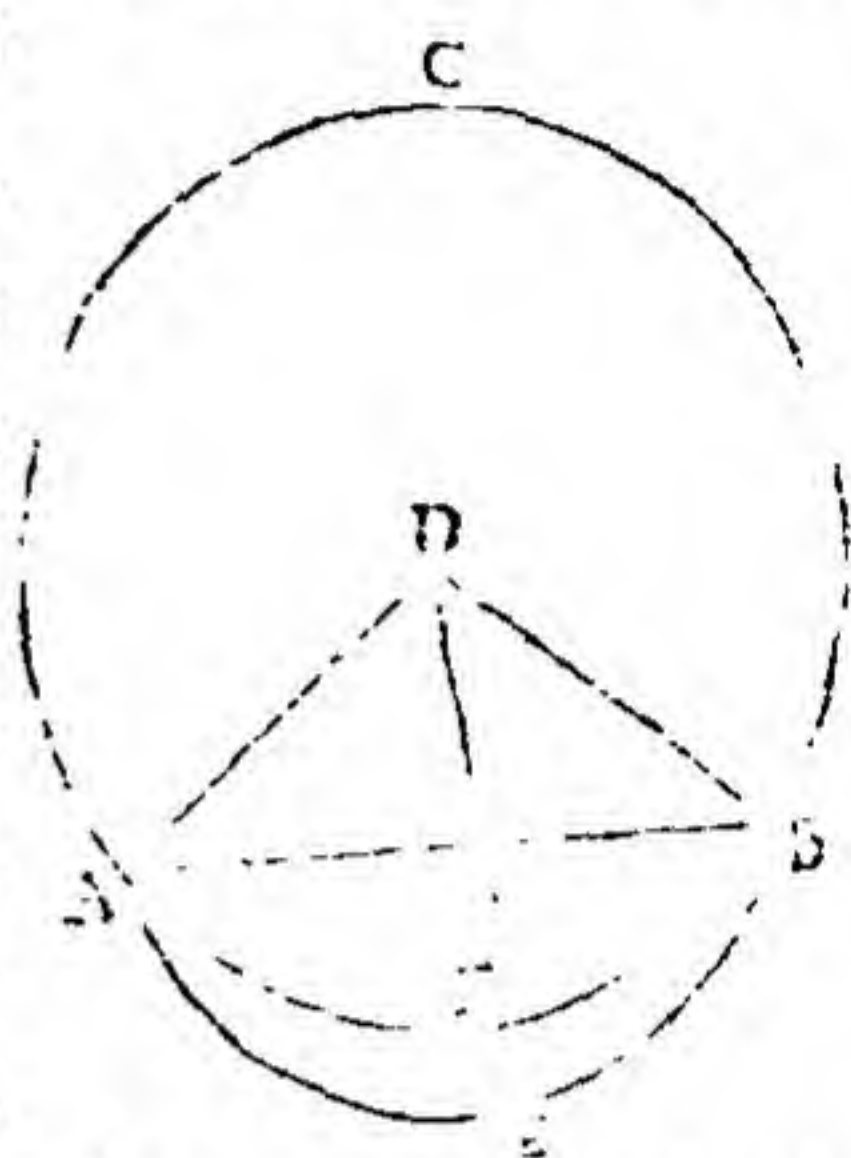
From hence it is manifest, that if in a Circle a strait line cuts a strait line into halves, and at right angles; the center of the Circle is in the cutting line.

PROPOSITION II.

I *F in the circumference of a Circle any two points be taken, the strait line joyning the same points shall fall within the Circle.*

Let the Circle be ABC , and in the circumference thereof let be taken any two points A, B . I say, that the strait line drawn from A to B falls within the Circle. For if not: let it, if possible, fall without, as AEB , and let the center of the Circle ABC be taken, and let it be D . Then let AD, DB , be joyn'd, and let DF be produced to E .

Now forasmuch as DA is equal to DB , therefore the angle DAE is equal to the angle DBE [Prop. 5. El. I.]. And because of the Triangle DAE , one side AEB is produced; therefore the angle DEB is greater than the angle DAE [Prop. 16. El. I.]. But the angle DAE is equal to the angle DBE [Prop. 5. El. I.]; therefore DEB is greater than DBE . But under the greater angle is subtended the greater side [Prop. 19. El. I.]; therefore BD is greater than DE . But DB is equal to DF , wherefore DF is greater than DE , the less than the greater, which is impossible; therefore the strait line drawn from A to B , shall not fall without the Circle. In like manner we may shew that it shall not fall upon the circumference: therefore it shall fall within.



If therefore in the circumference of a Circle any two points be taken, the strait line joyning the same points shall fall within the Circle. Which was to be demonstrated.

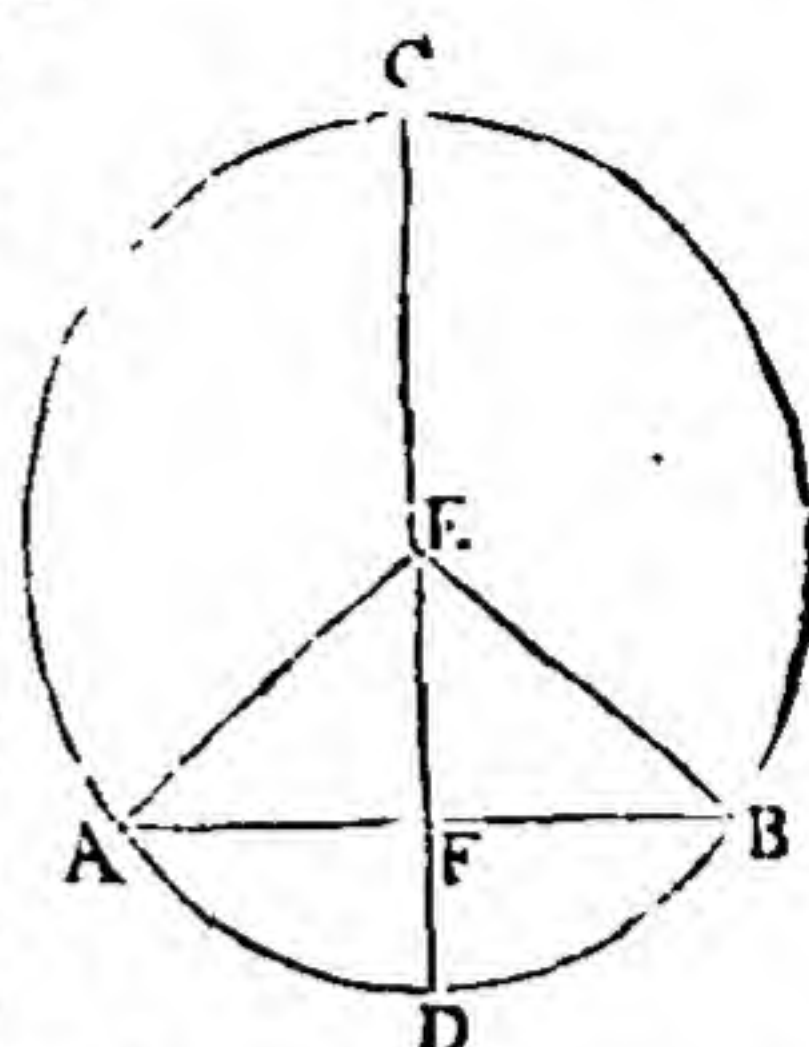
PROPOSITION III.

I *F in a Circle a strait line drawn through the center cuts a strait line not drawn through the center into halves; it shall also cut the same at right angles. And if it cuts it at right angles, it shall also cut the same into halves.*

Let the Circle be ABC , and in the same let the strait line CD , drawn through the center, cut the strait line AB not drawn through the center, into halves in the point F . I say, that it also cuts the same at right angles. Let the center of the Circle ABC be taken, and let it be E , and let EA, EB , be joyn'd. Now because AF is equal to FB [by Supposition], and FE is common; therefore two to two are equal, and the base EA is equal to the base EB ; wherefore the angle AFE is equal to the angle BFE [Prop. 8. El. I.]. But when a strait line standing upon a strait line, makes the angles on each side
equal

equal to one another, each of the equal angles is a right angle; wherefore each of the angles $\angle AFE$, $\angle BFE$, is a right angle. Therefore CD drawn through the Center cutting AB not drawn through the Center into halves, does also cut it at right angles.

But again, let the strait line CD cut the strait line AB at right angles. I say, that it also cuts the same into halves: which is, that AF is equal to FB . For the same construction being made, because in the Triangle EAB , EA , a line from the Center is equal to EB ; therefore the angle EAF is equal to the angle EBF [Prop. 5. El. I.]; but the right angle $\angle AFE$ is equal to the right angle



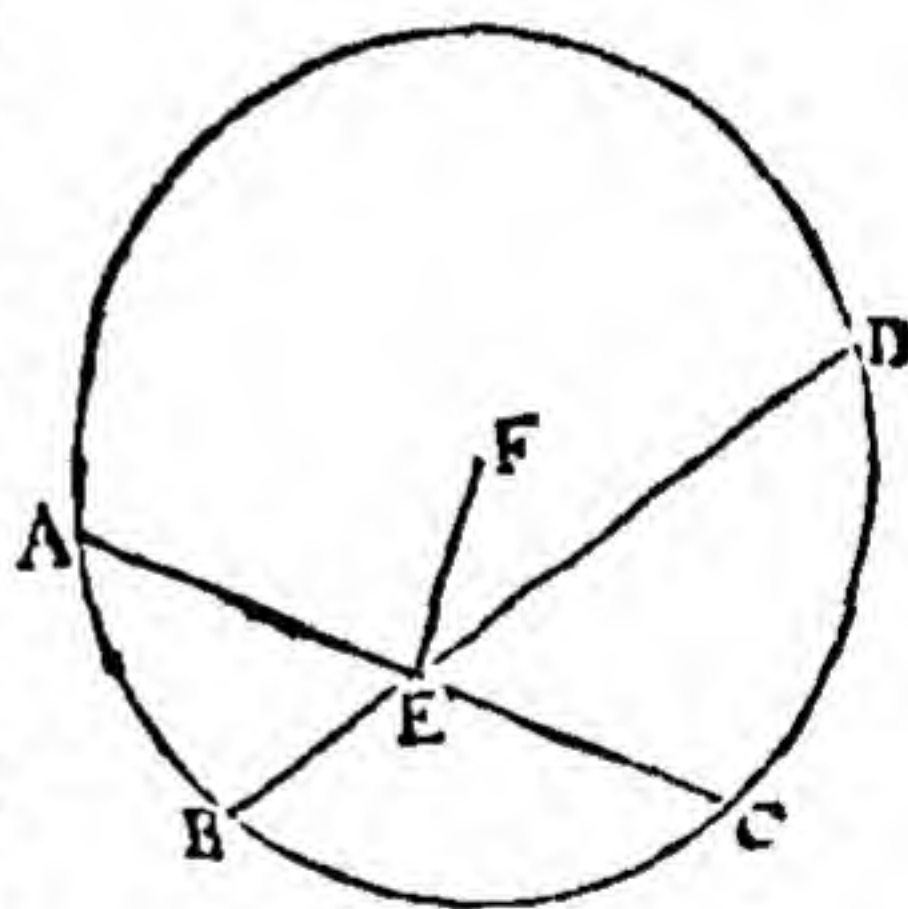
$\angle BFE$: therefore there are the Triangles EFA , EFB , having two angles equal to two angles, and one side equal to one side, namely EF common to both, which subtends one of the equal angles: Therefore they shall have the remaining sides equal to the remaining sides [Prop. 26. El. I.]; wherefore AF is equal to BF .

If therefore in a circle a strait line drawn through the Center cuts a strait line not drawn through the Center into halves; it shall also cut the same at right angles. And if it cuts it at right angles, it shall also cut it into halves. Which was to be demonstrated.

PROPOSITION IV.

In a circle two strait lines not drawn through the center, cut one another, they shall not cut one another into halves.

Let the circle be $ABCD$, and in the same let the strait lines AC , BD , not drawn through the Center, cut one another in the point E . I say, they do not cut one another into halves; so that AE be equal to EC , and BE to ED . Let the Center of the circle $ABCD$ be taken, and be it F , and let EF be joyn'd. Now forasmuch as the strait line FE drawn through the Center, cuts the strait line AC not drawn through the Center into halves, it shall also cut the same at right angles [Prop. 3. El. III.]; therefore $\angle FEA$ is a right angle. Again, because FE cuts the strait line BD not drawn through the Center into halves; it shall also cut the same at right angles; therefore $\angle FEB$ is a right angle. But it has been proved, that $\angle FEA$ is a right angle; therefore $\angle FEA$ is equal to $\angle FEB$; the less to the greater, which is impossible; wherefore AC , BD , do not cut one another into halves.

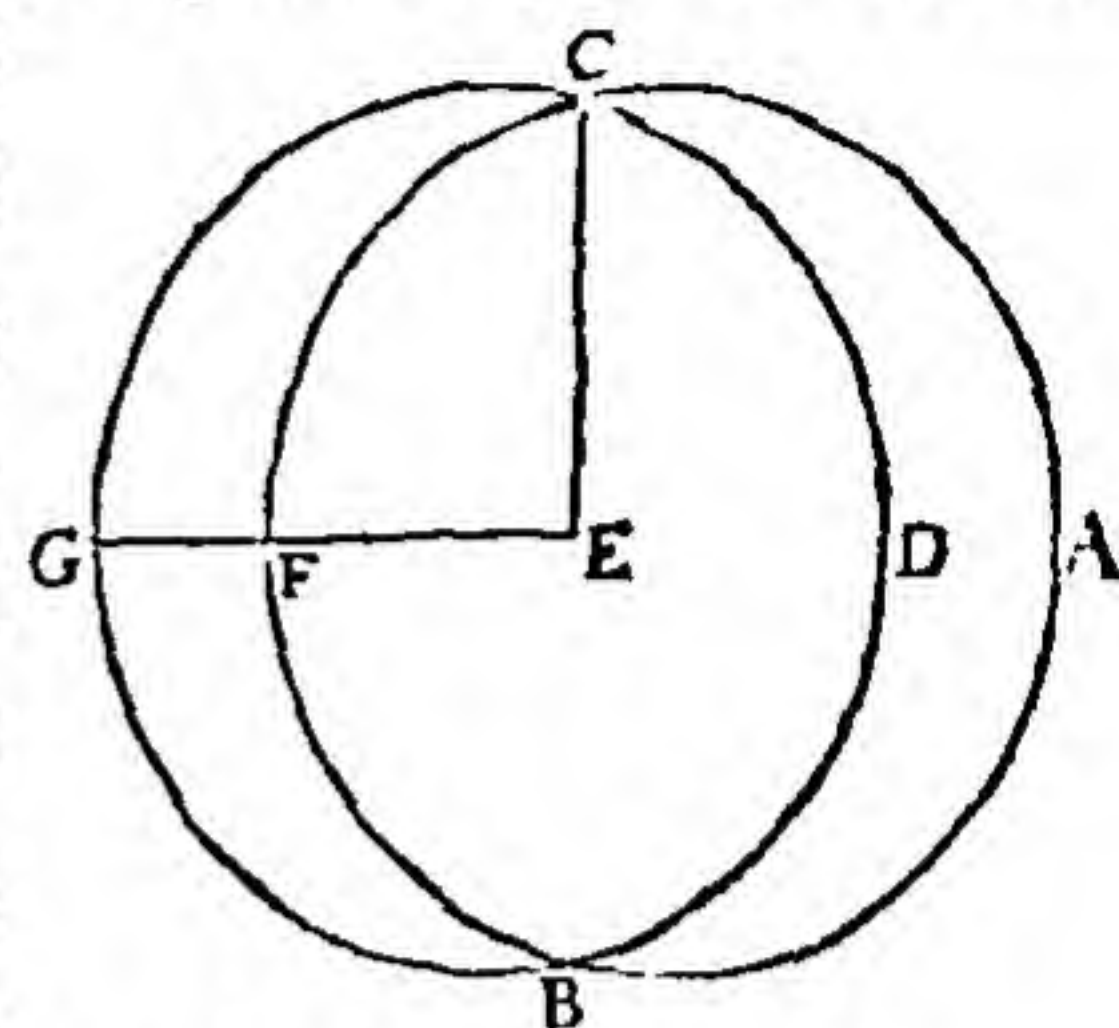


If therefore in a circle two strait lines not drawn through the Center, cut one another, they shall not cut one another into halves. Which was to be demonstrated.

PROPOSITION V.

I *F two circles cut one another, they shall not have the same Center.*

Let the two circles ABC , CDG , cut one another in the points B , C . I say, they have not the same Center. For, if possible, let E be the Center, and let EC be joyn'd; also let EF be drawn at adventure. Now because E is the Center of the circle ABC , therefore EC is equal to EF . Again, because E is the Center of the circle CDG , therefore EC is equal to EG . But it has been proved, that EC is equal to EF ; therefore EF is equal to EG ; the less to the greater, which is impossible.



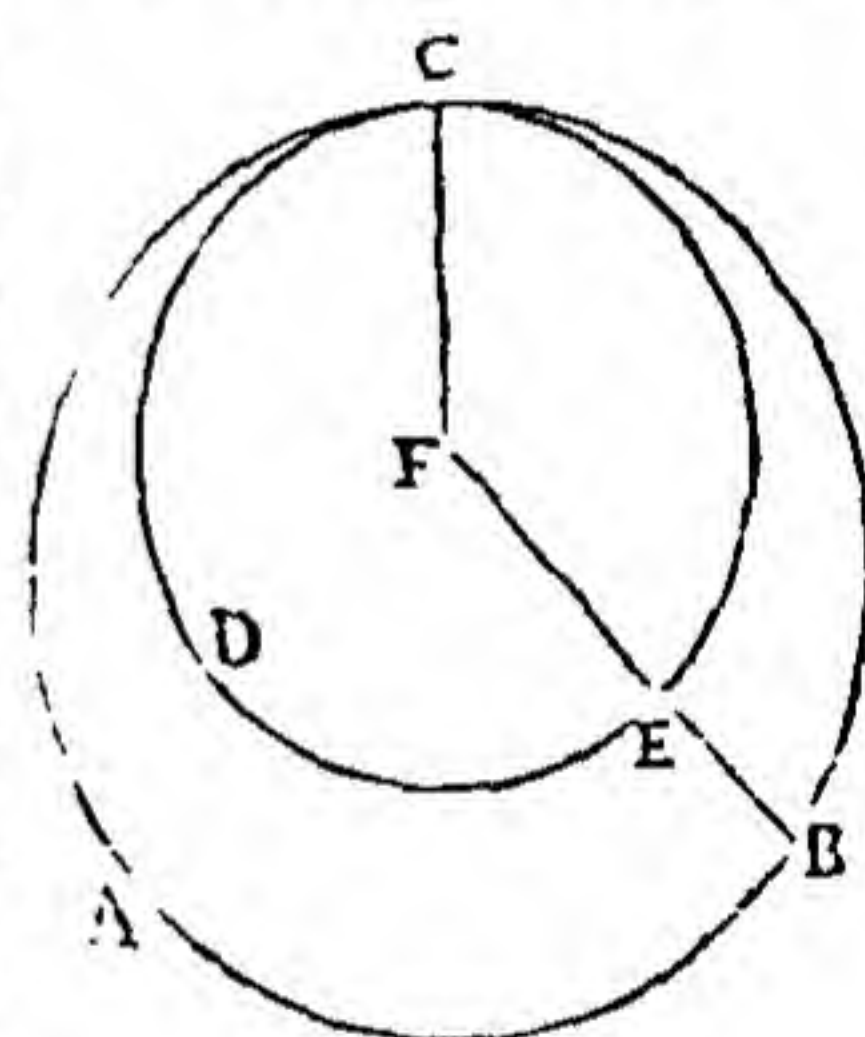
Therefore the point E is not the Center of the circles ABC , CDG .

Wherefore if two circles cut one another, they shall not have the same Center. Which was to be demonstrated.

PROPOSITION VI.

I *F two circles touch one another within, they shall not have the same Center.*

Let the two circles ABC , CDE , touch one another within in the point C . I say, they have not the same Center. For if possible, let the Center be F , and let FC be joyned; also let FE be drawn at adventure. Now forasmuch as F is the Center of the circle ABC , therefore FC is equal to FB . Again, because F is the Center of the circle CDE , therefore FC is equal to FE . But it has been proved that FC is equal to FB ; therefore FE is equal to FB ; the less to the greater, which is impossible. Therefore the point F is not the Center of the circles ABC , CDE .



Wherefore if two circles touch one another within, they shall not have the same Center. Which was to be demonstrated.

PROPOSITION VII.

I *F in the Diameter of a circle be taken any point, which is not the Center of the circle; and from that point do fall upon the circle any strait lines: the greatest shall be that, in which the Center is; and the remaining part shall be the least. Of the others the nearer to the line through the Center, is always greater than the more remote.*

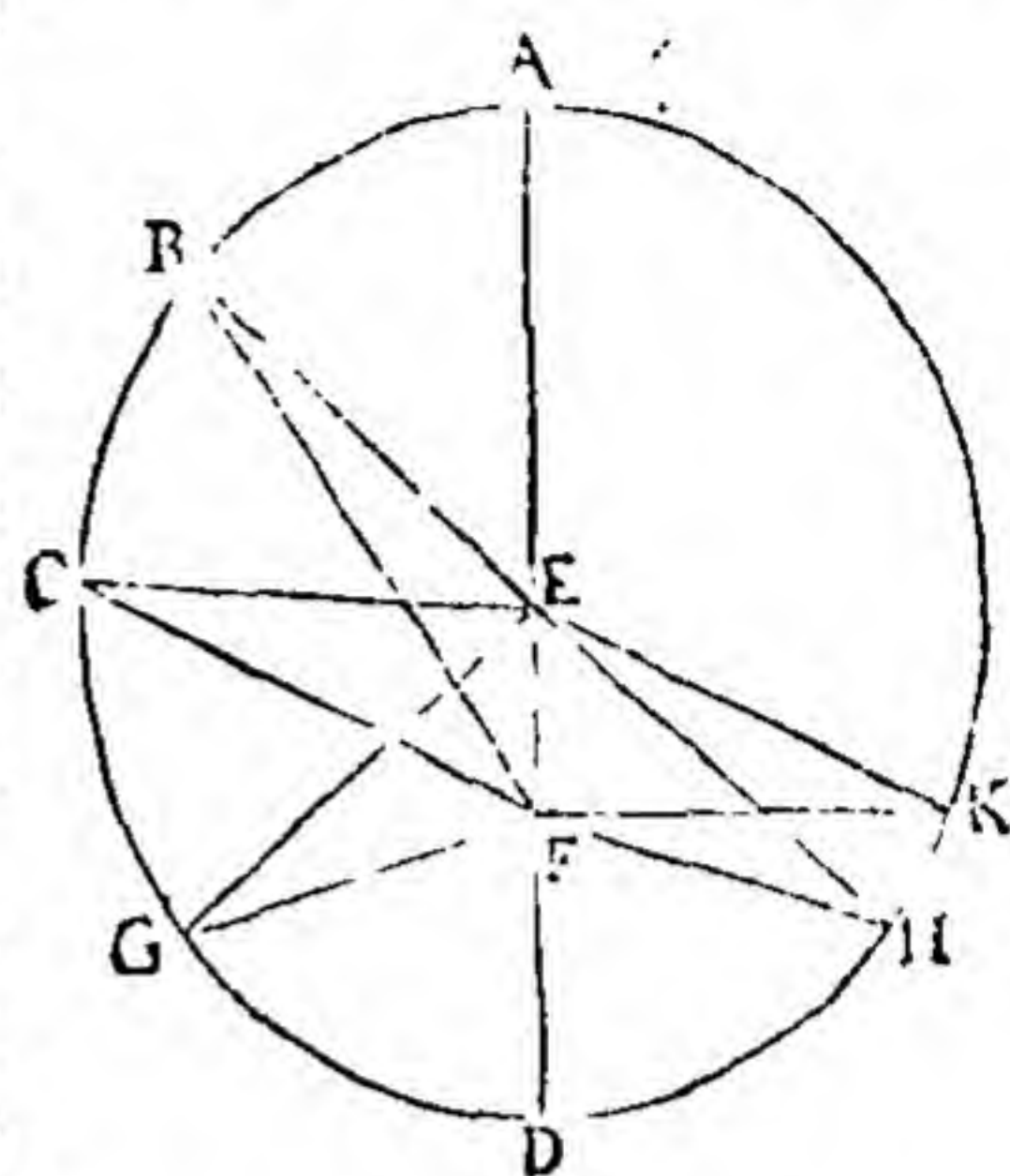
And two only equal lines can from the same point fall upon the circle, on each side of the least line.

Let the circle be $ABCD$, and the Diameter thereof be AD , and in
 AD ,

AD, let be taken any point as F, which is not the center of the Circle. But let the center of the Circle be E; and from F let the strait lines FB, FC, FG, fall upon the circle ABCD.

I say, that the greatest is FA, and the least FD: Of the others FB is greater than FC, and FC than FG. For let BE, CE, GE be joyn'd.

Now because two sides of every Triangle are greater than the remaining side [Prop. 20. El. I.]; therefore BE, EF, are greater than FB; but AE is equal to BE, therefore BE, EF, are equal to AF; therefore AF is greater than BF. Again, because BE is equal to CE, and FE common, therefore there are the two lines BE, EF, equal to the two lines CE, EF. But the angle BEF is greater than the angle CEF, therefore the base BF, is greater than the base CF [Prop. 4. El. I.]. And by the same reason CF is greater than FG. Again, because GF, FE, are greater than EG, and EG is equal to ED; therefore GF, FE, are greater than ED. Let EF common be taken away, therefore the remaining line GF, is greater than the remaining line FD; Wherefore FA is the greatest line, and FD the least: And FB is greater than FC, and FC than FG.



I say also, that from the point F, two only equal lines can fall upon the Circle ABCD, on each side of the least line FD.

For to the line EF, and to the point E in the same, let be constituted the angle FEH, equal to the angle GEF [Prop. 23. El. I.], and let EH be drawn. Forasmuch then as GE is equal to EH, and EF common; therefore there are the two lines GE, EF, equal to the two lines HE, EF, and the angle GEF is equal to the angle HEF: therefore the base FG is equal to the base FH [Prop. 4. El. I.].

I say now, that from the point F there cannot fall upon the Circle any other line equal to FG.

For if possible, let FK so fall, and because FK is equal to FG, FH is equal to FG; therefore FK is equal to FH: The nearer to the line through the Center, equal to the more remote: which is impossible.

Or also thus. Let EK be joyn'd, and because GE is equal to EK, FE common, and the base GF equal to the base FK; therefore the angle GEF is equal to the angle KEF. But the angle GEF is equal to the angle HEF; therefore the angle HEF is equal to the angle KEF: The less to the greater, which is impossible. Wherefore from the point F any other line cannot fall upon the Circle equal to FG; therefore one only.

If therefore in the Diameter of a Circle be taken any point, &c. Which was to be demonstrated.

PROPOSITION VIII.

If without a Circle be taken any point, and from that point be drawn to the Circle any strait lines, of which one is through the Center, and the rest at adventure. Of those that fall upon the Concave circumference, the greatest is that through the Center. Of the others the nearer to the line through the Center, shall be always greater than the more remote.

But of those lines which fall upon the Convex circumference, the least is that between the point and the Diameter. Of the others the nearer to the least line, is always less than the more remote.

And two only equal lines can from that point fall upon the Circle on each side of the least line.

Let the Circle be ABC , and without the Circle ABC let be taken any point as D , and from the same let be drawn to the Circle the strait lines DA, DE, DF, DC , and let DA be through the Center.

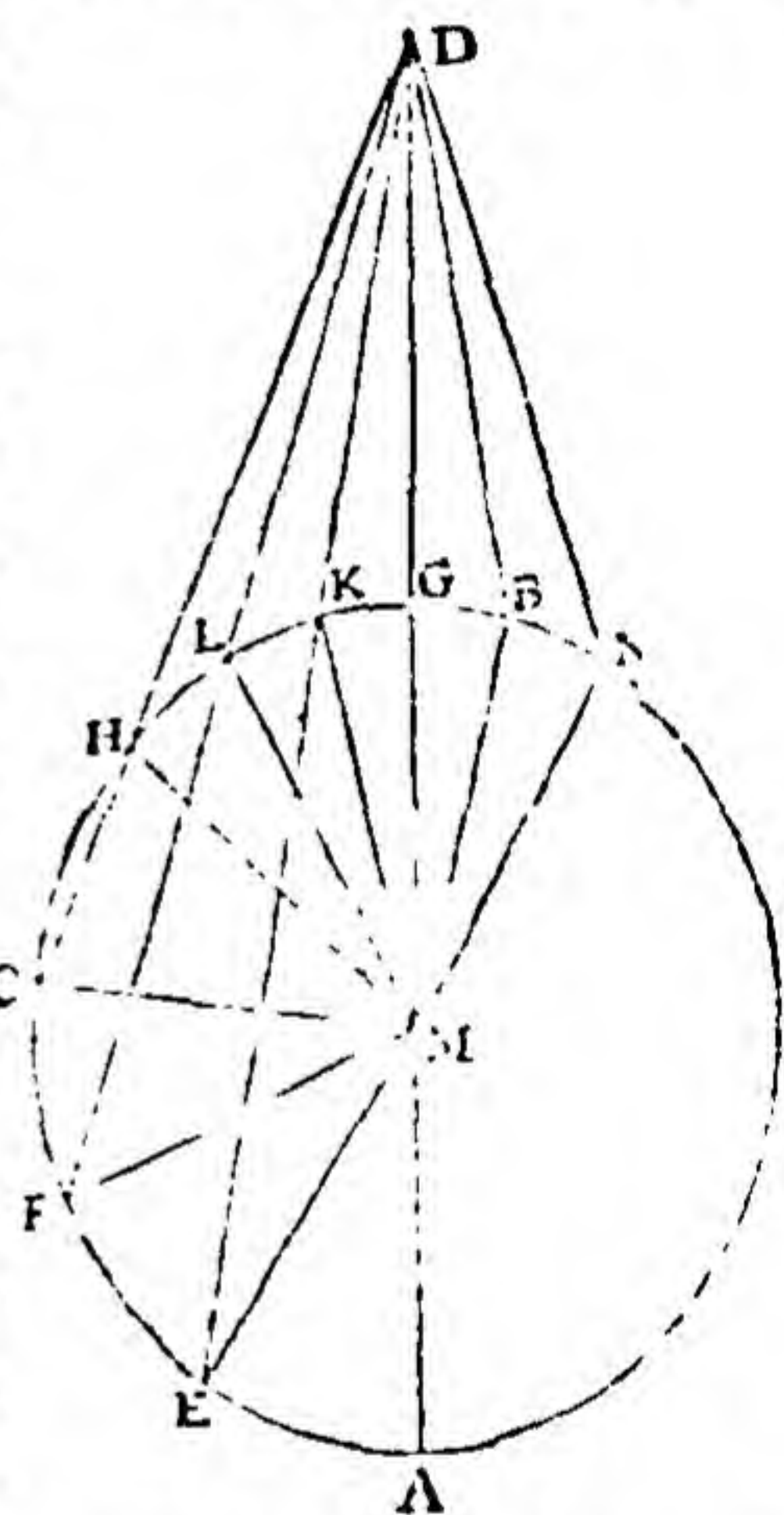
I say, that of those which fall upon AEC , the Concave circumference, the greatest is DA , the line through the Center. And always the nearer to the line through the Center shall be greater than the more remote; that is DE , than DF , and DF , than DC .

But of those lines which fall upon $HLKG$, the convex circumference the least is DG , between the point D and the Diameter AG . And the nearer to DG the least line, is always less than the more remote; that is, DK than DL , and DL than DH .

Let the Center of the Circle ABC be taken, and be it M , and let be joyned ME, MF, MC, MK, ML, MH . Now forasmuch as AM is equal to ME , let MD common be added; therefore AD is equal to EM, MD . But EM, MD , are greater than ED [Prop. 20. El. I.]; therefore AD is also greater than ED . Again, because ME is equal to MF , and MD common, therefore EM, MD , are equal to FM, MD , and the angle EMD , is greater than the angle FMD ; therefore the base ED is greater than the base FD [Prop. 24. El. I.]. And in like manner may we prove, that FD is greater than CD . Therefore DA is the greatest, and DE is greater than DF , and DF than DC .

And again, because MK, KD , are greater than MD , and MG is equal to MK ; therefore the remaining line KD , is greater than the remaining line GD , so that GD is less than KD , therefore GD is the least. And because of the Triangle MLD , upon one of the sides MD are constituted within the same Triangle two strait lines MK, KD ; therefore MK, KD , are less than ML, LD [Prop. 21. El. I.]; of which MK

is



is equal to ML ; therefore the remaining line DK , is less than the remaining line DL . In like manner we may show that DL is less than DH ; therefore DG is the least: and DK is less than DL , and DL than DH .

I say also that two only equal lines can from the point D fall upon the Circle on each side of the least line DG .

For to the line MD , and in it to the point M , let be constituted the angle DMB , equal to the angle KMD , and let DB be joyned. Now because MK is equal to MB , and MD common; therefore there are the two lines KM, MD , equal to the two lines BM, MD , each to each, and the angle KMD is equal to the angle BMD ; therefore the base DK is equal to the base DB [Prop. 4. El. I.].

I say now, that from the point D there cannot fall upon the Circle any other line equal to DK .

For if possible let a line so fall, and be it DN . Now forasmuch as DK is equal to DN , and DK is equal to DB ; therefore DB is equal to DN : The nearer to DG the least, equal to the more remote, which has been proved to be impossible.

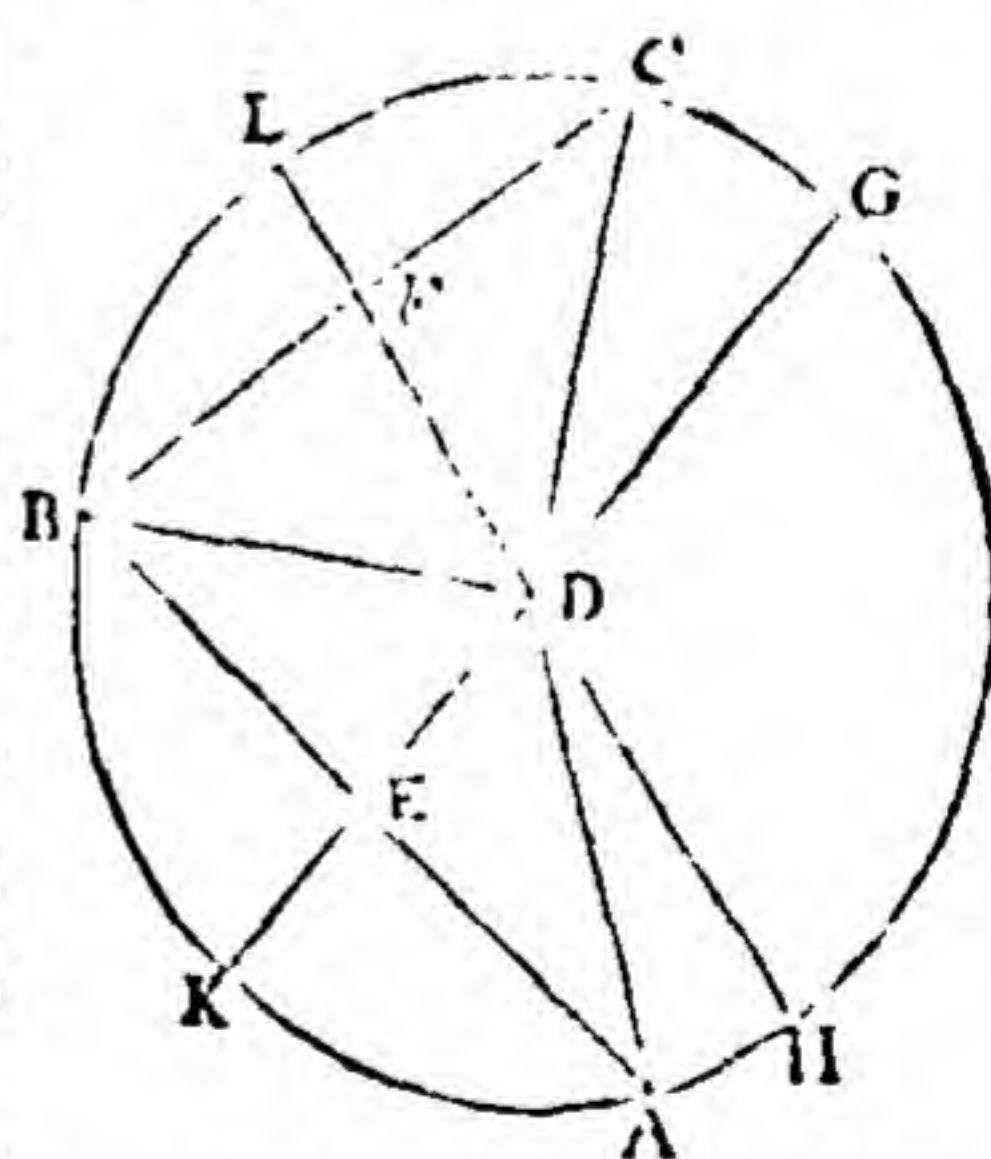
Or otherwise, let MN be joyned: and because KM is equal to MN , and MD common, and the base DK is equal to the base DN , therefore the angle KMD is equal to the angle DMN [Prop. 8. El. I.]; but the angle KMD is equal to the angle BMD ; therefore the angle BMD is equal to the angle NMD : the less to the greater, which is impossible. Wherefore not more than two equal strait lines can from the point D fall upon the Circle ABC , on each side of the least line DG .

If therefore without a Circle be taken any point, &c. Which was to be demonstrated.

PROPOSITION IX.

If within a Circle be taken any point, and from that point do fall upon the Circle more than two equal strait lines, that point taken is the Center of the Circle.

Let the Circle be ABC , and within the same the point be D : and from D let upon the Circle ABC fall more than two equal strait lines, as DA, DB, DC . I say, that the point D is the Center of the Circle ABC . For let AB, BC , be joyned, and be cut into halves at the points E, F . And DE, DF , being joyned, let them be produced to the points $G, K; H, L$. Now forasmuch as AE is equal to EB , and ED common; therefore there are the two lines AE, ED , equal to the two lines BE, ED , and the base DA is equal to the base DB (by Supposition); therefore the angle AED is equal to the angle BED : wherefore each of the angles AED, BED , is a right angle; therefore GK cutting AB into halves,

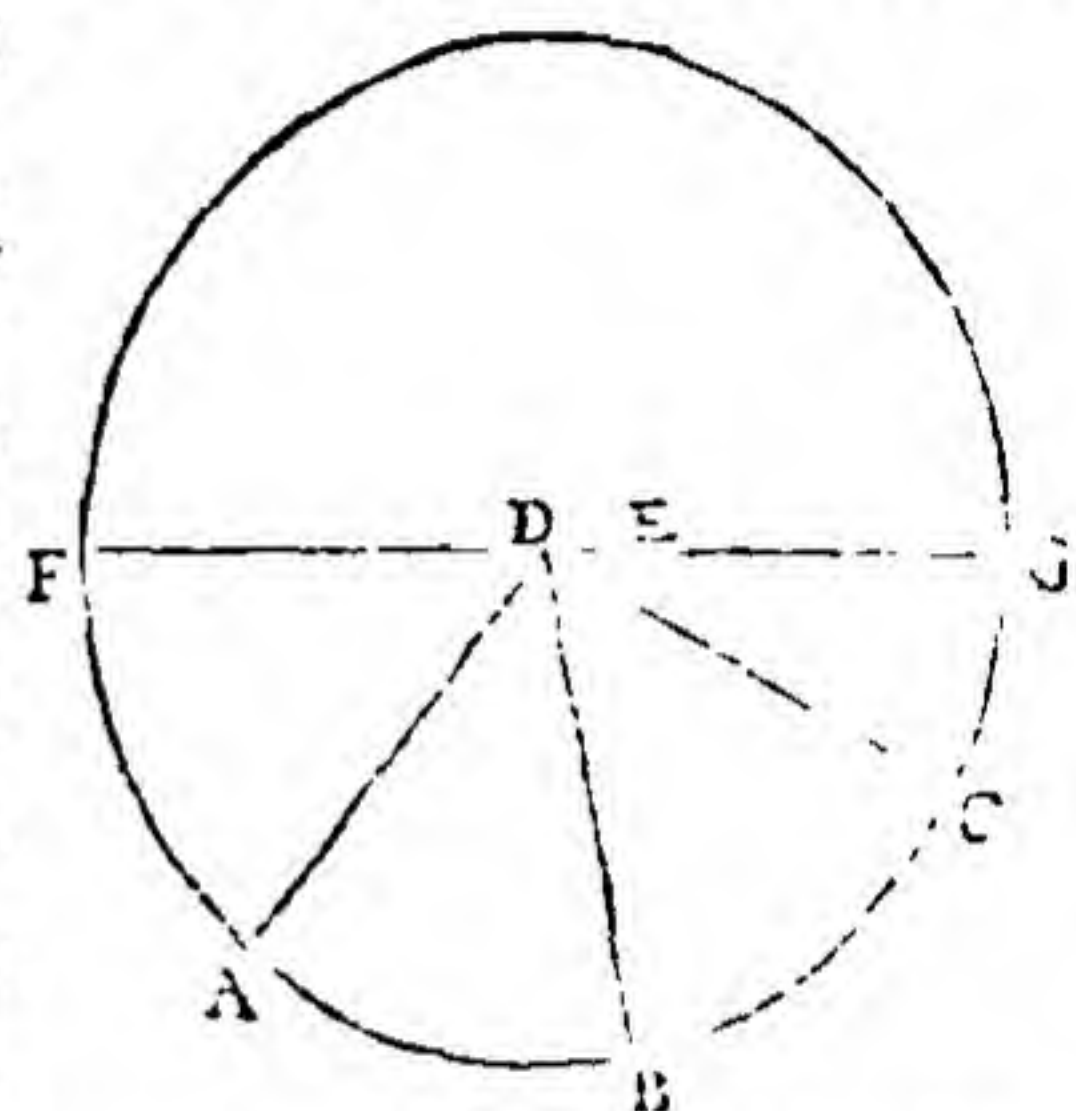


does also cut it at right angles. And because if in a Circle a strait line cuts a strait line into halves, and at right angles, the center of the Circle is in the cutting line; therefore in the line GK is the center of the Circle ABC [by Coroll. Prop. I. El. III.]. And by the same reason also the center of the Circle ABC , is in the line HL . But the lines GK, HL , have no other common point than D ; therefore the point D is the center of the Circle ABC .

If therefore in a Circle be taken any point, &c. Which was to be demonstrated.

Otherwise.

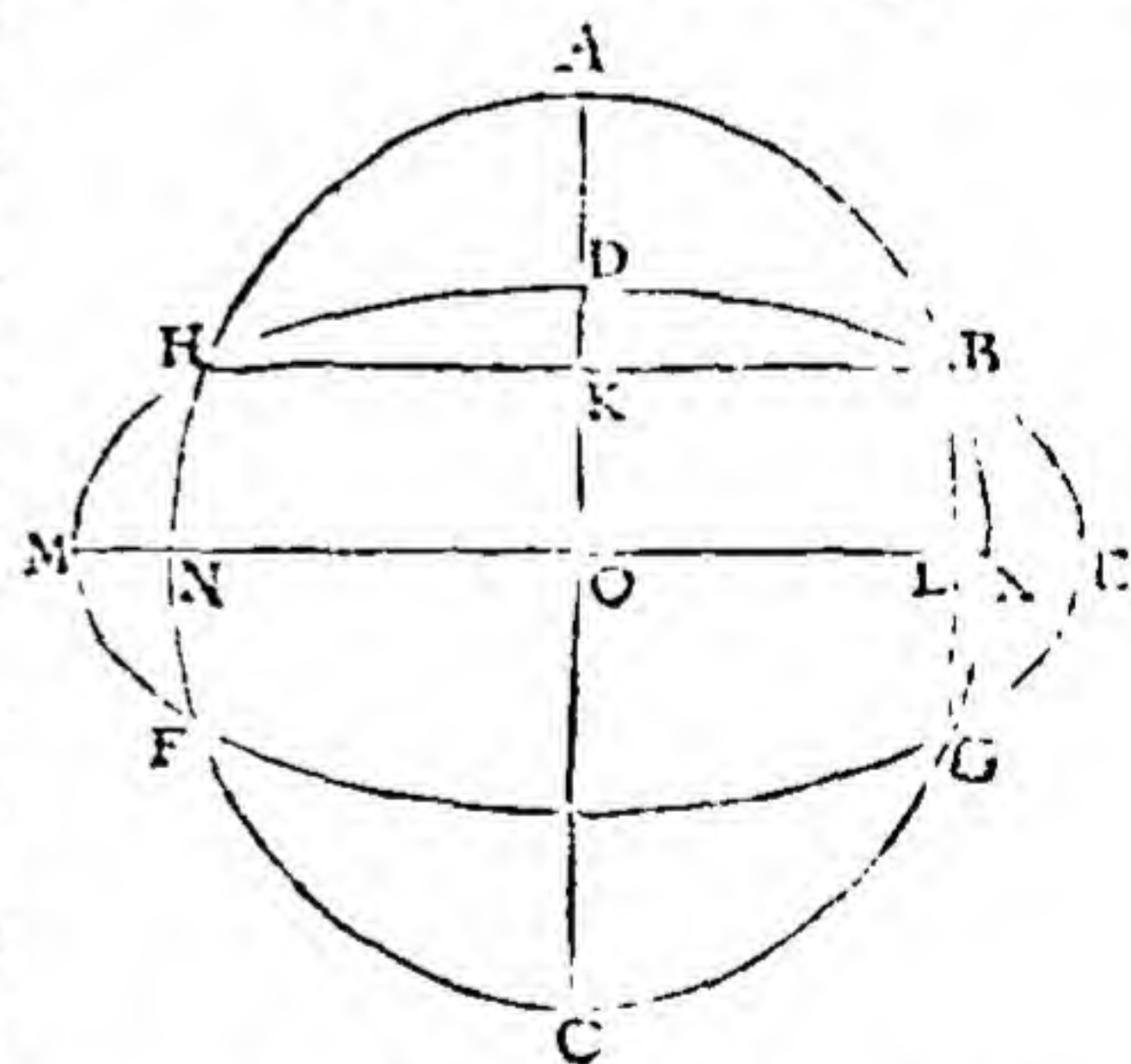
For within the Circle ABC let be taken any point as D : and from D let upon the Circle ABC , fall more than two equal strait lines, as DA, DB, DC . I say, that D the point taken is the center of the Circle ABC . For if not; let it, if possible, be E , and DE being joyned, let it be produced to the points F, G ; then the line FG is the Diameter of the Circle ABC . Now forasmuch as in FG the Diameter of the Circle ABC , is taken a point D , which is not the center of the Circle, the greatest shall be DG , and DC greater than DB , and DB than DA [Prop. 7. El. III.]. But they are also equal, which is impossible: Therefore E is not the center of the Circle ABC . In like manner may we prove, that no other point can be the center besides D : therefore D is the center of the Circle ABC .



PROPOSITION X.

A Circle does not cut a Circle in more points than two.

For if it be possible let the Circle ABC , cut the Circle DEF in more points than two, as, B, G, F, H , and BG, BH , being joyned, let them be cut into halves in the points K, L , and from K, L , to the lines BG, BH , let be drawn at right angles KC, LM , [Prop. 11. El. I.] and produce them to A, E . Now because in the Circle ABC , the strait line AC cuts the strait line BH into halves, and also at right angles; therefore in AC is the center of the circle ABC [Coroll. Prop. I. El. III.]. Again, because in the same Circle ABC , the strait line NX cuts the strait line BG into halves, and also at right angles; therefore in NX is the center of the Circle ABC . But it has been proved to be also in AC , and in no other point do the strait lines AC, NX , agree than in O : therefore the point O is the center of the Circle



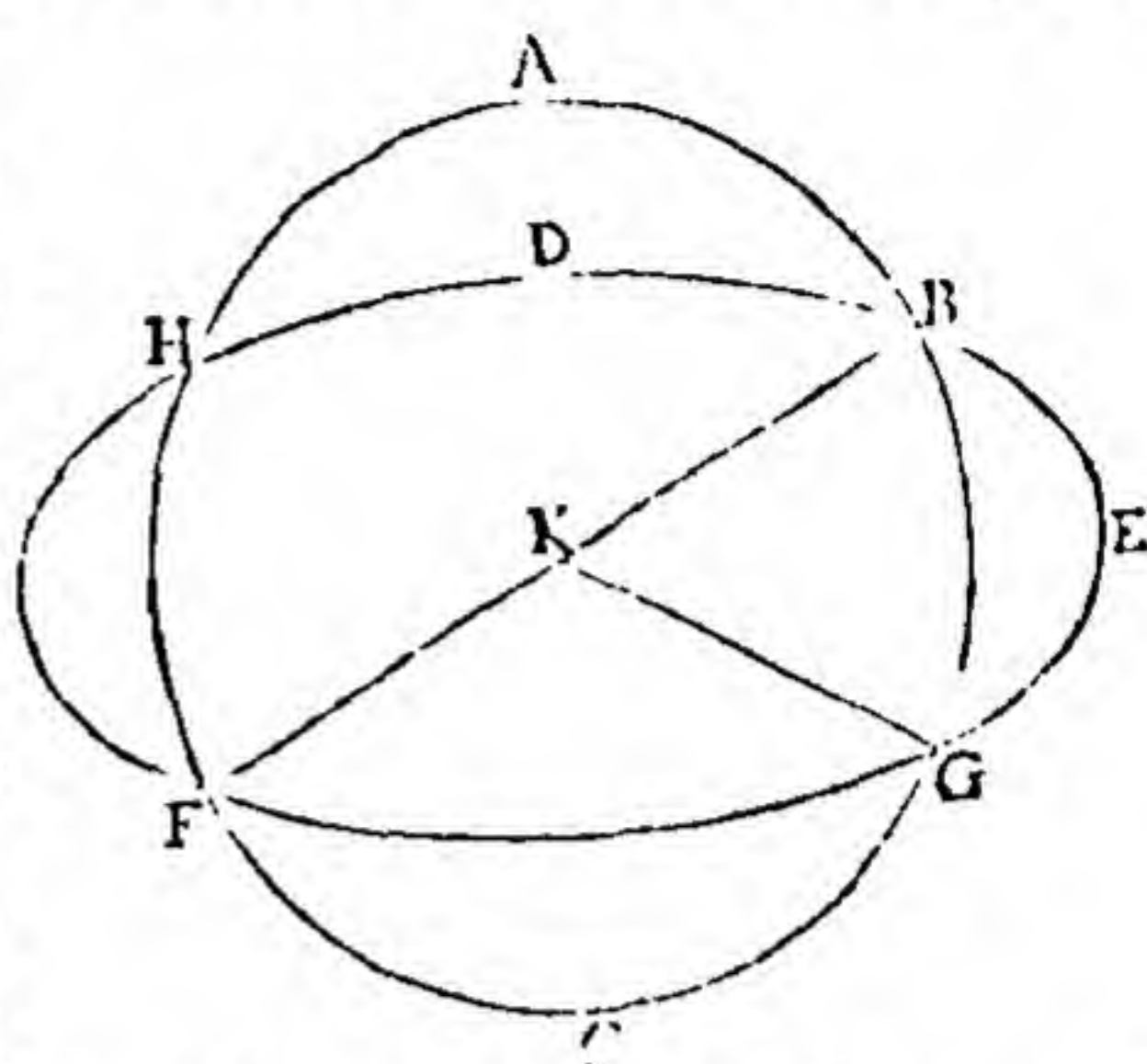
ABC .

ABC. In like manner may we prove, that o is the center of the Circle DEF. Therefore of two Circles cutting one another, there is the same center o , which is impossible [Prop. 5. El. III.].

Therefore a Circle does not cut a Circle in more points than two. Which was to be demonstrated.

Otherwise.

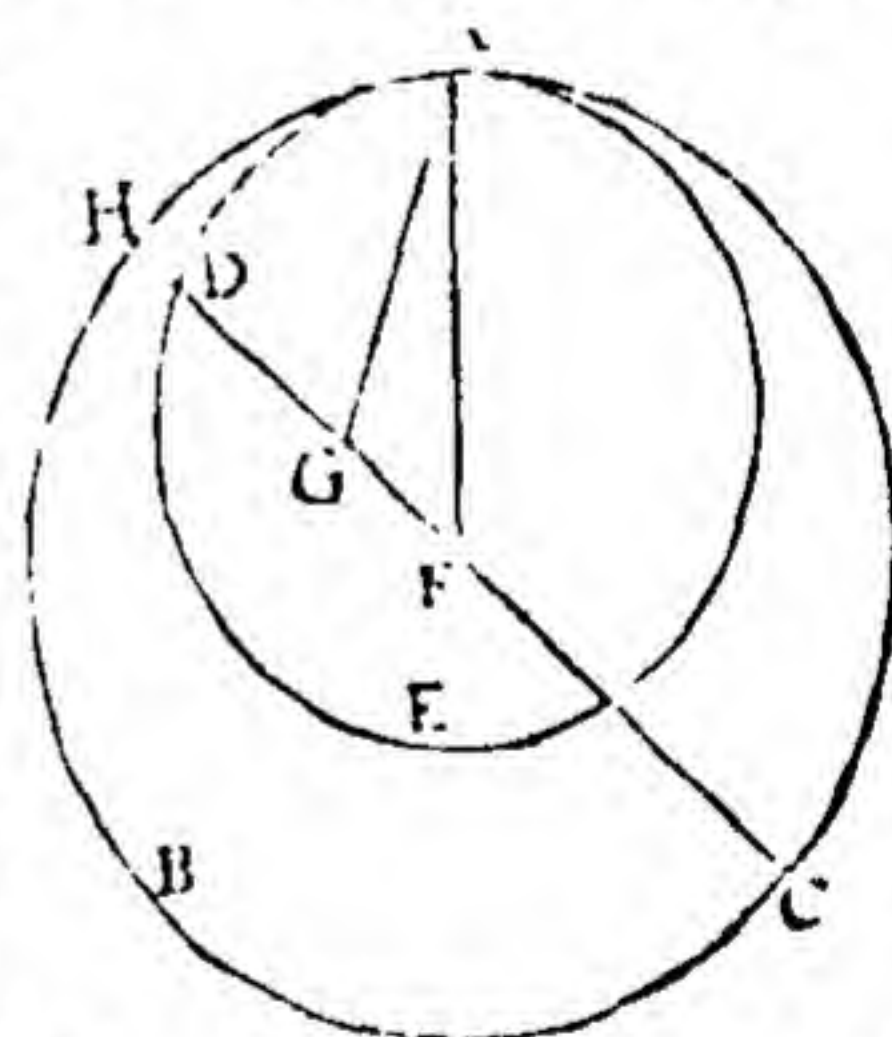
Let again the Circle ABC cut the Circle DEF in more points than two, as, B, G, F, H, and of the Circle ABC let the center be taken, as K, and let KB, KG, KF, be joyned. Now because within the Circle DEF, is taken the point K, and from K do fall upon the Circle DEF, more than two equal strait lines, namely, KB, KG, KF; therefore the point K is the center of the Circle DEF [Prop. 9. El. III.]; but K is also the center of the Circle ABC: wherefore of two Circles cutting one another, there is the same center K; which is impossible. Therefore a Circle does not cut a Circle in more points than two. Which was to be demonstrated.



PROPOSITION XI.

If two Circles touch one another within, and the centers of the same be taken, the strait line joyning their centers being produced, shall fall upon the Contact of the Circles.

Let the two Circles ABC, ADE, touch one another within in the point A. And of the Circle ABC, let be taken the center F: also of the Circle ADE, the center G. I say, that the strait line drawn from G to F, being produced falls upon the point A. For if not, let it, if possible, fall as FGDH, and let AF, AG be joyned. Now forasmuch as AG, GF, are greater than FA, that is, than FH (for FA is equal to FH, both being from the center) let FG common be taken away; therefore the remaining line AG is greater than the remaining line GH. But AG is equal to GD, therefore GD is greater than GH; the less than the greater, which is impossible. Wherefore the strait line drawn from F to G being produced, does not fall beside the point of Contact A, therefore it must fall upon it.

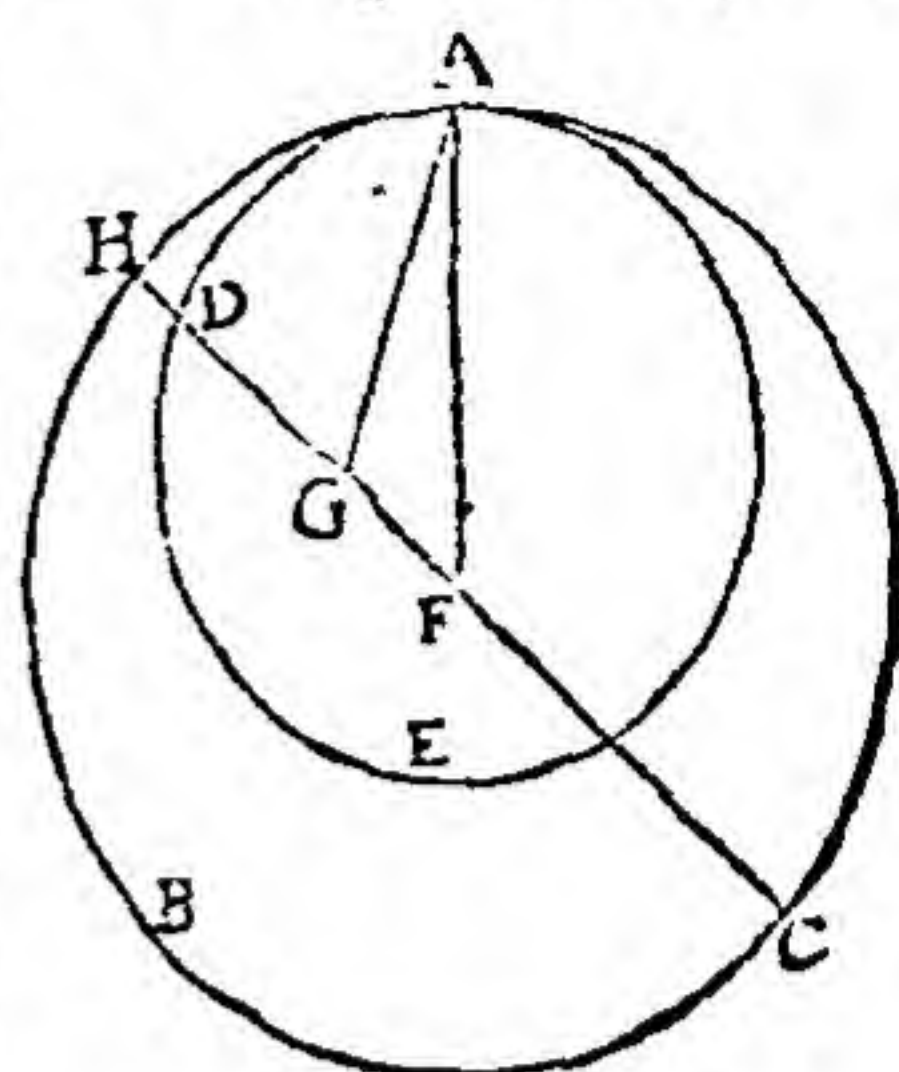


If therefore two Circles touch one another within, and the centers of the same be taken, the strait line joyning their centers being produced does fall upon the Contact of the Circles. Which was to be demonstrated.

Otherwise.

Otherwise.

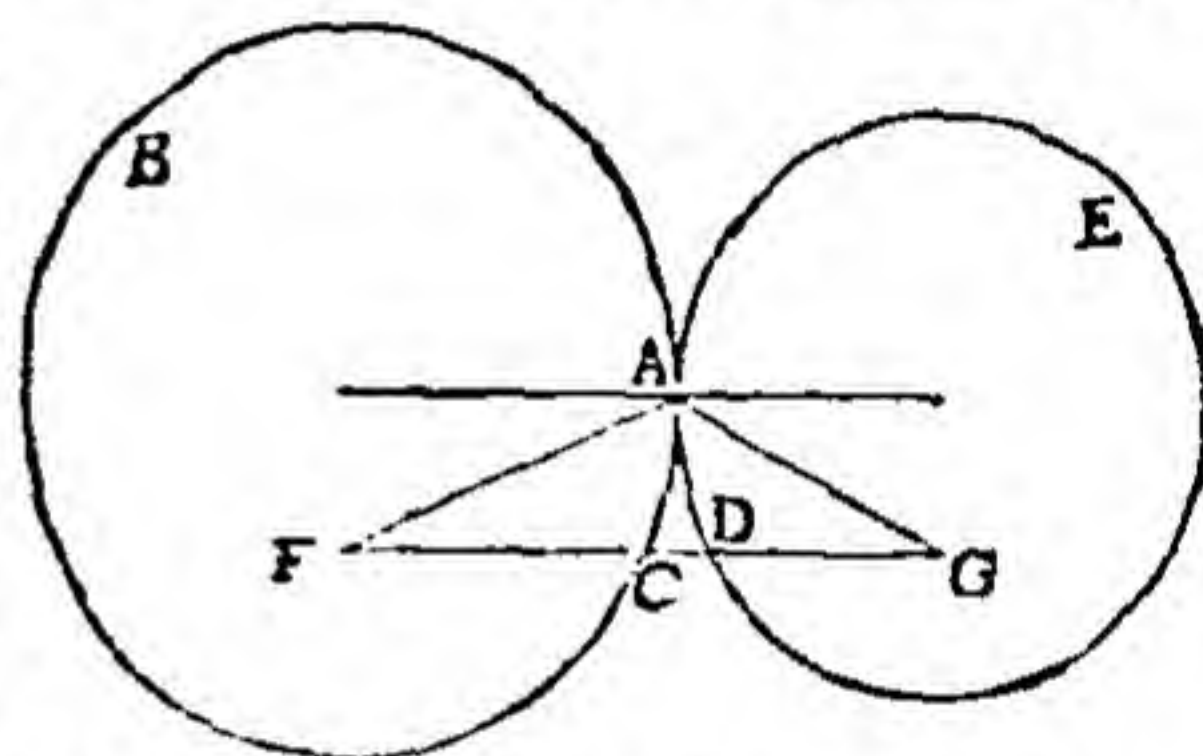
But let it fall as GFC , and produce directly CFG to the point H ; and let AG , AF , be joyned. Now forasmuch as AG , GF , are greater than AF , and AF is equal to FC , that is, to FH , let FG common be taken away; therefore the remaining line AG , is greater than the remaining line GH , that is, GD greater than GH ; the less than the greater, which is impossible. In like manner may we prove the same absurdity, if the center of the greater Circle be without the less.



PROPOSITION XII.

I *f two Circles touch one another without, the strait line joyning their centers shall pass through the Contact.*

Let the two Circles ABC , ADE , touch one another without in the point A . And of the Circle ABC , let the center F be taken: also of the Circle ADE the center G . I say, that the strait line drawn from F to G , shall pass through the Contact in A . For if not, let it, if possible, pass as $FCDG$, and let AF , AG , be joyned. Now forasmuch as the point F is the center of the Circle ABC ; therefore FA is equal to FC . Again, because the point G is the center of the Circle ADE , therefore GA is equal to GD . But it has been proved, that FA is equal to FC , therefore FA , AG , are equal to FC , DG . So that the whole FG is greater than FA , AG ; but it is less [Prop. 20. El. I.], which is impossible. Therefore the strait line drawn from F to G , cannot pass otherways but through the Contact in A : wherefore it passes through the Contact.



If therefore two Circles touch one another without, the strait line joyning their centers shall pass through the Contact. Which was to be demonstrated.

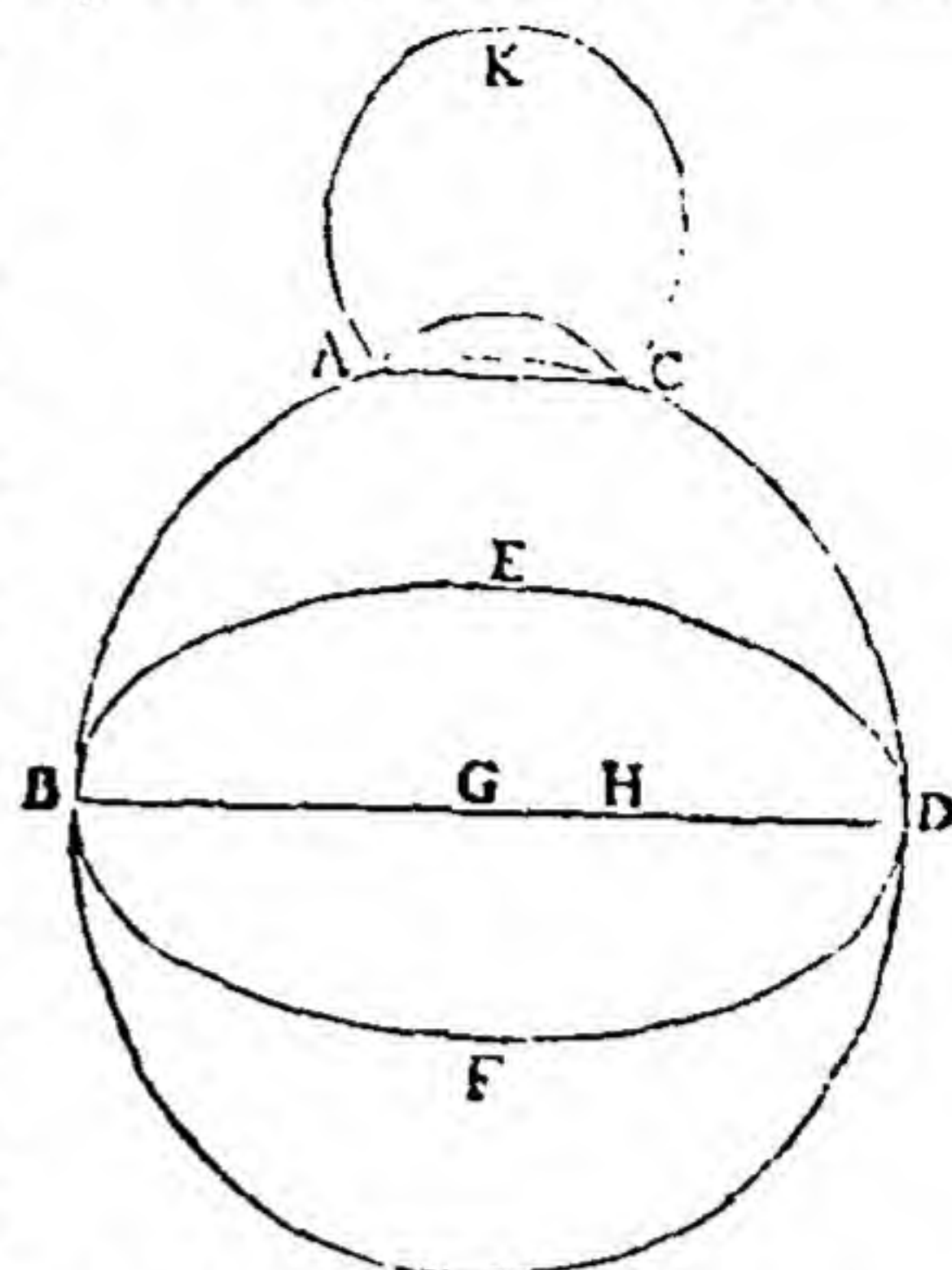
PROPOSITION XIII.

A *Circle touches not a Circle in more points than one, whether it touches within or without.*

For if it be possible, let the Circle $ABDC$, touch the Circle $EBFD$, first within, in more points than one, as in B, D . And of the Circle $ABDC$, let the center G be taken: also of the Circle $EBFD$ the center H ; wherefore the strait line drawn from G to H , shall fall upon the points B, D [Prop. 11. El. III.]. Let it fall as $BGHD$. Now forasmuch as the point G is the center of the Circle $ABDC$, therefore BG is equal to GD ; wherefore BG is greater than HD , and BH much greater than HD . Again, because the point H is the center of the Circle

Circle $EBFD$, therefore BH is equal to HD . But it has been proved to be much greater, which is impossible; therefore a Circle touches not a Circle within, in more points than one.

I say also that neither without. For if possible, let the Circle ACK touch the Circle $ABDC$ without, in more points than one, as in A, C : and let AC be joyned. Now forasmuch as in the circumference of each of the Circles $ABDC$, ACK , are taken any two points A, C ; the strait line AC joyning the same points shall fall within each of the Circles [by Pr. 2. El. III.]; but AC falling within the Circle $ABDC$, must fall without the Circle ACK ; which is absurd, (viz. to fall within and without the same Circle ACK ;) therefore a Circle touches not a Circle without, in more points than one. And it has been proved that it touches not within.

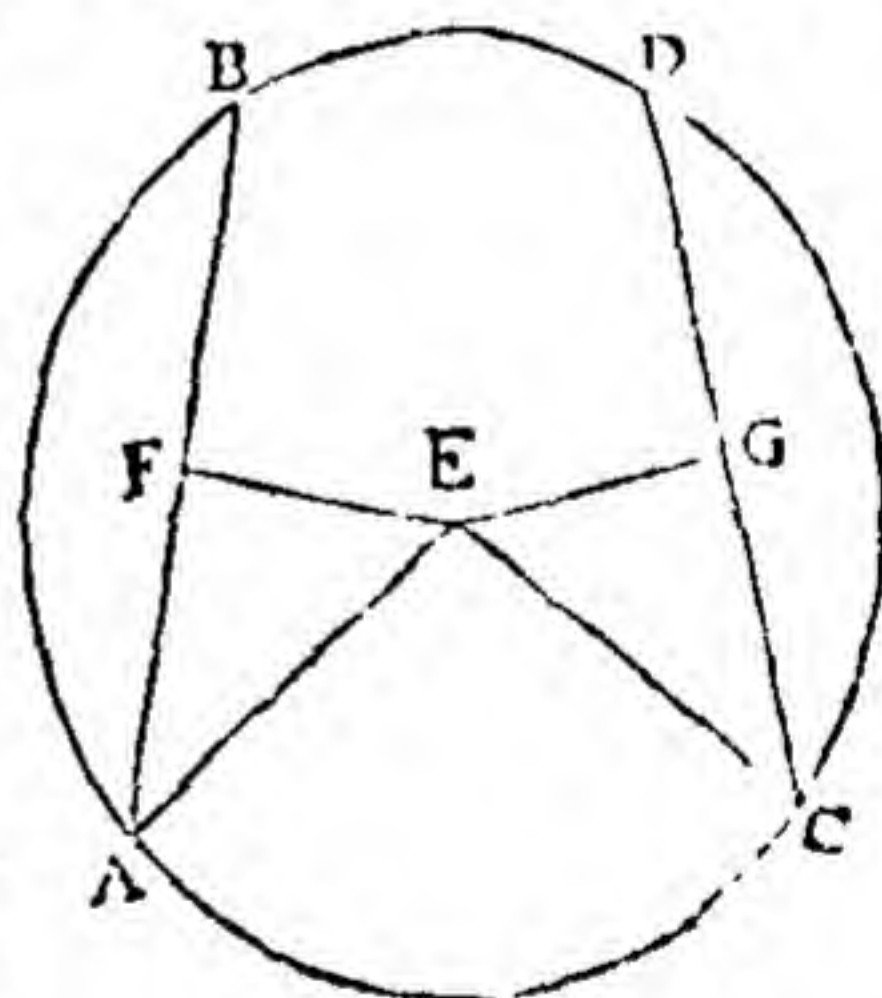


Therefore a Circle touches not a Circle in more points than one, whether it touches within, or without. Which was to be demonstrated.

PROPOSITION XIV.

IN a Circle equal strait lines are equally distant from the center. And lines equally distant from the center, are equal to one another.

Let a Circle be $ABDC$, and in the same let the equal strait lines be AB, CD . I say, that they are equally distant from the center. For of the Circle $ABDC$ let the center be taken, and be it E ; then from E let be drawn to AB, CD , the perpendiculars EF, EG , and let AE, CE , be joyned. Now forasmuch as the strait line EF drawn through the center, cuts the strait line AB not drawn through the center, at right angles, it also cuts the same into halves [Prop. 3. El. III.]; wherefore AF is equal to FB ; therefore AB is the double of AF . And by the same reason also CD is the double of CG . But AB is equal to CD [by Supposition], therefore AF is equal to CG [Ax. 7.].



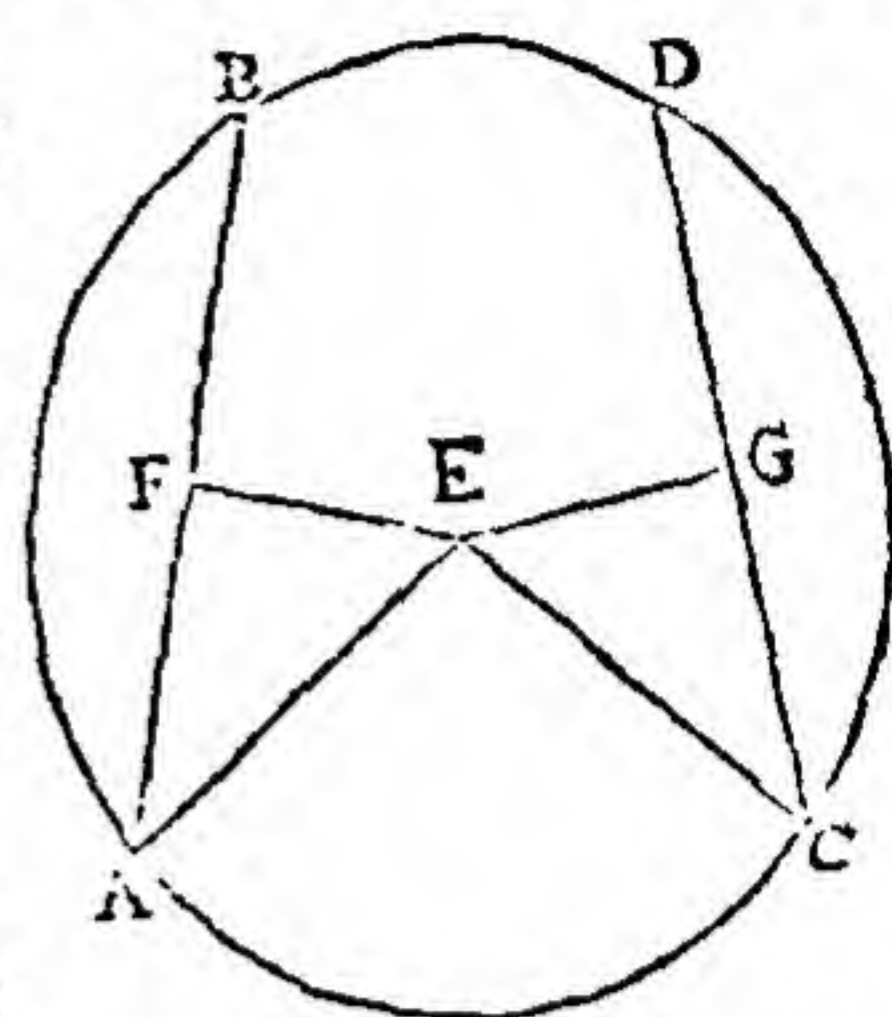
And because AE is equal to EC , therefore the square of AE is also equal to the square of EC . But the squares of AF, FE , are equal to the square of AE [Prop. 47. El. I.]: for the angle at F is a right angle. But also the squares of EG, GC , are equal to the square EC ; for the angle at G is a right angle. Therefore the squares of AF, FE , are equal to the squares of CG, GE , of which the square of AF is equal to the square of CG ; for the line AF is equal to the line CG : Therefore the remaining square of FE , is equal to the remaining square of EG ; therefore EF is equal to EG . But in a Circle strait

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lines

lines are said to be equally distant from the center, when the perpendiculars drawn from the center to the same lines are equal [Def. 4. El. III.]; therefore AB, CD , are equally distant from the center.

But now let the lines AB, CD , be equally distant from the center, that is, let EF be equal to EG . I say, that AB is also equal to CD . For the same construction being made, we shall in like manner prove that AB is the double of AF , and CD the double of CG . And because AE is equal to EC , therefore the square of AE is also equal to the square of EC . But the squares of EF, FA , are equal to the square of AE , and the squares of EG, GC , are equal to the square of EC . Therefore the squares of EF, FA , are equal to the squares of EG, GC , of which the square of EG is equal to the square of EF , for EF is equal to EG ; and therefore the remaining square of AF , is equal to the remaining square of CG : wherefore AF is equal to CG . But AB is the double of AF , and CD the double of CG ; wherefore AB is equal to CD [Ax. 6.].

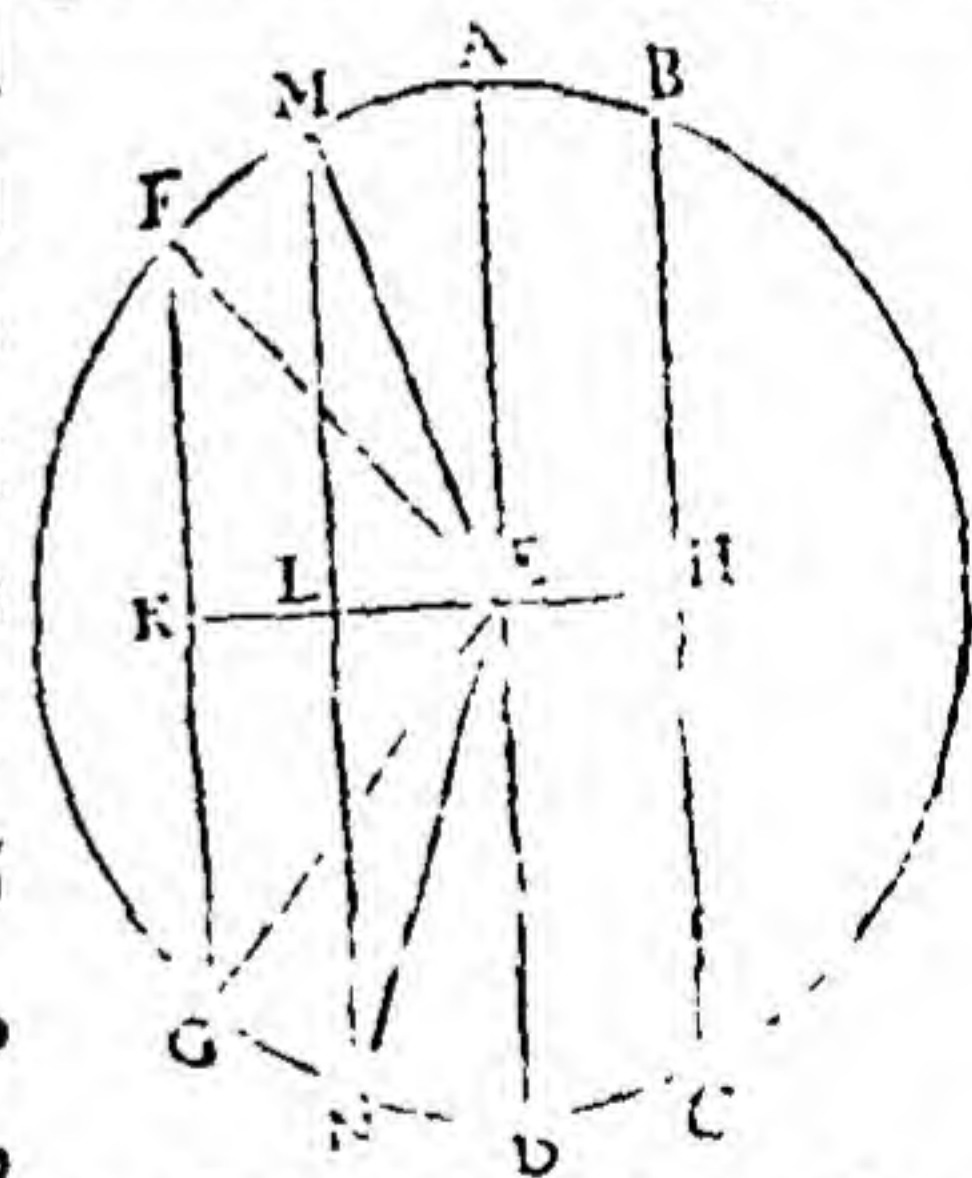


Therefore in a Circle equal strait lines are equally distant from the center. And lines equally distant from the center, are equal to one another. Which was to be demonstrated.

PROPOSITION XV.

In a Circle the greatest line is the Diameter. And of the others, always the nearer to the center is greater than the more remote.

Let the Circle be $ABCD$, and the Diameter thereof be AD , and the center E . Now to the center E let BC be nearer, and FG more remote. I say, that AD is the greatest, and BC greater than FG . Let from the center be drawn to BC, FG , the perpendiculars EH, EK . Now because BC is nearer to the center, and FG more remote, therefore EK is greater than EH [Def. 5. El. III.]. Let EL be put equal to EH , and through L , let LM be drawn at right angles to EK , and produced to N : then let be joyned EM, EN, EF, EG . Now forasmuch as EH is equal to EL , therefore BC is equal to MN [Prop. 14.]. Again, because AE is equal to EM , and DE to EN , therefore AD is equal to ME, EN ; but ME, EN , are greater than MN [Prop. 20. El. I.]; therefore AD is greater than MN , but MN is equal to BC , therefore AD is greater than BC . And because the two lines ME, EN , are equal to the two lines FE, EG , and the angle MEN is greater than the angle FEG , therefore the base MN is greater than the base FG [Prop. 24. El. I.]. But MN has been proved equal to BC , wherefore BC is greater than FG . The greatest therefore is AD the Diameter, and BC is greater than FG . There-



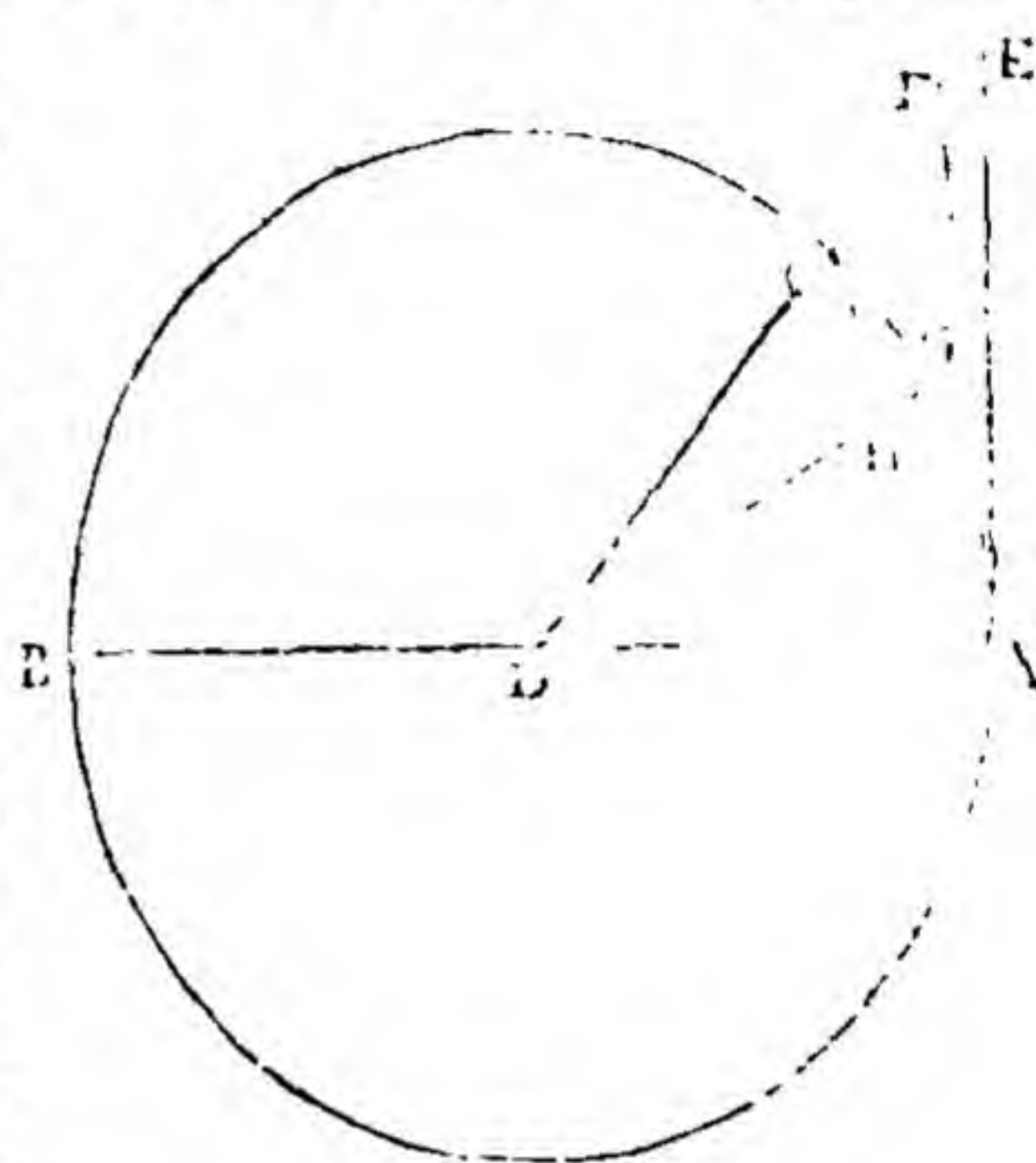
Therefore in a Circle the greatest line is the Diameter. And of the others, always the nearer to the center is greater than the more remote. Which was to be demonstrated.

PROPOSITION XVI.

A Strait line drawn at right angles to the Diameter of a Circle, from an extremity thereof shall fall without the Circle: And in the place between the strait line and the circumference, no other strait line shall not fall.

And the angle of the Semicircle is greater than any Acute strait-lined angle; and the remaining angle is less.

Let the Circle be ABC about the center D , and the Diameter be AB . I say, that the strait line drawn at right angles to AB from the point A the extremity of the Diameter, shall fall without the Circle. For if not, let it, if possible, fall within as AC , and let DC be joyned. Now forasmuch as DA is equal to DC , therefore the angle DAC is equal to the angle ACD [Prop. 5. El. I.]. But DAC is a right angle [by Supposition], therefore ACD is also a right angle, wherefore DAC , ACD , are equal to two right angles, which is impossible [Prop. 17. El. I.]. Therefore a strait line drawn from the point A at right angles to AB , shall not fall within the Circle. In like manner we may shew that it shall not fall upon the circumference; therefore it must fall without, as AE .



I say, that in the place between the strait line AE , and the circumference CHA , another strait line shall not fall. For if possible, let it fall as FA , and from the point D to the line FA , let be drawn the perpendicular DG . Now because AGD is a right angle, and DAG is less than a right angle; therefore AD is greater than DG [Prop. 19. El. I.]. But DA is equal to DH , therefore DH is greater than DG , the less than the greater, which is impossible. Therefore in the place between the strait line and the circumference, another strait line shall not fall.

I say moreover, that the angle of the Semicircle, which is contained by the strait line BA , and the circumference CHA , is greater than any Acute strait-lined angle; and the remaining angle contained by the circumference CHA , and the strait line AE , is less than any Acute strait-lined angle. For if there be any Acute strait-lined angle greater than the angle contained by the strait line BA , and the circumference CHA , and any other less than the angle contained by the circumference CHA ; and the strait line AE , then in the place between the circumference CHA and the strait line AE , shall fall a strait line, which shall make one strait-lined angle greater than the

angle contained by the strait line BA , and the circumference CHA , and another strait-lined angle less than the angle contained by the circumference CHA and the strait line AE . But such a line cannot fall; therefore there shall not be any Acute strait-lined angle greater than the angle contained by the strait line BA , and the circumference CHA , nor any less than the angle contained by the circumference CHA and the strait line AE . Which was to be demonstrated.

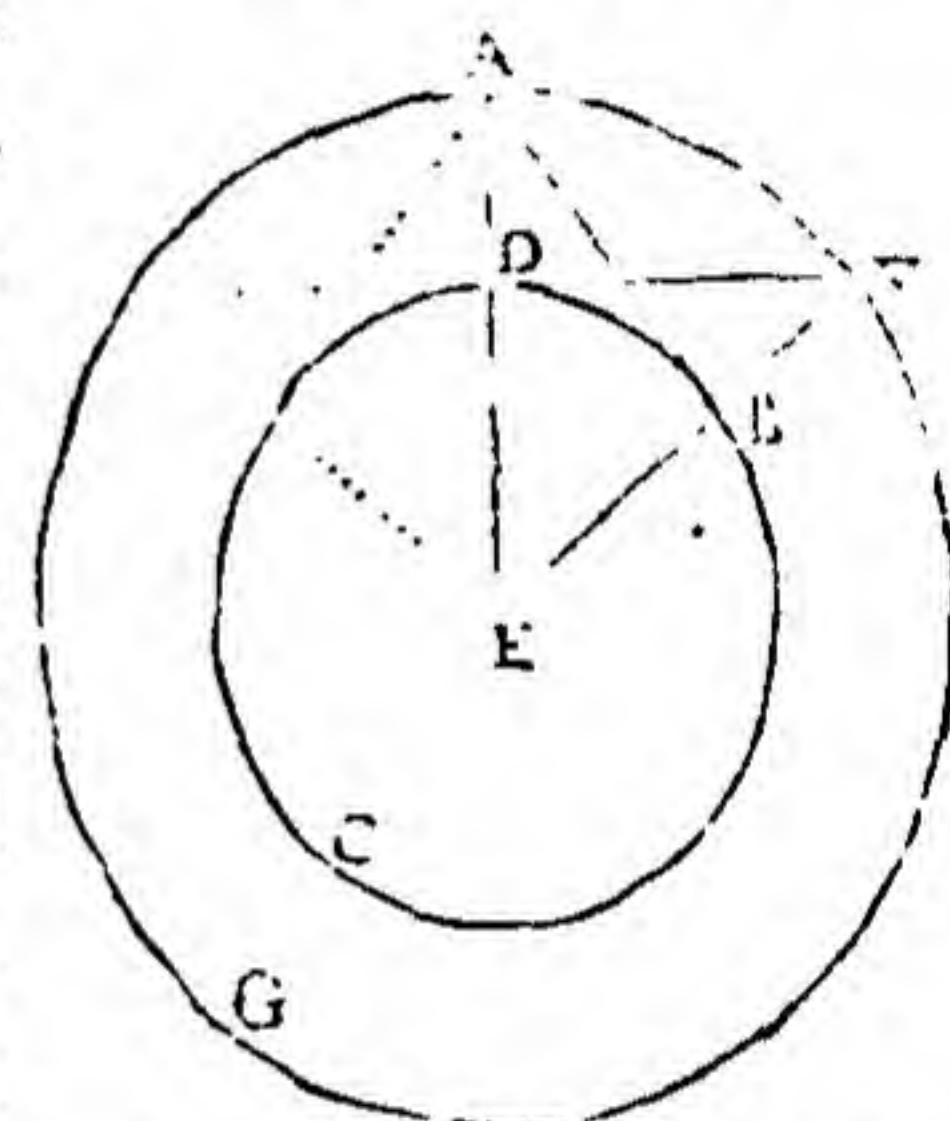
Corollary.

From hence it is manifest, that a strait line drawn at right angles to the Diameter of a Circle, from the extremity thereof doth touch the Circle [Def. 2.]. And that a strait line touches a Circle in one point only: Because a strait line concurring with a Circle in two points, has been proved to fall within the same [Prop. 2.]

PROPOSITION XVII.

From a given point to draw a strait line, which shall touch a given Circle.

Let the given point be A , and the given Circle BCD . It is required from the point A , to draw a strait line which shall touch the Circle BCD . Let the center of the Circle be taken as E , and draw AE ; then the center E and distance EA , let the Circle AFG be described; and from the point D to the line EA , let DF be drawn at right angles, and let EBF , and AB be joyned. I say, that from the point A , is drawn the line AB , which touches the Circle BCD . Now forasmuch as E is the center of the Circles BCD , AFG , therefore EA is equal to EF , and ED to EB . There are then the two lines AE , EB , equal to the two lines FE , ED , and they contain a common angle at E ; wherefore the base DF is equal to the base AB , and the Triangle DEF equal to the Triangle BEA , and the remaining angles to the remaining angles; therefore the angle EBA is equal to the angle EDF : but EDF is a right angle, therefore EBA is also right; and EB is from the center. But what is drawn at right angles to the Diameter of a Circle from an extremity thereof does touch the Circle: Wherefore AB does touch the Circle.



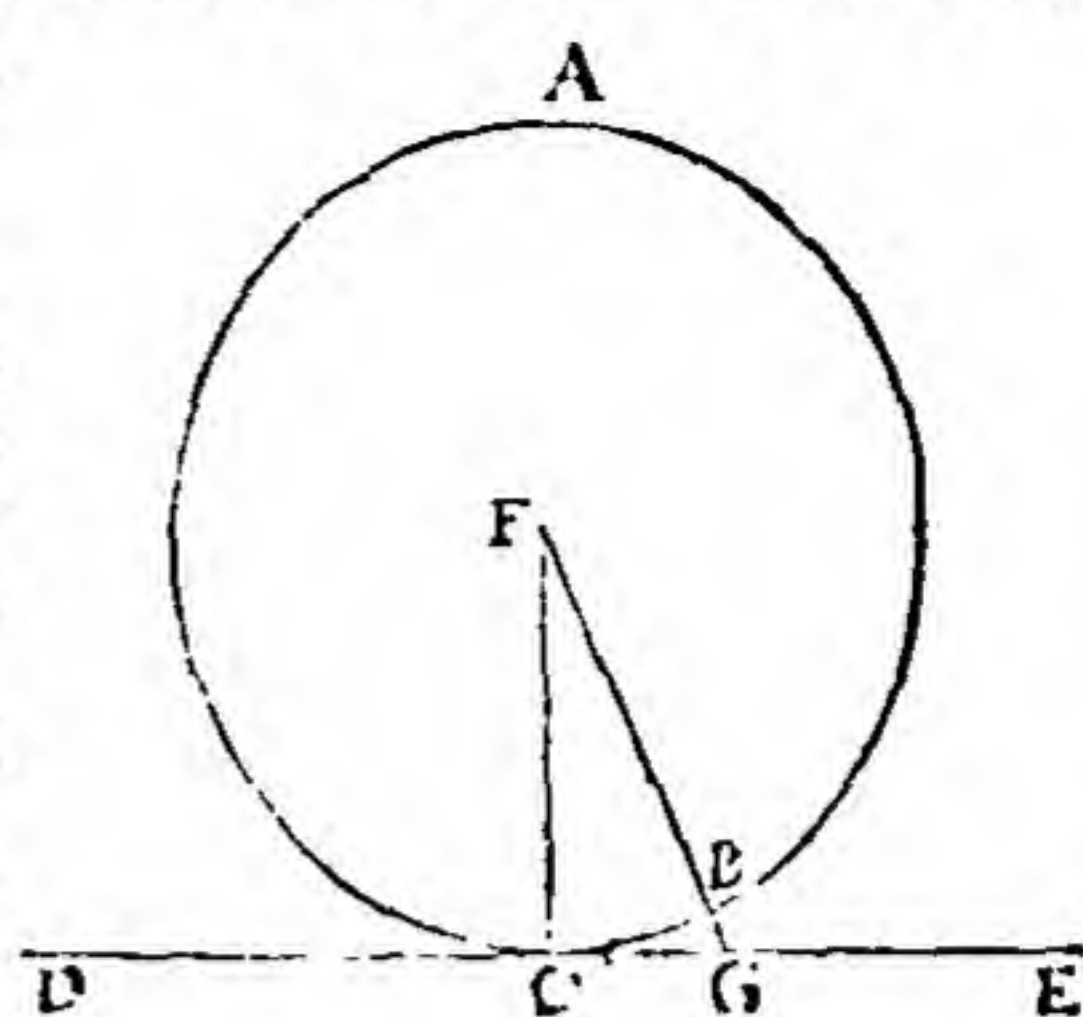
Therefore from the given point A is drawn a strait line AB , touching the given Circle BCD . Which was to be done.

PROPOSITION XVIII.

If a strait line touches a Circle, and from the center be drawn a strait line to the Contact, that line shall be perpendicular to the Tangent.

Let the strait line DE touch the Circle ABC in the point c ; and of the Circle ABC , let the center F be taken; then from F to c let
 bc

be drawn FC . I say, that FC is perpendicular to DE . For if not; let from the point F , be drawn FG perpendicular to DE . Now because the angle FGC is a right angle, therefore GCF is an acute angle [Prop. 32. El. I.]. But under the greater angle is subtended the greater side [Prop. 19. El. I.]; therefore FC is greater than FG ; but FC is equal to FB , therefore FB is greater than FG ; the less than the greater, which is impossible. Therefore FG is not perpendicular to DE .



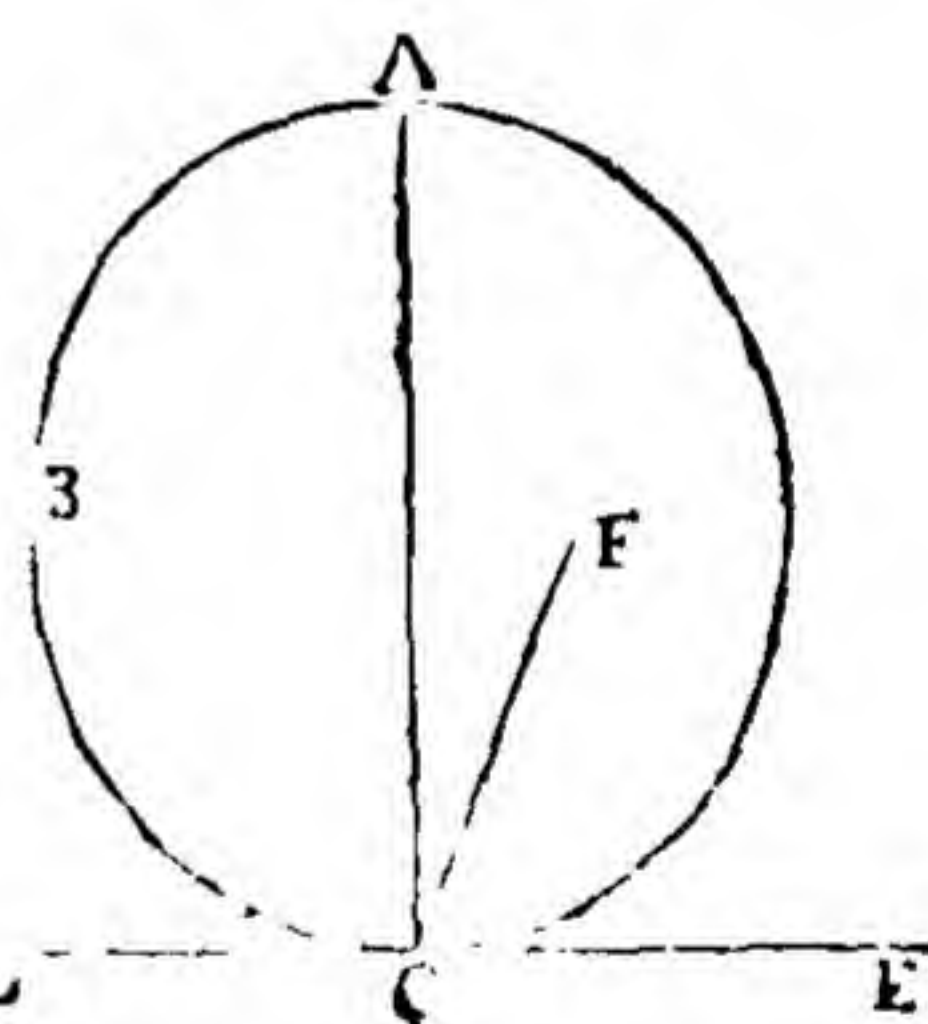
In like manner may we prove, that no other line can be besides FC ; therefore FC is perpendicular to DE .

If therefore a straight line touches a Circle, and from the center be drawn a straight line to the Contact, That is perpendicular to the Tangent. Which was to be demonstrated.

PROPOSITION XIX.

If a straight line touches a Circle, and from the Contact be drawn a straight line at right angles to the Tangent, the Center of the Circle shall be in the same line.

Let the straight line DE touch the Circle ABC , in the point C , and from C let CA be drawn at right angles to DE . I say, that the center of the Circle is in the line CA . For if not, let it, if possible, be F , and let CF be joyned. Now forasmuch as the straight line DE touches the Circle ABC , and from the center is drawn to the Contact the line FC : wherefore FC is perpendicular to DE [Prop. 18. El. III.]; and therefore FCE is a right angle; but ACE is also a right angle [by Supposition]; wherefore FCE is equal to ACE ; the less to the greater, which is impossible. Therefore F is not the center of the Circle ABC . And in like manner may we prove, that no other can be besides a point in AC .



If therefore a straight line touches a Circle, and from the Contact be drawn a straight line at right angles to the Tangent, the center of the Circle shall be in the same line. Which was to be demonstrated.

PROPOSITION XX.

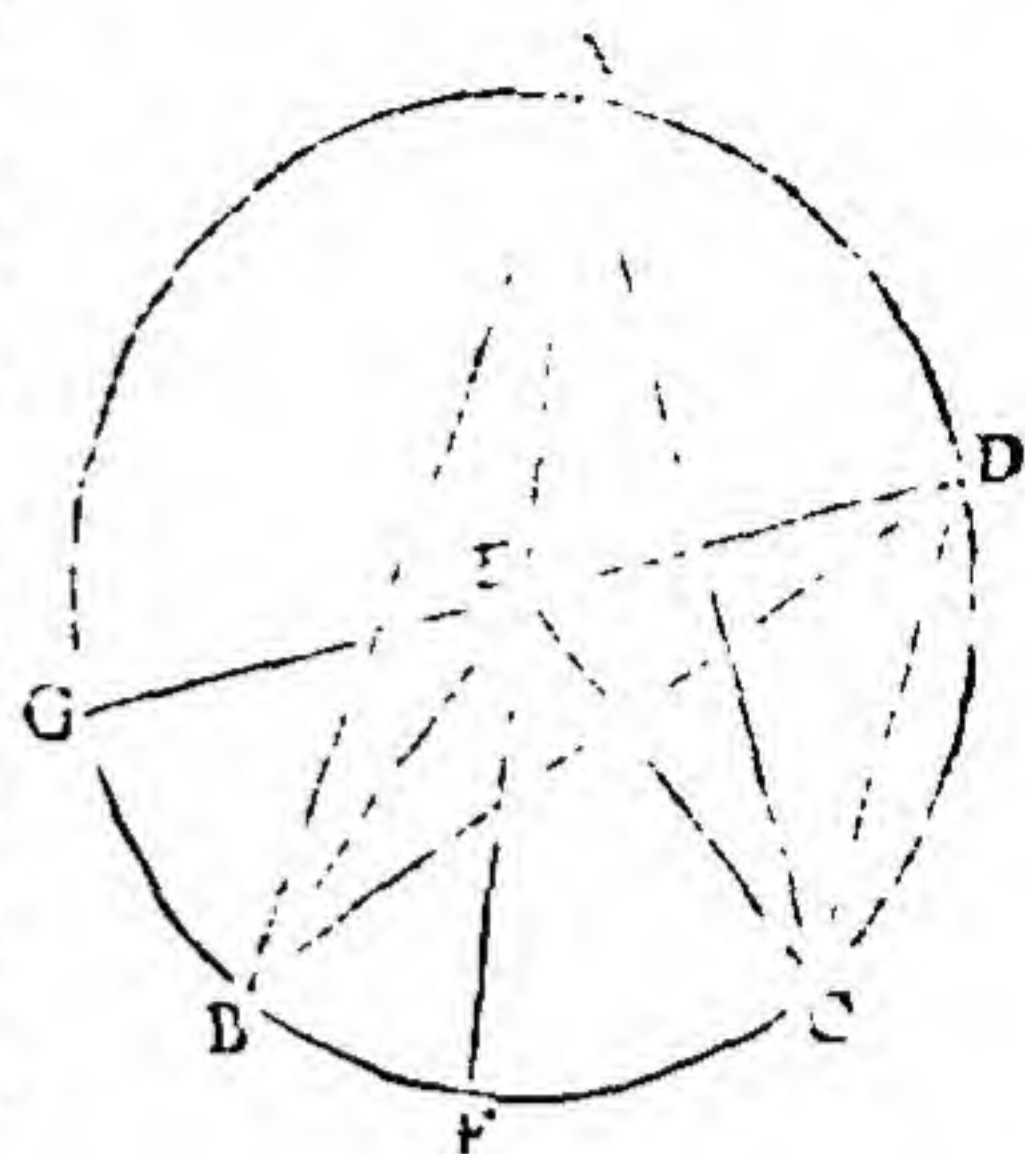
In a Circle, the angle at the center is double to the angle at the circumference, when the angles have the same circumference for their base.

Let the Circle be ABC , the angle at the center BEC , and at the circumference BAC ; and let them have the same circumference

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BC

BC for their base. I say, that the angle BEC is double to the angle BAC. For drawing the line AE, let it be produced to F. Now forasmuch as EA is equal to EB, therefore the angle EAB, is equal to the angle EBA [Prop. 5. El. I.]: wherefore the angles EAB, EBA, are double to the angle EAB. But the angle BEF is equal to the angles EAB, EBA [Prop. 32. El. I.]: therefore the angle BEF, is double to the angle EAB. By the same reason also the angle FEC is double to the angle EAC; therefore the whole angle BEC, is double to the whole angle BAC. Again, let the angle at the circumference be declined beyond the center E, and let it be BDC: then drawing DE, let it be produced to G. In like manner may we prove, that the angle GEC is double to the angle GDC; of which angles GEB is double to GDB: wherefore the remaining angle BEC, is double to the remaining angle BDC.

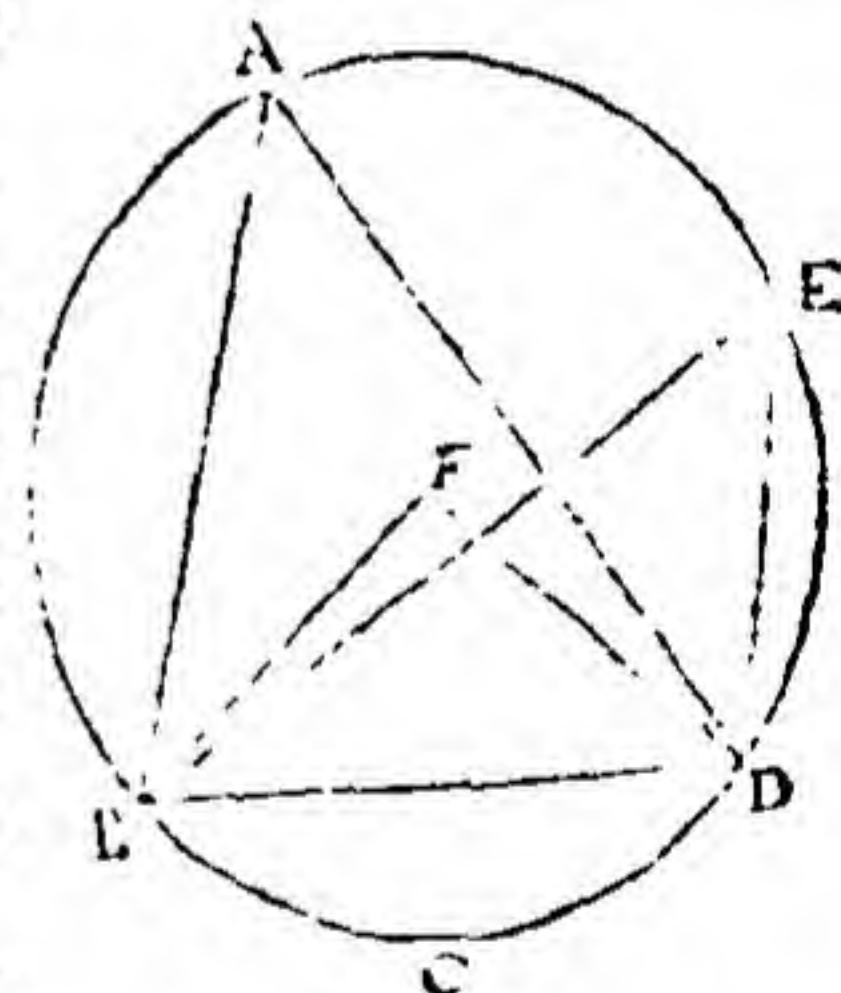


Therefore in a Circle, the angle at the center is double to the angle at the circumference, when the angles have the same circumference for their base. Which was to be demonstrated.

PROPOSITION XXI.

IN a Circle angles in the same Segment are equal to one another.

Let the Circle be ABCD, and in the Segment BAED, let the angles be BAD, BED. I say, that the angles BAD, BED, are equal to one another. For let the center of the Circle be taken, and be it F; and let BF, FD, be joyned. Now forasmuch as the angle BFD is at the center, and the angle BAD at the circumference, and they have the same circumference for a base BCD; therefore the angle BFD is double to the angle BAD [Prop. 20 El. III.]. By the same reason the angle BFD is also double to the angle BED; therefore the angle BAD is equal to the angle BED [Ax. 7.]



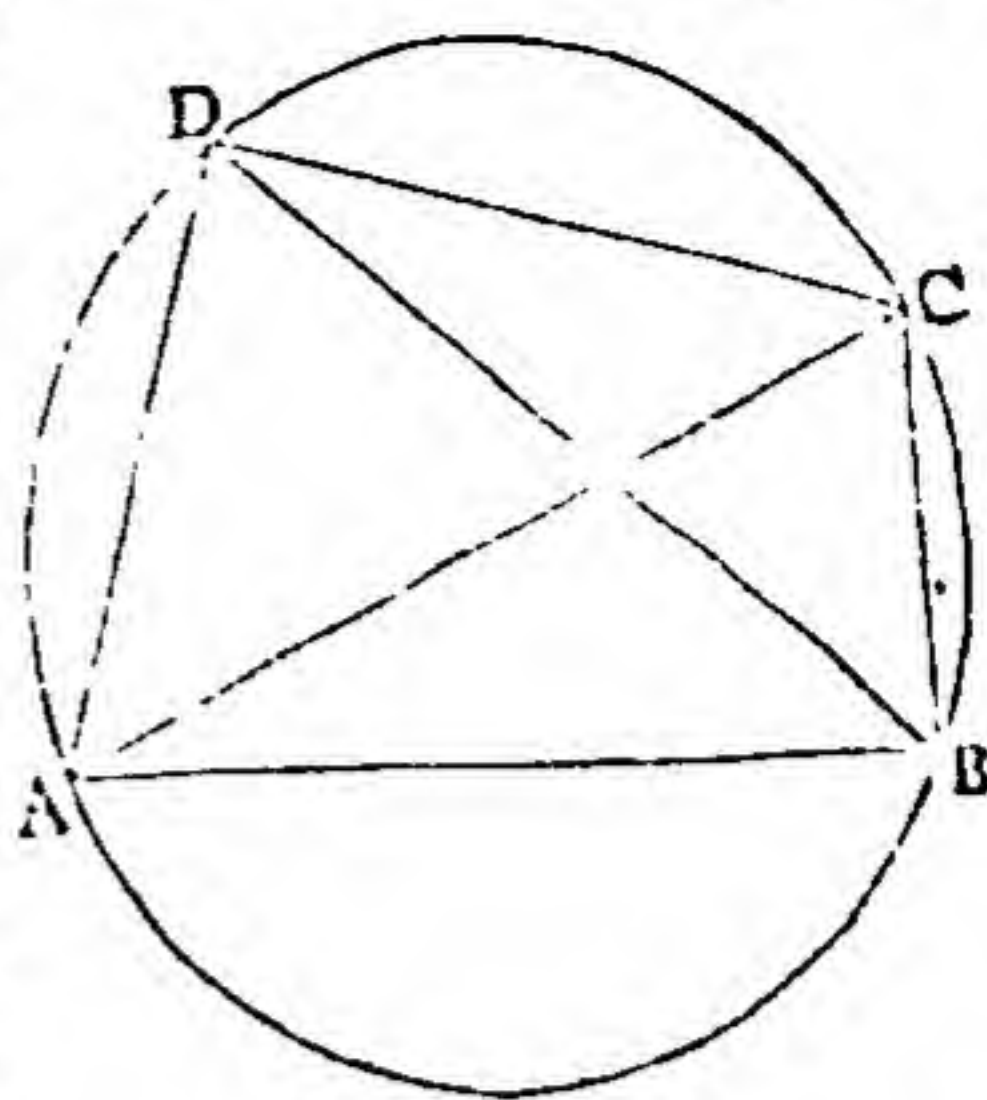
Therefore in a Circle angles in the same Segment are equal to one another. Which was to be demonstrated.

PROPOSITION XXII.

OF four sided Figures in Circles, the opposite angles are equal to two right.

Let the Circle be ABCD, and in the same the four sided Figure be ABCD. I say, that the opposite angles are equal to two right angles. Let AC, BD, be joyned. Now forasmuch as, of every Triangle the three angles are equal to two Right [Prop. 32. El. I.]; therefore of the

the Triangle ABC , the three angles CAB, ABC, BCA , are equal to two Right. But the angle CAB , is equal to the angle BDC , for they are in the same Segment $BADC$. And the angle ACB is equal to the angle ADB , for they are in the same Segment $ADCB$. Therefore the whole angle ADC , is equal to the angles BAC, ACB ; let the common angle ABC be added; therefore the angles ABC, BAC, ACB , are equal to the angles ABC, ADC . But the angles ABC, BAC, ACB , are equal to two Right: therefore ABC, ADC , are also equal to two Right. In like manner may we prove that BAD, DCB , are also equal to two Right angles.

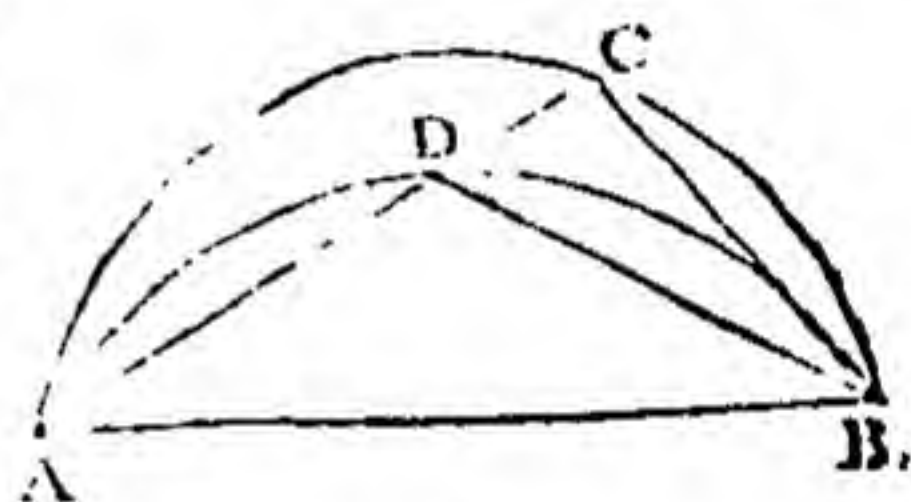


Therefore of four sided Figures in Circles, the opposite angles are equal to two Right angles. Which was to be demonstrated.

PROPOSITION XXIII.

Upon the same strait line two Segments of Circles like, and unequal, cannot be constituted the same way.

For if possible, upon the same strait line AB let two like, and unequal Segments of Circles ACB, ADB , be constituted the same way: then let ADC be drawn, and CB, BD , be joyned. Now forasmuch as the Segment ACB is like to the Segment ADB ; and like Segments of Circles are such, which receive equal angles [Def. 11. El. III.]; therefore the angle ACB , is equal to the angle ADB ; the outward to the inward, which is impossible.



Therefore upon the same strait line two Segments of Circles like, and unequal, cannot be constituted the same way. Which was to be demonstrated.

PROPOSITION XXIV.

Upon equal strait lines like Segments of Circles, are equal to one another.

For upon equal strait lines AB, CD , let like Segments of Circles be AEB, CFD . I say, that the Segment AEB , is equal to the Segment CFD . For the Segment AEB being apply'd to the Segment CFD , and the point A put upon the point C , and the line AB upon CD ; then shall the point B agree with the point D ; for that AB is equal to CD . Now the strait line AB agreeing with the strait line CD , the Segment AEB shall also agree with the Segment CFD . For if AB shall agree with CD , and the Segment AEB shall not agree with the Segment CFD , then shall it differ from CFD , as $CHGD$. But a Circle cuts not a Circle in more points than two, yet here



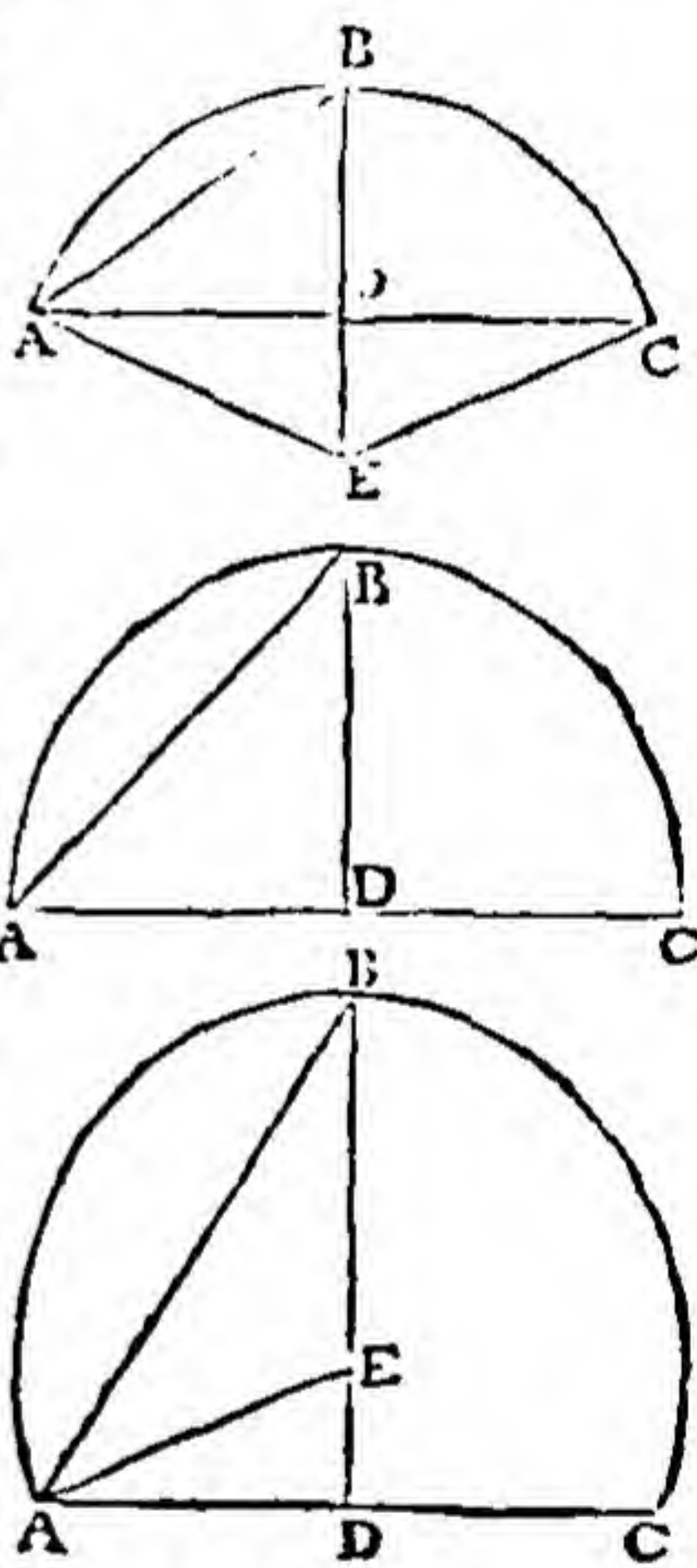
here the Circle $CHGD$, cuts the Circle CFD , in more points than two, namely in c, G, D , which is impossible [Prop. 10. El. III.]. Wherefore the strait line AB , agreeing with the strait line CD , the Segment AEB cannot disagree with the Segment CFD : wherefore it shall agree, and be equal to it.

Therefore upon equal strait lines like Segments of Circles, are equal to one another. Which was to be demonstrated.

PROPOSITION XXV.

A Segment of a Circle being given to describe unto it the Circle, of which it is a Segment.

Let the given Segment of a Circle be ABC . It is required to describe unto it the Circle of which ABC is a Segment. Let AC be cut into halves in D , and from the point D let DB be drawn at right angles to AC , and let AB be joyned. Now then the angle ABD is either greater than the angle BAD , or equal to it, or less. First, let it be greater, and to the strait line BA , and to the point A in the same, let the angle BAE be constituted equal to the angle ABD [Prop. 23. El. I.], and let BD be produced to E , and EC be joyned. Now forasmuch as the angle ABE , is equal to the angle BAE ; therefore the strait line EB is equal to the strait line EA . And because AD is equal to DC , and DE common, therefore there are the two lines AD, DE , equal to the two lines CD, DE , each to each, and the angle ADE is equal to the angle CDE , for each of them is a right angle; therefore the base EA is equal to the base EC . But it has been proved, that EA is equal to EB , therefore EB is also equal to EC , wherefore the three lines EA, EB, EC , are equal to one another; therefore the center E , and distance any one of the lines EA, EB, EC , a Circle being described shall pass through the other points, and the Circle shall be described to the given Segment. Therefore a Segment of a Circle being given, there is described unto it the Circle of which it is a Segment. And it is manifest that the Segment ABC is less than a Semicircle, for that E the center of the same falls without.



In like manner if the angle ABD , be equal to the angle BAD , and so AD be equal to either of the lines BD, DC ; therefore the three lines AD, DB, DC , are equal to one another, and D shall be the center of the completed Circle; and the Segment ABC shall be a Semicircle.

But if the angle ABD , be less than the angle BAD , and to the line BA , and to the point A in the same an angle be constituted equal to ABD ; the center shall fall within the Segment ABC , and in the line

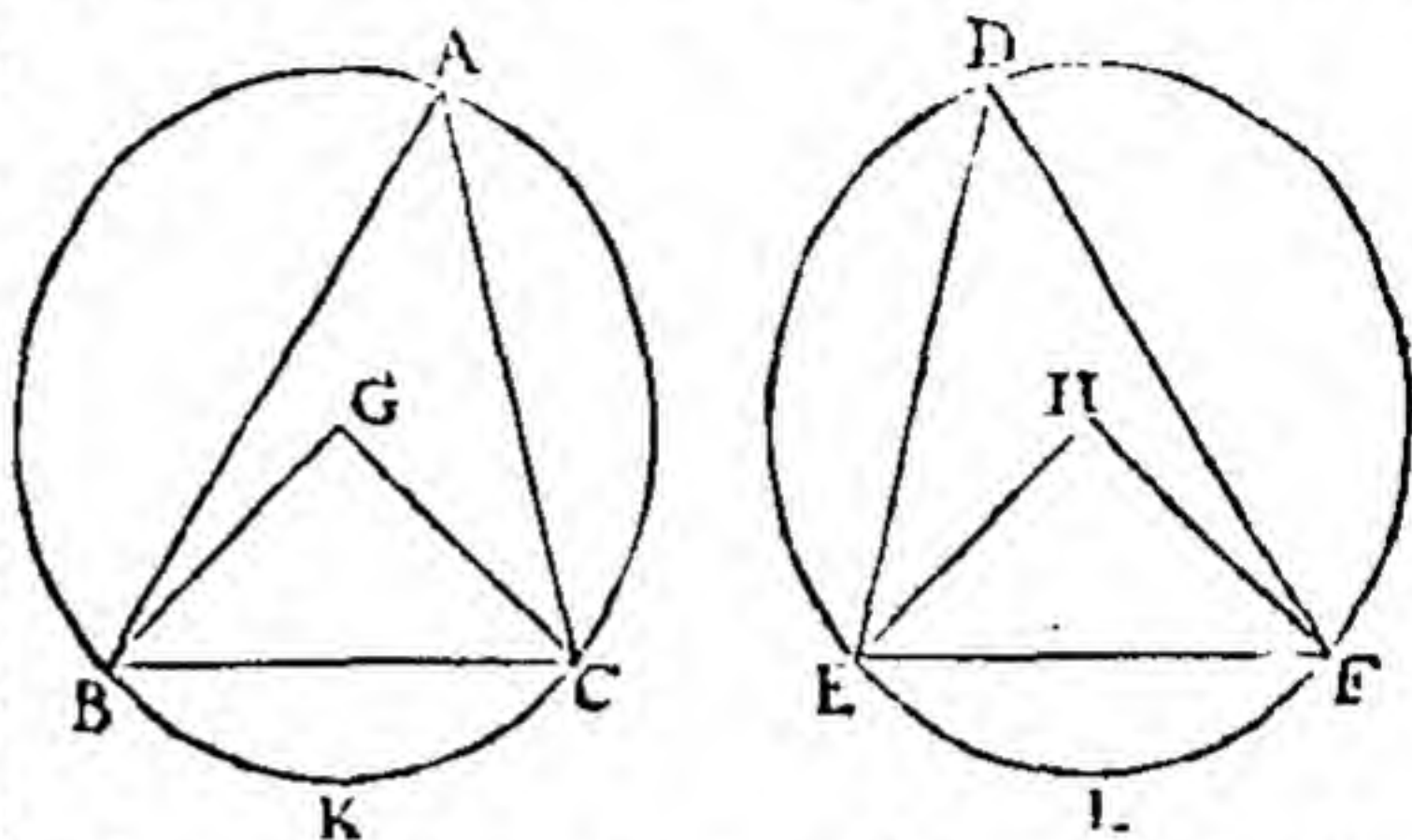
line BD ; and the Segment shall be greater than a Semicircle.

Therefore a Segment of a Circle being given there is described unto it, the Circle, of which it is a Segment. Which was to be done.

PROPOSITION XXVI.

IN equal Circles, equal angles insist upon equal circumferences, whether the insisting angles be at the centers, or at the circumferences.

Let the equal Circles be ABC , DEF , and in the same let the equal angles at the centers be BGC , EHF , and the circumferences BAC , EDF . I say, that the circumference BKC , is equal to the circumference ELF . For let BC , EF , be joyned. Now forasmuch as the Circles ABC , DEF , are equal; therefore the lines from the centers are equal: wherefore there are the two lines BG , GC , equal to the two lines EH , HF , and the angle at G is equal to the angle at H : therefore the base BC is equal to the base EF [Prop. 4. El. I.]. And because the angle at A is equal to the angle at D , therefore the Segment BAC is like to the Segment EDF : and they are upon equal strait lines BC , EF . But upon equal strait lines like Segments of Circles are equal to one another [Prop. 24. El. III.]; therefore the Segment BAC , is equal to the Segment EDF . But also the whole Circle ABC , is equal to the whole Circle DEF ; wherefore the remaining Segment BKC , is equal to the remaining Segment ELF , and therefore the circumference BKC , is equal to the circumference ELF .

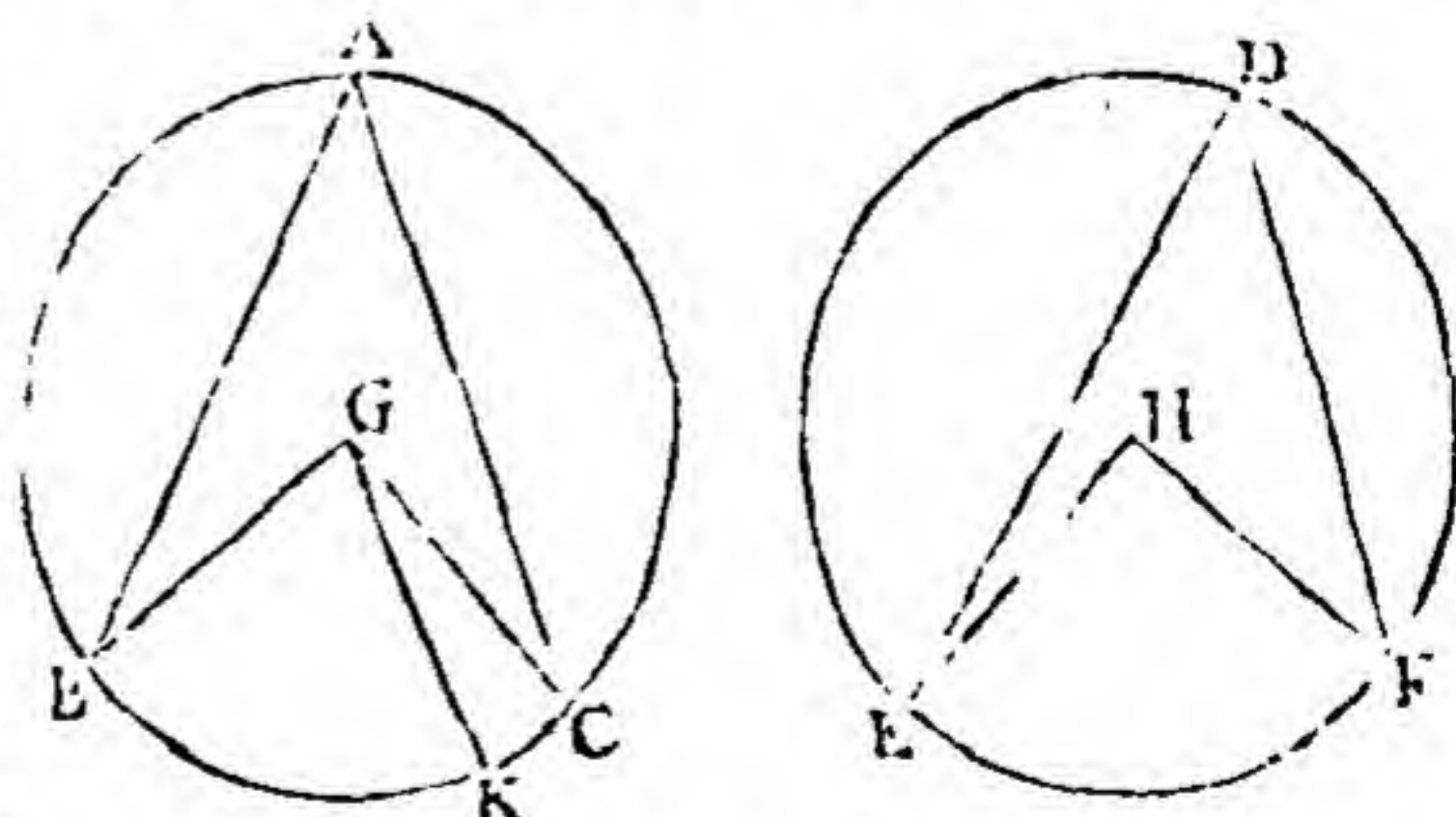


Wherefore in equal Circles, equal angles insist upon equal circumferences, whether the insisting angles be at the centers, or at the circumferences. Which was to be demonstrated.

PROPOSITION XXVII.

IN equal Circles, angles insisting upon equal circumferences are equal to one another; whether the insisting angles be at the centers, or at the circumferences.

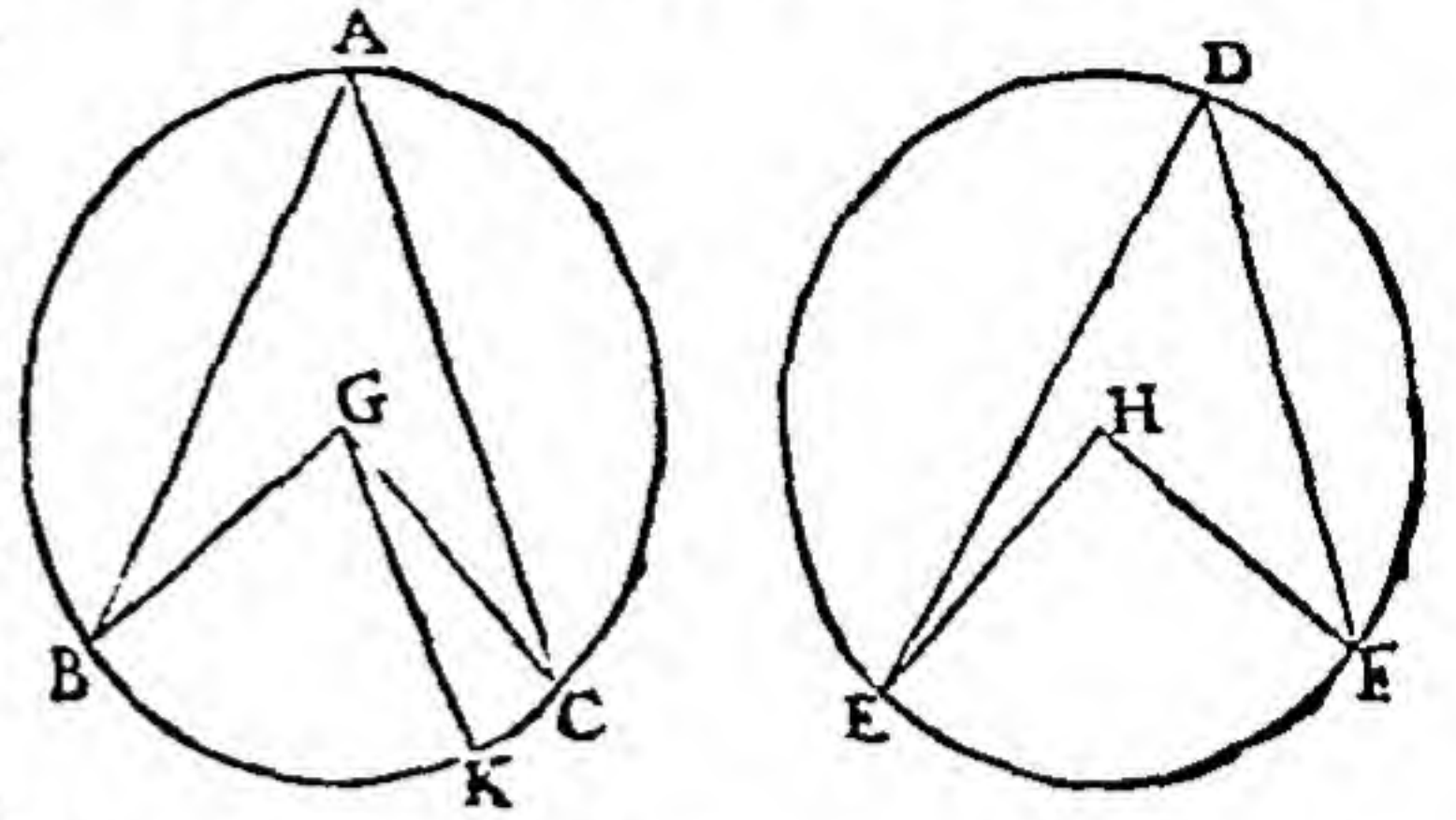
For in the equal Circles ABC , DEF , and upon the equal circumferences BC , EF ; let insist the angles BGC , EHF , at the centers G , H ; and at the circumferences the angles BAC , EDF . I say, that the angle BGC is equal to the angle EHF , and the angle BAC , to the angle EDF . For if the angle BGC be equal to the angle EHF , it is manifest that also the angle BAC , is equal to the angle EDF [Prop. 20. El. III.]. But if not, one of them is the greater.



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Let the greater be BGC ; and to the line BG , and in the same to the point G , let be constituted the angle BGK equal to the angle EHF [Prop. 23. El. I.]. But equal angles insist upon equal circumferences, when they are at the center [Prop. 26. El. III.]; therefore the circumference BK is equal to the circumference EF . But EF is equal to BC [by Supposition]; therefore also BK is equal to BC ; the less to the greater, which is impossible; wherefore the angle BGC , is not unequal to the angle EHF ; therefore it is e-



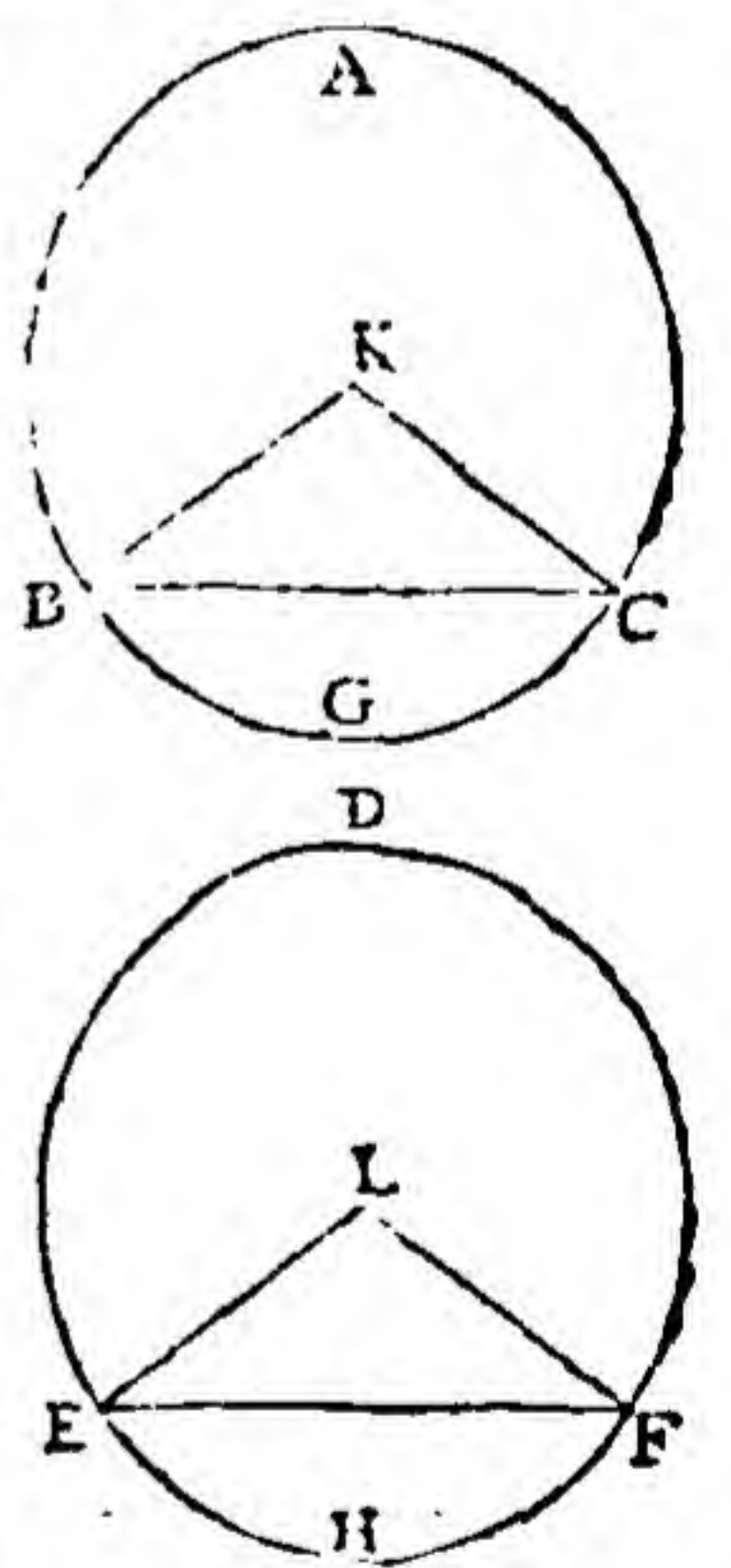
qual. Now the angle at A is the half of the angle BGC , and the angle at D half of the angle EHF [Prop. 20. El. III.]; wherefore the angle at A is equal to the angle at D [Ax. 7.].

Therefore in equal Circles, angles which insist upon equal circumferences are equal to one another, whether the insisting angles be at the centers, or at the circumferences. Which was to be demonstrated.

PROPOSITION XXVIII.

In equal Circles equal strait lines take off equal circumferences, the greater equal to the greater, the less to the less.

Let the equal Circles be ABC , DEF ; and in the same let the equal strait lines be BC , EF , taking off the greater circumferences BAC , EDF , and the less BGC , EHF . I say, that BAC the greater circumference, is equal to EDF the greater circumference, and BGD the less circumference, is equal to EHF the less. For let the centers of the Circles be taken K , L ; and let be joyned KB , KC ; LE , LF . Now forasmuch as the Circles are equal, therefore the lines from the centers are equal. There are then the two lines BK , KC , equal to the two lines EL , LF , and the base BC equal to the base EF ; therefore the angle BKC is equal to the angle ELF [Prop. 8. El. I.]. Now equal angles insist upon equal circumferences when they are at the centers [Prop. 26. El. III.]: wherefore the circumference BGC , is equal to the circumference EHF . But also the whole Circle ABC , is equal to the whole Circle DEF ; wherefore the remaining circumference BAC , is equal to the remaining circumference EDF .



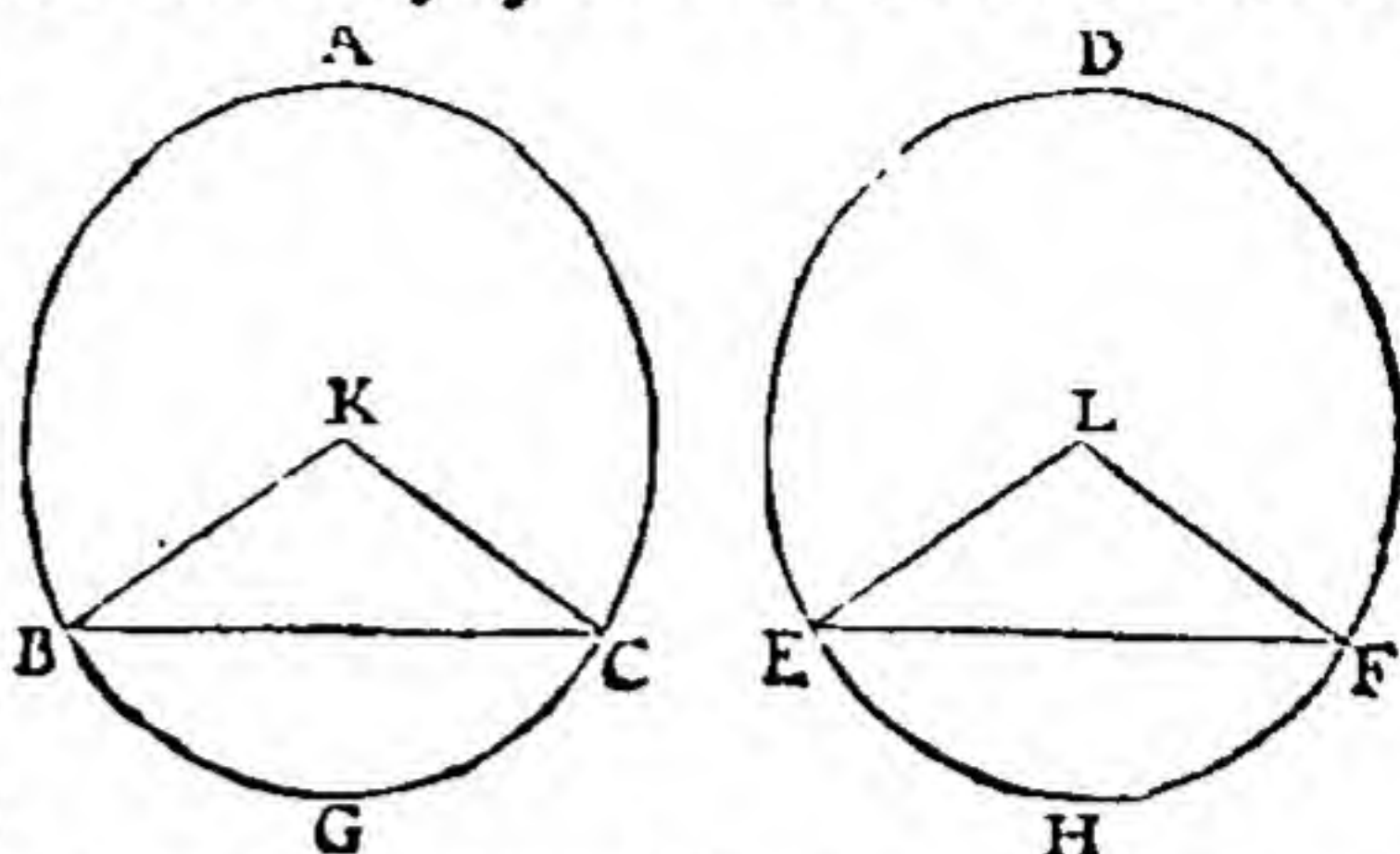
Therefore in equal Circles equal strait lines take off equal circumferences, the greater equal to the greater, the less to the less. Which was to be demonstrated.

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PROPOSITION XXIX.

IN equal Circles, under equal circumferences are subtended equal straight lines.

Let the equal Circles be ABC , DEF , and in the same let be taken equal circumferences BGC , EHF : and let be joyned the straight lines BC , EF . I say, that BC is equal to EF . Let the centers of the Circles be taken K , L , and let be joyned KB , KC ; LE , LF . Now forasmuch as the circumference BGC , is equal to the circumference EHF ; therefore the angle BKC , is equal to the angle ELF [Prop. 27. El. III.]. And because the Circles ABC , DEF , are equal, therefore the lines from the centers are equal. There are then the two lines BK , KC , equal to the two lines EL , LF , and they contain equal angles, wherefore the base BC , is equal to the base EF [Prop. 4. El. I.].

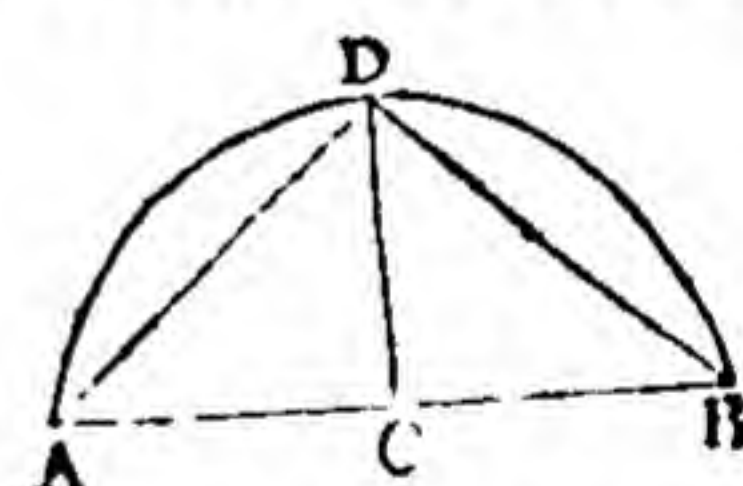


Therefore in equal Circles, under equal circumferences are subtended equal straight lines. Which was to be demonstrated.

PROPOSITION XXX.

TO cut a given circumference into halves.

Let the given circumference be ADB ; it is required to cut the circumference ADB into halves. Let AB be joyned, and cut into halves in the point C , and from the point C to the line AB , let be drawn at right angles CD , and let be joyned AD , DB . Now forasmuch as AC is equal to CB , and CD common: therefore there are the two lines AC , CD , equal to the two lines BC , CD , and the angle ACD equal to the angle BCD , for each of them is a right angle; therefore the base AD is equal to the base DB [Prop. 4. El. I.]. Now equal straight lines take off equal circumferences, the greater to the greater, the less to the less [Prop. 28. El. III.]; and each of the circumferences AD , DB , is less than a Semicircle; wherefore the circumference AD is equal to the circumference DB .



Therefore the given circumference is cut into halves. Which was to be demonstrated.

PROPOSITION XXXI.

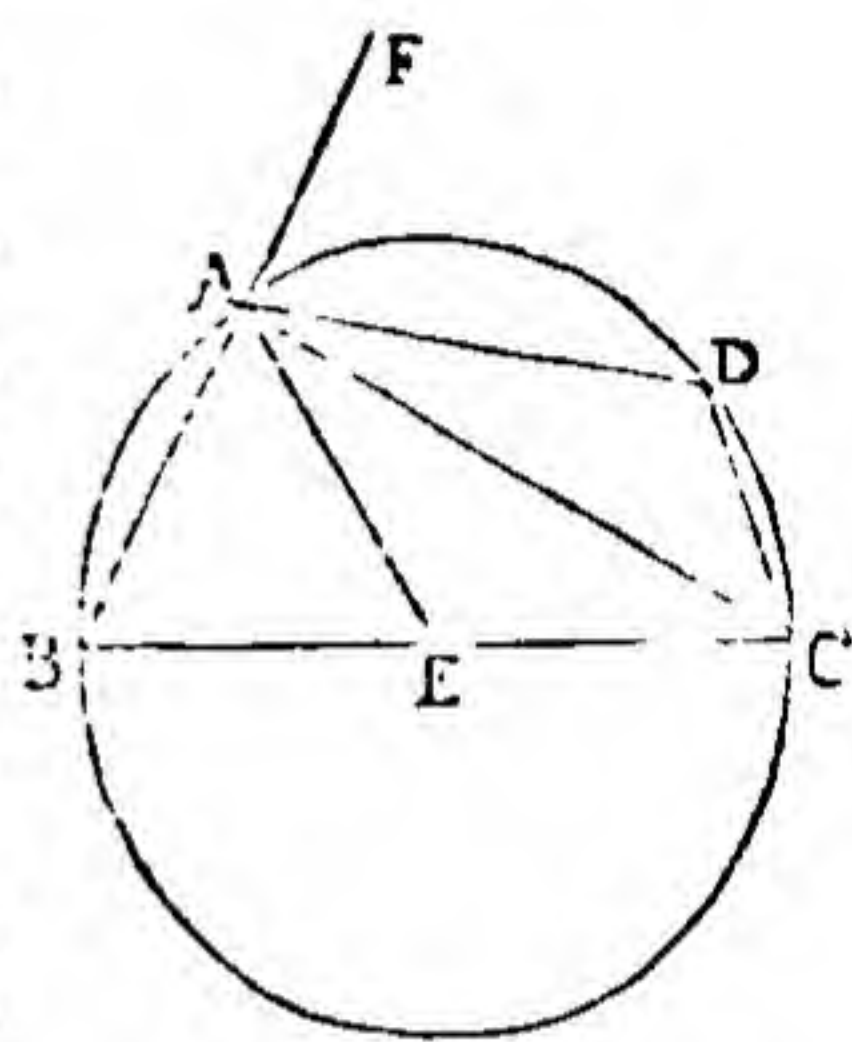
IN a Circle the angle in the Semicircle is a right angle. But the angle in the greater Segment is less than a right: And the angle in the less Segment is greater than a right angle. And moreover the angle of the greater Segment is greater than a right angle: and the angle of the less Segment is less than a right angle.

Let the Circle be $ABCD$, and the Diameter thereof be BC , and

the center E. Then taking any point in the circumference as A, let be joyned BA, AC, AD, DC. I say, that the angle in the Semicircle BAC is a right angle: And the angle in the Segment ABC greater than the Semicircle, namely, the angle ABC is less than a right angle: And the angle in the Segment ADC less than the Semicircle, namely, the angle ADC is greater than a right angle. Let AE be joyned, and BA be produced to F. Now forasmuch as BE is equal to EA, therefore the angle EAB is equal to the angle EBA [Prop. 5. El. I.]. Again, because EA is equal to EC, therefore the angle ACE is equal to the angle CAE: wherefore the whole angle BAC is equal to the two angles ABC, ACB. But the angle FAC without the Triangle ABC, is also equal to the two angles ABC, ACB [Prop. 32. El. I.]; wherefore the angle BAC is equal to the angle FAC, each therefore of them is a right angle [Def. 10. El. I.]. Therefore in the Semicircle BAC the angle CAB is a right angle.

And because of the Triangle ABC the two angles ABC, BAC, are less than two right [Prop. 17. El. I.], and BAC is a right angle; therefore the angle ABC is less than a right angle: and it is in the Segment ABC greater than the Semicircle.

And because in a Circle the Figure ABCD is quadrilateral, and of quadrilateral Figures in Circles, the opposite angles are equal to two right angles [Prop. 22. El. III.]. Therefore the angles ABC, ADC, are equal to two right angles, and ABC is less than a right angle; therefore the remaining angle ADC, is greater than a right angle: and it is in the Segment ADC less than the Semicircle.



I say, moreover, that the angle of the greater Segment contained by the circumference ABC, and the strait line AC is greater than a right angle: And the angle of the less Segment contained by the circumference ADC and the strait line AC, is less than a right angle. This is of it self very manifest. For because the angle contain'd by the strait lines CA, AB, is a right angle, therefore the angle contain'd by the strait line CA, and the circumference ABC, is greater than a right angle. Again, because the angle contain'd by the strait lines CA, AF, is a right angle, therefore the angle contain'd by the strait line CA, and the circumference ADC, is less than a right angle. In a Circle therefore the angle in the Semicircle is a right angle: but the angle in the greater Segment is less than a right angle: and the angle in the less Segment is greater than a right angle. And moreover the angle of the greater Segment is greater than a right angle; and the angle of the less Segment, is less than a right angle.

Otherwise.

That the angle BAC is a right angle. Because the angle AEC is double of the angle BAE, for it is equal to the two inward and opposite

posite angles [Prop. 32. El. I.]: and also the angle AEB is double of the angle EAC [Prop. 32. El. I.]; therefore the angles AEB, AEC are double of the angle BAC . But the angles AEB, AEC , are equal to two right angles [Prop. 13. El. I.]; therefore the angle BAC is a right angle. Which was to be demonstrated.

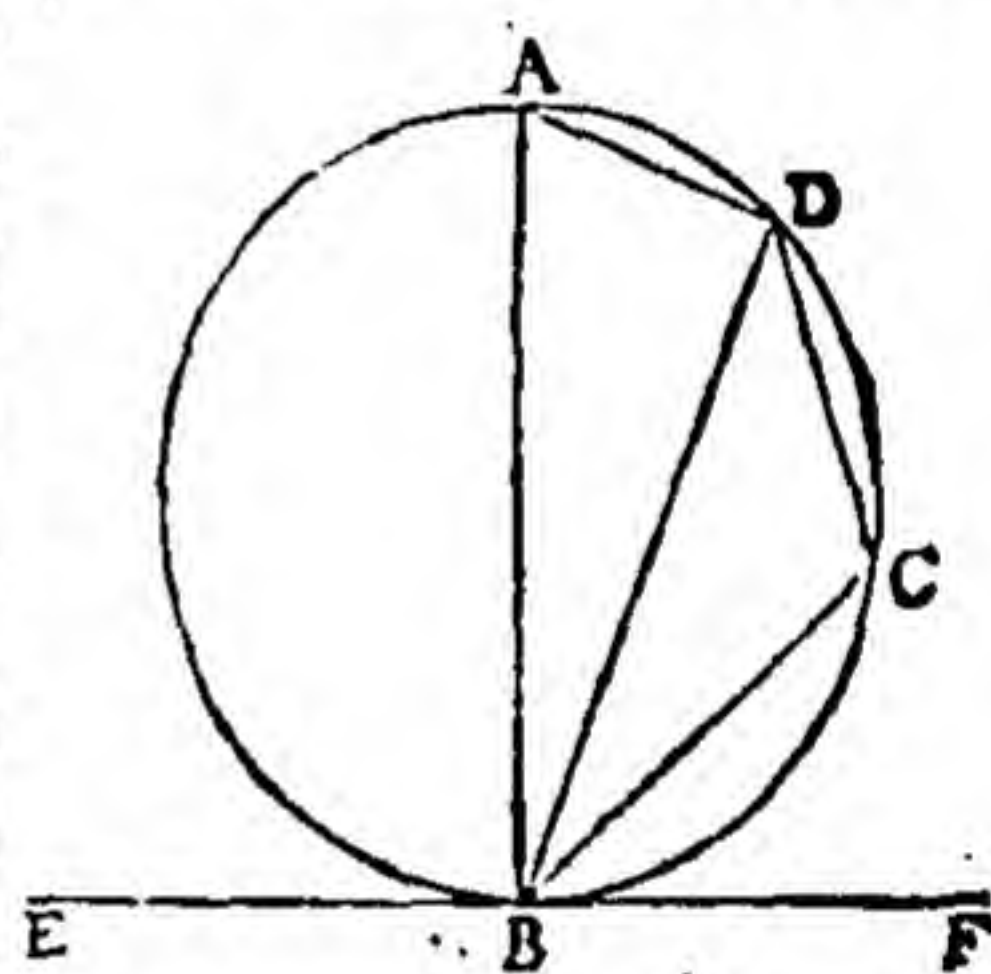
Corollary.

From hence it is manifest, that if of a Triangle one angle be equal to two it is a right angle: For that its consequent angle is equal to the same; and when consequent angles are equal, they are right angles.

PROPOSITION XXXII.

IF a strait line touches a Circle, and from the Contact to the Circle be drawn a strait line cutting the Circle, the angles which it makes with the Tangent line, shall be equal to the angles in the alternate Segments of the Circle.

Let the strait line EF touch the Circle $ABCD$ in the point B : and from B to the Circle $ABCD$ let be drawn any strait line as BD cutting the Circle. I say, that the angles which the line BD makes with the Tangent line EF , shall be equal to the angles in the alternate Segments of the Circle, that is, the angle FBD is equal to the angle constituted in the Segment DAB : and the angle EBD is equal to the angle in the Segment DCB . From the point B to the line EF , let be drawn at right angles the line BA ; and in the circumference BD , let be taken any point, as C , and let be joyned AD, DC, CB . Now forasmuch as the strait line EF touches the circle $ABCD$, in the point B ; and from the Contact at B is drawn the strait line BA at right angles to the Tangent, the center of the Circle $ABCD$ is in BA [Prop. 19. El. III.]; therefore BA is the Diameter of the Circle $ABCD$; and the angle ADB in the Semicircle is a right angle [Prop. 31. El. III.]; therefore the remaining angles BAD, ABD , are equal to one right angle. But the angle ABF is a right angle; wherefore the angle ABF is equal to the angles BAD, ABD . Let the common angle ABD be taken away: therefore the remaining angle DBF is equal to the angle BAD in the alternate Segment of the Circle. And because in a Circle the Figure $ABCD$ is quadrilateral, therefore the opposite angles are equal to two right angles [Prop. 22. El. III.]: wherefore the angles DBF, DBE , are equal to the angles BAD, BCD , of which BAD has been prov'd equal to DBF ; therefore the remaining angle DBE , is equal to the angle DCB in the alternate Segment of the Circle.



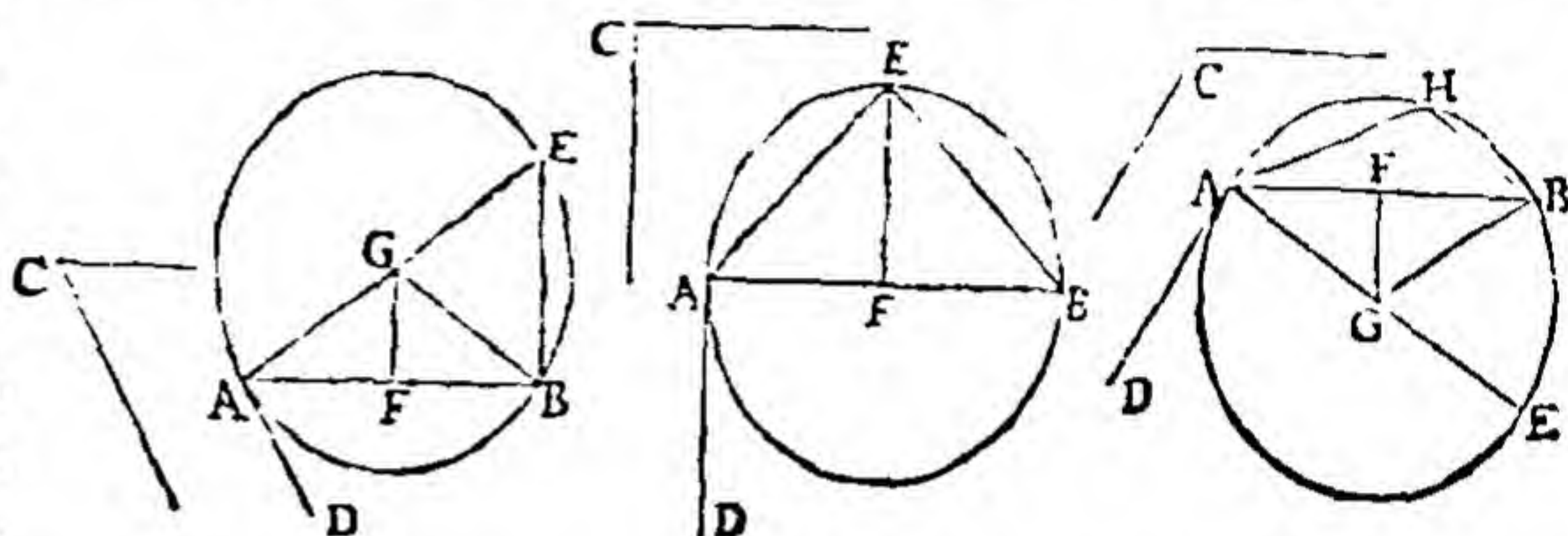
If therefore a strait line touches a Circle, and from the Contact to the Circle be drawn a strait line cutting the Circle, the angles which it makes with the Tangent line, shall be equal to the angles in the alternate Segments of the Circle. Which was to be demonstrated.

PROPOSITION XXXIII.

UPon a given strait line to describe a Segment of a Circle, which may receive an angle equal to a given strait-lin'd angle.

Let the given strait line be AB , and the given strait-lin'd angle be at c . It is required upon the strait line AB to describe a Segment of a Circle, which may receive an angle equal to the angle at c . Now the angle at c , is either an Acute, or a Right, or an Obtuse angle.

First, let it be Acute, as in the first Figure. And to the strait line AB , and to the point A , let the angle BAD be constituted equal to the angle c [by Prop. 23. El. I.]; therefore the angle BAD is an Acute angle. Now from the point A to AD , let AE be drawn at right angles; and let AB be cut into halves in the point F [by Prop. 11. El. I.]: then from the point F to AB let FG be drawn at right angles, and let GB be joyned. Now forasmuch as AF is equal to FB , and FG common; therefore there are the two lines



AF , FG , equal to the two lines BF , FG , and the angle AFG , is equal to the angle GFB ; wherefore the base AG , is equal to the base BG [Prop. 4. El. I.]; therefore the center G , and distance GA , a Circle described shall pass also by B . Let it be described, and be it ABE , and let be joyn'd EB . Now forasmuch as from the extremity of the Diameter AE , namely from the point A to AE , is drawn at right angles AD , therefore AD does touch the Circle [Prop. 16. El. III.]. And because the strait line AD touches the Circle ABE , and from the Contact at A to the Circle ABE , is drawn the line AB ; therefore the angle DAB , is equal to the angle AEB in the alternate Segment. But the angle DAB , is equal to the angle at c [by Construction]; wherefore the angle at c is equal to the angle AEB . Therefore upon the given strait line AB is described a Segment of a Circle ABE , receiving an angle AEB , equal to the given angle at c .

But now let the angle at c be a right angle. And again, let it be required upon the strait line AB , to describe a Segment of a Circle, which may receive an angle equal to the right angle at c .

Let again the angle BAD be constituted equal to the right angle at c , as in the second Figure. And let AB be cut into halves in the point F ; then from the center F , and to the distance of either FA , or FB , let the Circle ABE be describ'd; therefore the strait line AD touches the Circle ABE , for that the angle at A is a right angle; and the angle BAD is equal to the angle in the Segment ABE : for being in a Semicircle, it is also a right angle. But the angle BAD is equal

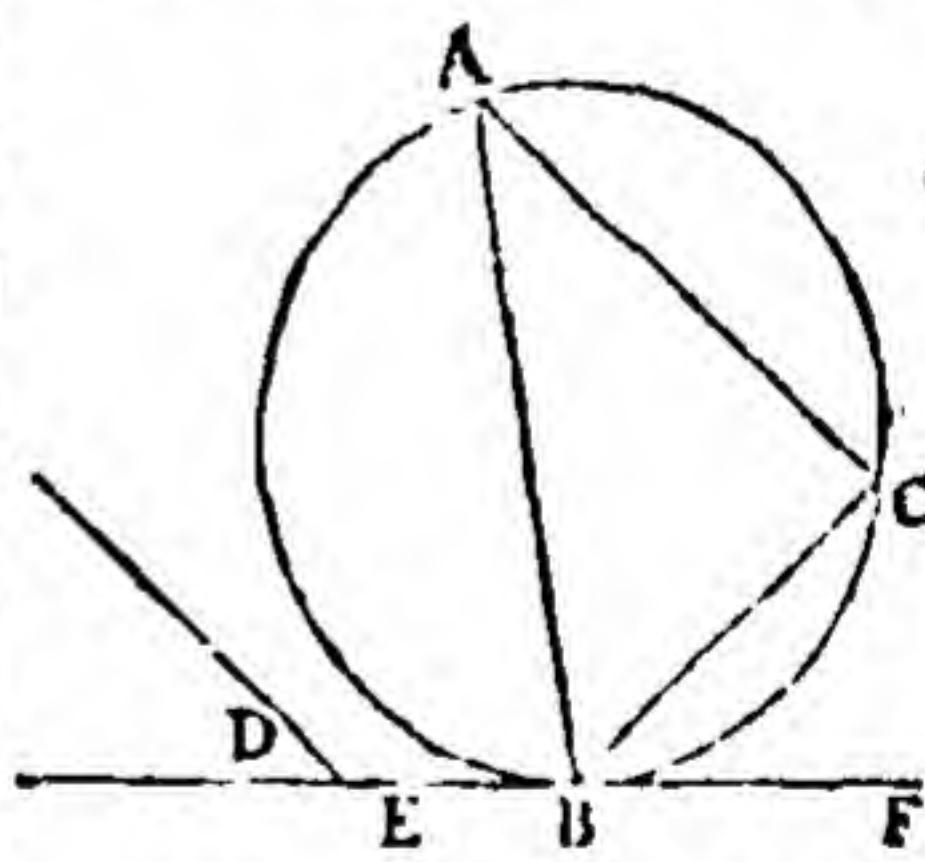
equal to the angle at c ; wherefore the angle AEB in the Segment, is equal to the angle at c . Therefore again upon the strait line AB , is described a Segment of a Circle AEB , receiving an angle equal to the angle at c .

But again, let the angle at c be an Obtuse angle, and to the strait line AB , and to the point A , let the angle BAD be constituted equal to c , as in the third Figure. Then to AD let AE be drawn at right angles; and again, let AB be cut into halves in the point F ; and from F let FG be drawn at right angles to AB , and let GB be joyned. Because again, AF is equal to FB , and FG common; therefore there are the two lines AF, FG , equal to the two lines BF, FG , and the angle AFG is equal to the angle BFG ; wherefore the base AG , is equal to the base BG : therefore the center G , and distance GA , a Circle described shall pass also by B . Let it pass as AEB . Now because to the Diameter AE , and from the extremity thereof is drawn at right angles AD , therefore AD does touch the Circle AEB , and from the Contact at A is drawn AB ; therefore the angle BAD is equal to the angle constituted in the alternate Segment of the Circle AHB . But the angle BAD is equal to the angle at c ; wherefore the angle in the Segment AHB , is equal to the angle at c . Therefore upon the given strait line AB , is described a Segment of a Circle AHB , receiving an angle equal to the angle at c . Which was to be done.

PROPOSITION XXXIV.

From a given Circle to take off a Segment, which may receive an angle equal to a given strait-lin'd angle.

Let the given Circle be ABC , and the given strait-lin'd angle be at D . It is required from the Circle ABC , to take off a Segment, which may receive an angle equal to the angle at D . Let be drawn EF touching the Circle ABC in the point B [Prop. 17. El. III.], and to the strait line EF , and to the point in it B , let the angle FBC be constituted equal to the angle at D [Prop. 31. El. I.]. Now forasmuch as the strait line EF touches the Circle ABC , in the point B ; and from the Contact at B is drawn the line BC : therefore the angle FBC , is equal to the angle constituted in the alternate Segment BAC . But the angle FBC is equal to the angle at D ; wherefore the angle in the Segment BAC , is equal to the angle at D .



Therefore from the given Circle ABC is taken off the Segment BAC , receiving an angle equal to the given strait-lin'd angle at D . Which was to be done.

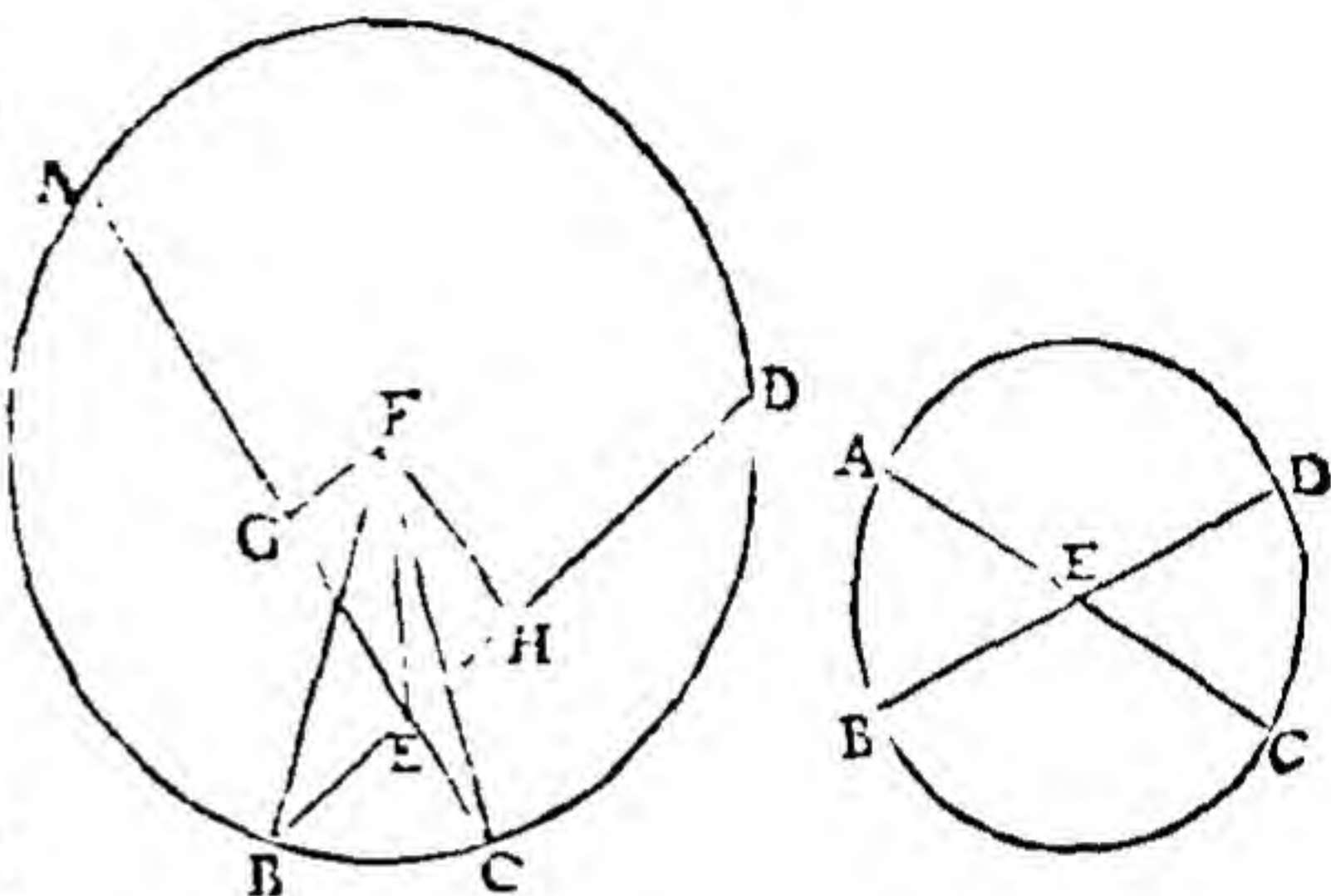
PROPO.

PROPOSITION XXXV.

I *In a Circle two strait lines cut one another, the Rectangle contained by the Segments of the one, is equal to the Rectangle contained by the Segments of the other.*

For in the Circle $ABCD$, let the two strait lines AC , BD , cut one another in the point E . I say, that the Rectangle contained by AE , EC , is equal to the Rectangle contained by DE , EB . If AC , BD , pass through the center, so that E be the center of the Circle $ABCD$, then it is manifest that the lines AE , EC ; DE , EB , being equal, the Rectangle also contained by AE , EC , is equal to the Rectangle contained by DE , EB .

But now let the strait lines AC , DB , not pass through the center. And let the center of the Circle $ABCD$ be taken, and be it F , and from F to the strait lines AC , DB , let perpendiculars be drawn FG , FH : and let be joyned FB , FC , FE . Now forasmuch as the strait line GF drawn through the center



cuts the strait line AC , not drawn through the center, at right angles, it shall also cut the same into halves [Prop. 3. El. III.]; therefore AG is equal to GC . And because the strait line AC is cut into equal parts in G , and into unequal parts in E : therefore the Rectangle contained by AE , EC , together with the square of EG is equal to the square of GC [Prop. 5. El. II.], let be added in common the square of GF : therefore the Rectangle contained by AE , EC , together with the squares of EG , GF , is equal to the squares of CG , GF . But the square of FE is equal to the squares of EG , GF [Prop. 47. El. I.]; and the square of FC is equal to the squares of CG , GF ; therefore the Rectangle contained by AE , EC , together with the square of EF , is equal to the square of FC . But FC is equal to FB ; therefore the Rectangle under AE , EC , together with the square of FE , is equal to the square of FB . By the same reason the Rectangle under DE , EB , together with the square of FE , is equal to the square of FB . But it has been proved, that the Rectangle under AE , EC , together with the square of FE , is equal to the square of FB ; therefore the Rectangle under AE , EC , together with the square of FE , is equal to the Rectangle under DE , EB , together with the square of FE . Let the square of FE common, be taken away; therefore the remaining Rectangle contained by AE , EC , is equal to the Rectangle contained by DE , EB .

If therefore in a Circle two strait lines cut one another, the Rectangle contained by the Segments of the one, is equal to the Rectangle contained by the Segments of the other. Which was to be demonstrated.

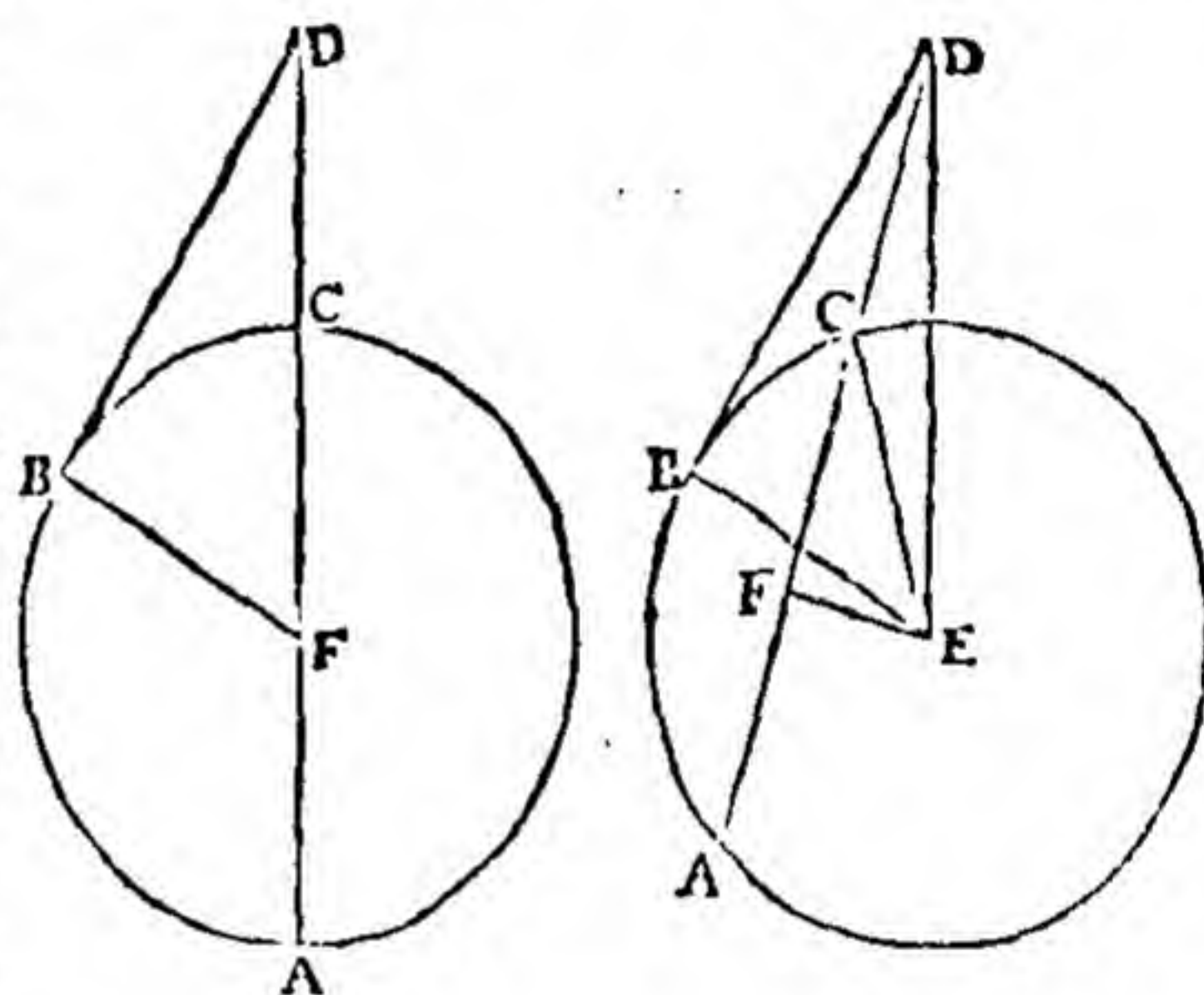
PRO.

PROPOSITION XXXVI.

If without a Circle be taken any point, and from the same do fall on the Circle two strait lines, of which one does cut the Circle, the other does touch it: The Rectangle contained by the whole Secant, and the outward Segment between the point and convex circumference, shall be equal to the square of the Tangent.

Let without the Circle ABC be taken any point as D , and from D let the two strait lines DCA , DB , fall on the Circle ABC : and let DCA cut the Circle ABC , and DB touch it. I say, that the Rectangle contained by AD , DC , is equal to the square of DB . Now DCA either passes through the center, or not.

First, let it pass through the center, and let F be the center of the Circle ABC . Then let be joyned FB ; therefore the angle FBD is a right angle [Prop. 18. El. III.]. Now forasmuch as the strait line AC is cut into halves in F , and to it is added CD ; therefore the Rectangle contained by AD , DC , together with the square of FC is equal to the square of FD [Prop. 6. El. II.]. But FC is equal to FB ; therefore the Rectangle of AD , DC , together with the square of FB , is equal to the square of FD . But the square of FD is equal to the squares of FB , BD , for the angle FBD , is a right angle: therefore the Rectangle contained by AD , DC , together with the square of FB , is equal to the squares of FB , BD . Let the square of FB common be taken away; therefore the remaining Rectangle under AD , DC , is equal to the square of the Tangent DB .



But now let DCA not pass through the center of the Circle ABC : and let the center E be taken, and from E to AC let be drawn a perpendicular EF ; and let be joyned EB , EC , ED . Now the angle EBD is a right angle [Prop. 18. El. III.]. And forasmuch as the strait line EF drawn through the center cuts the strait line AC , not drawn through the center at right angles; it shall also cut the same into halves [Prop. 3. El. III.]; therefore AF is equal to FC . And because the strait line AC is cut into halves in F , and to it is added CD , therefore the Rectangle contained by AD , DC , together with the square of FC , is equal to the square of FD [Prop. 6. El. II.]. Let be added in common the square of EF ; therefore the Rectangle under AD , DC , together with the squares of CF , FE , is equal to the squares of DF , FE . But the square of DE is equal to the squares of DF , FE , for EDF is a right angle; and the square of CE is equal to the squares of CF , FE : Therefore the Rectangle contain'd by AD , DC , together with the square of CE , is equal to the

A a

square

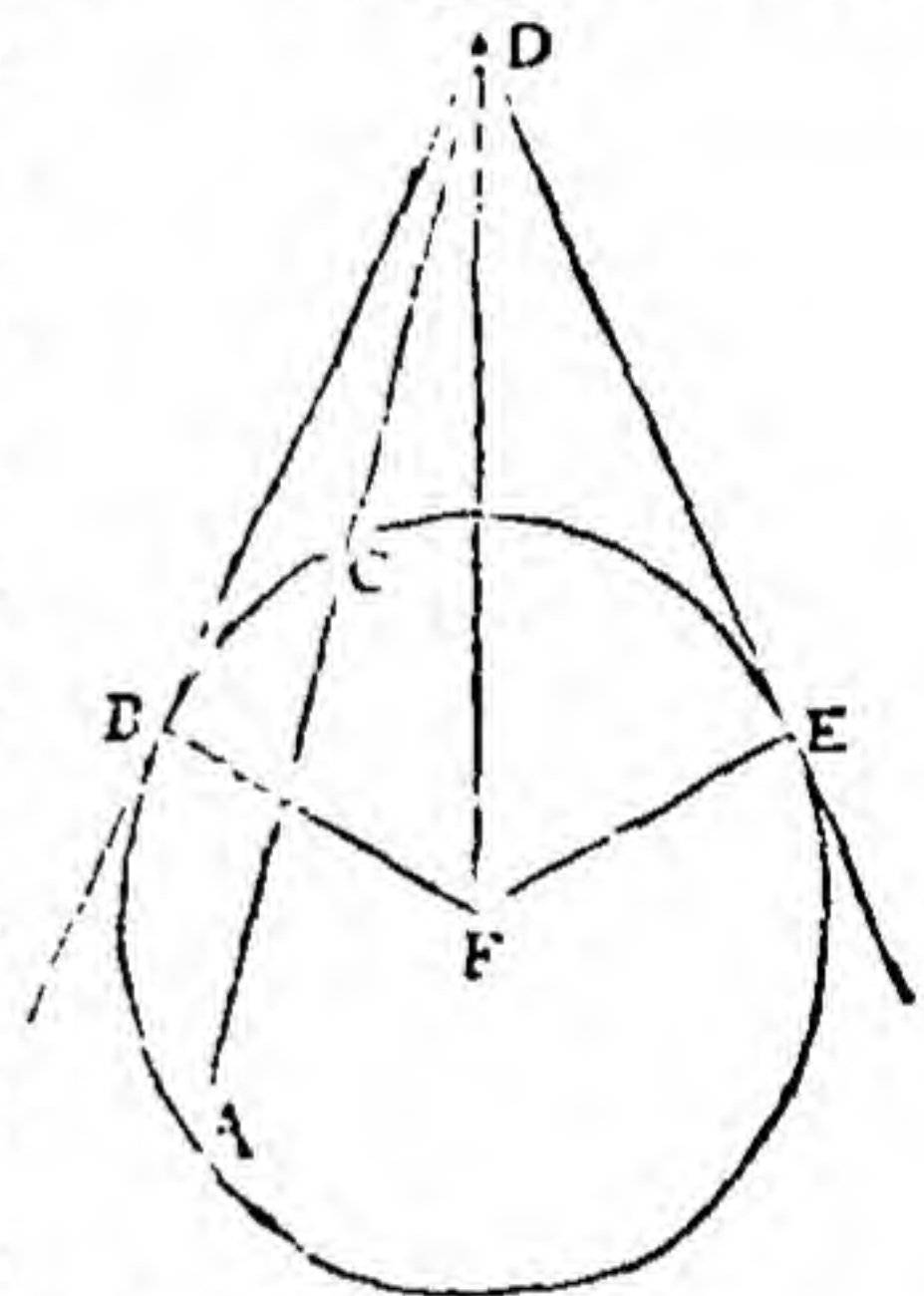
square of ED . But CE is equal to EB ; therefore the Rectangle under AD, DC , together with the square of EB , is equal to the square of ED . But the squares of EB, BD , are equal to the square of ED , for the angle EBD is a right angle: therefore the Rectangle contained by AD, DC , together with the square of EB , is equal to the squares of EB, BD . Let the square of EB common be taken away; therefore the Rectangle contained by AD, DC , is equal to the square of DB .

If therefore without a Circle be taken any point, &c. Which was to be demonstrated.

PROPOSITION XXXVII.

If without a Circle be taken any point, and from the same do fall upon the Circle two strait lines; of which one does cut the Circle; the other does fall upon it: and the Rectangle contained by the whole Secant and the outward Segment between the point and the convex circumference, be equal to the square of the incident line; the incident line shall touch the Circle.

Let without the Circle ABC be taken any point as D ; and from D let the two strait lines DCA, DB , fall upon the Circle ABC , and let DCA cut the Circle, and DB fall upon it: Also let the Rectangle contained by AD, DC be equal to the square of DB . I say, that DB touches the Circle ABC . For let the strait line DE be drawn touching the Circle ABC [Prop. 17. El. III.]: and let be taken F the center of the Circle ABC ; then let be joyned FE, FB, FD . Now the angle FED is a right angle [Prop. 18. El. III.]: And forasmuch as DE touches the Circle ABC , and DCA cuts it; therefore the Rectangle under AD, DC , is equal to the square of DE . But the Rectangle under AD, DC , is put equal to the square of DB ; wherefore the square of DE , is equal to the square of DB , and therefore DE is equal to DB . But FE is equal to FB ; there are then the two lines DE, FE , equal to the two lines DB, BF , and the base FD is common; therefore the angle DEF is equal to the angle DBF [Prop. 8. El. I.]. But DEF is a right angle, therefore DBF is also a right angle. Now BF being produced is the Diameter, but a strait line drawn at right angles to the Diameter, from the extremity thereof touches the Circle ABC . In like manner the same shall be demonstrated, if the center were in AC it self.



If therefore without a Circle be taken any point, &c. Which was to be demonstrated.

THE FOURTH ELEMENT.

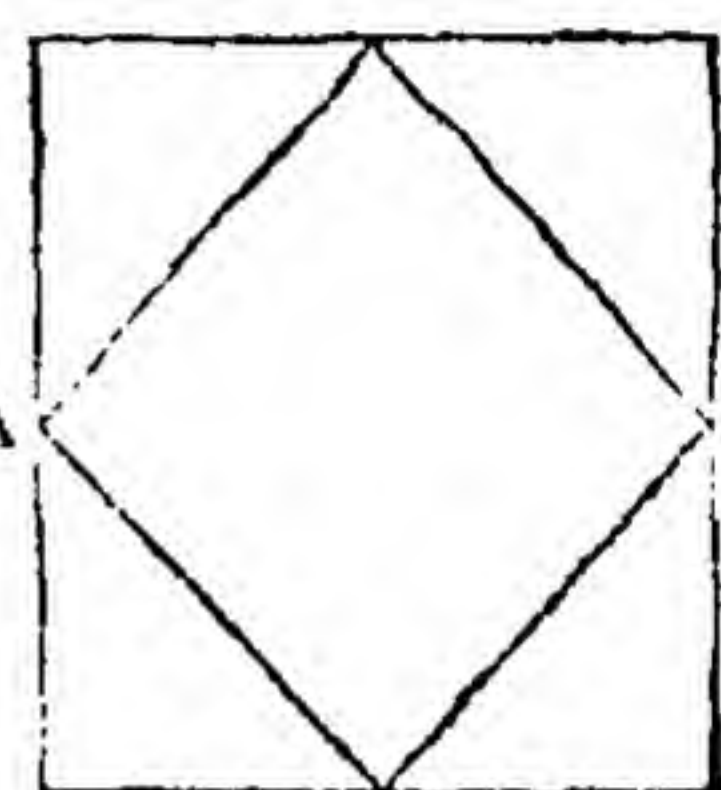
DEFINITIONS.

DEFINITION I.

A *Strait-lin'd Figure is said to be inscribed in a strait-lin'd Figure, when every angle of the inscribed Figure touches every side of the Figure, in which it is inscribed.*

DEFINITION II.

L *ikewise a Strait-lin'd Figure is said to be circumscribed about a strait-lin'd Figure, when every side of the circumscribed Figure touches every angle of the Figure, about which it is circumscribed.*



DEFINITION III.

A *Strait-lin'd Figure is said to be inscribed in a Circle, when every angle of the inscribed Figure touches the circumference of the Circle.*



DEFINITION IV.

A *Strait-lin'd Figure is said to be circumscribed about a Circle, when every side of the circumscribed Figure touches the circumference of the Circle.*



DEFINITION V.

L *ikewise a Circle is said to be inscribed in a strait-lin'd Figure, when the circumference of the Circle touches every side of the Figure, in which it is inscribed.*

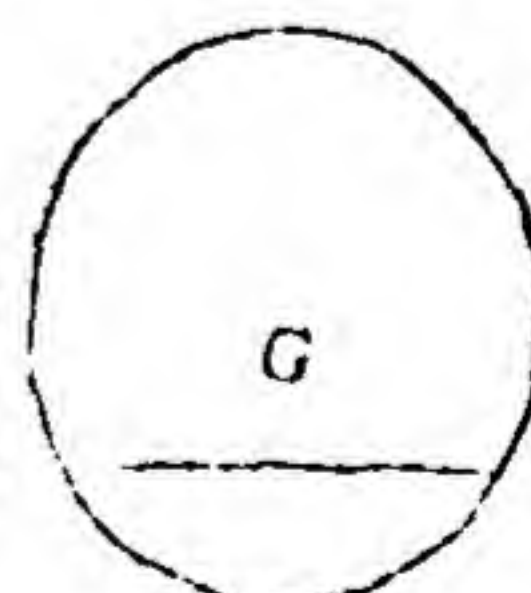
A

DEFINITION VI.

A *Circle is said to be circumscribed about a strait-lin'd Figure, when the circumference of the Circle touches every angle of the Figure, about which it is circumscribed.*

DEFINITION VII.

A *Strait line is said to be adapted in a Circle, when the extremes of the line are in the circumference of the Circle.*



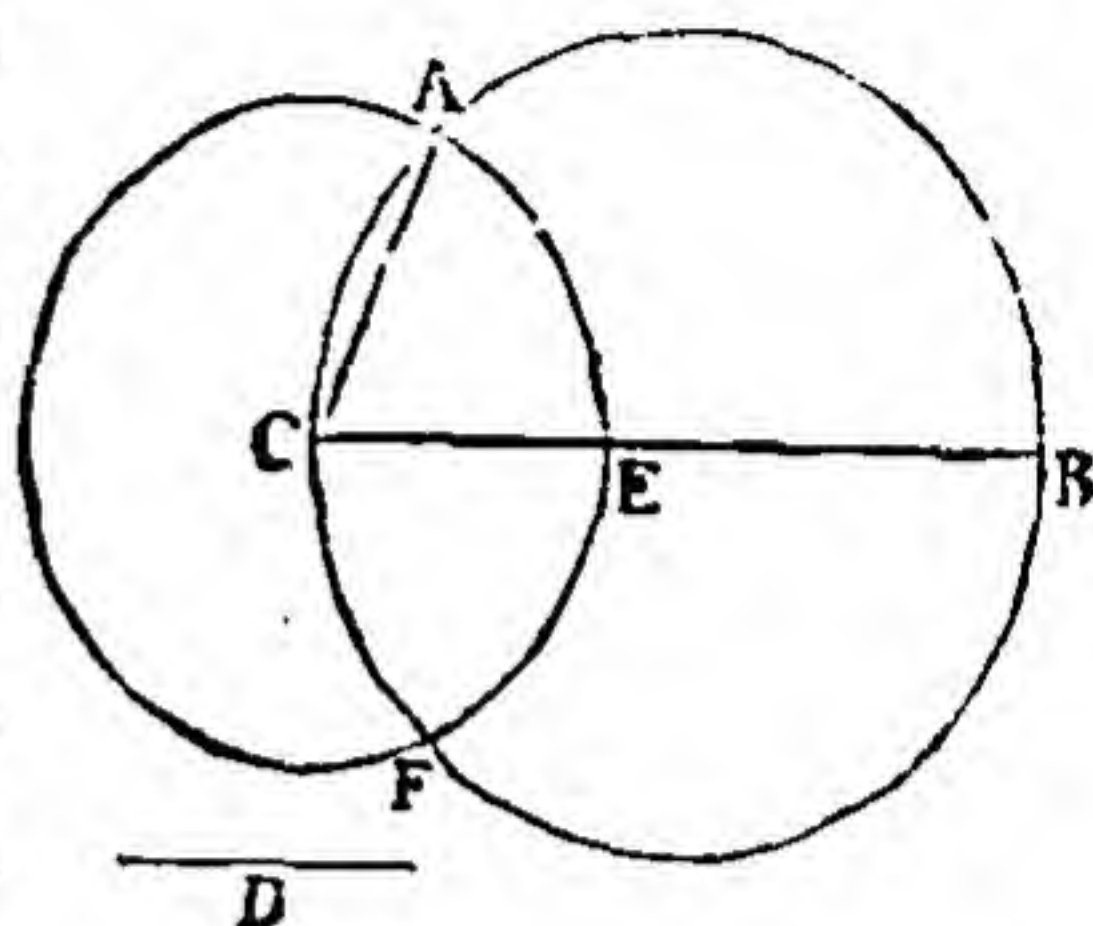
PROPOSITION I.

IN a given Circle to adapt a strait line equal to a given strait line, which is not greater than the Diameter of the Circle.

Let the given Circle be ABC , and the given strait line, which is not greater than the Diameter of the Circle, be D . It is required in the Circle ABC , to adapt a strait line equal to the strait line D .

Let there be drawn BC , the Diameter of the Circle ABC . Now if BC be equal to D , then that is done which was proposed. For in the Circle ABC is adapted the line BC , equal to the given line D .

But if not, then BC is greater than D [by Supposition]; and let there be put CE equal to D . Then to the center C , and distance CE , let the Circle EAF be described, and let CA be drawn. Now forasmuch as the point C is the center of the Circle EAF , therefore CA is equal to CE , but D is equal to CE ; wherefore also D is equal to CA .



Therefore in the given Circle ABC , is adapted a strait line CA , equal to given strait line D , which is not greater than the Diameter of the Circle. Which was to be done.

ANNOTATIONS.

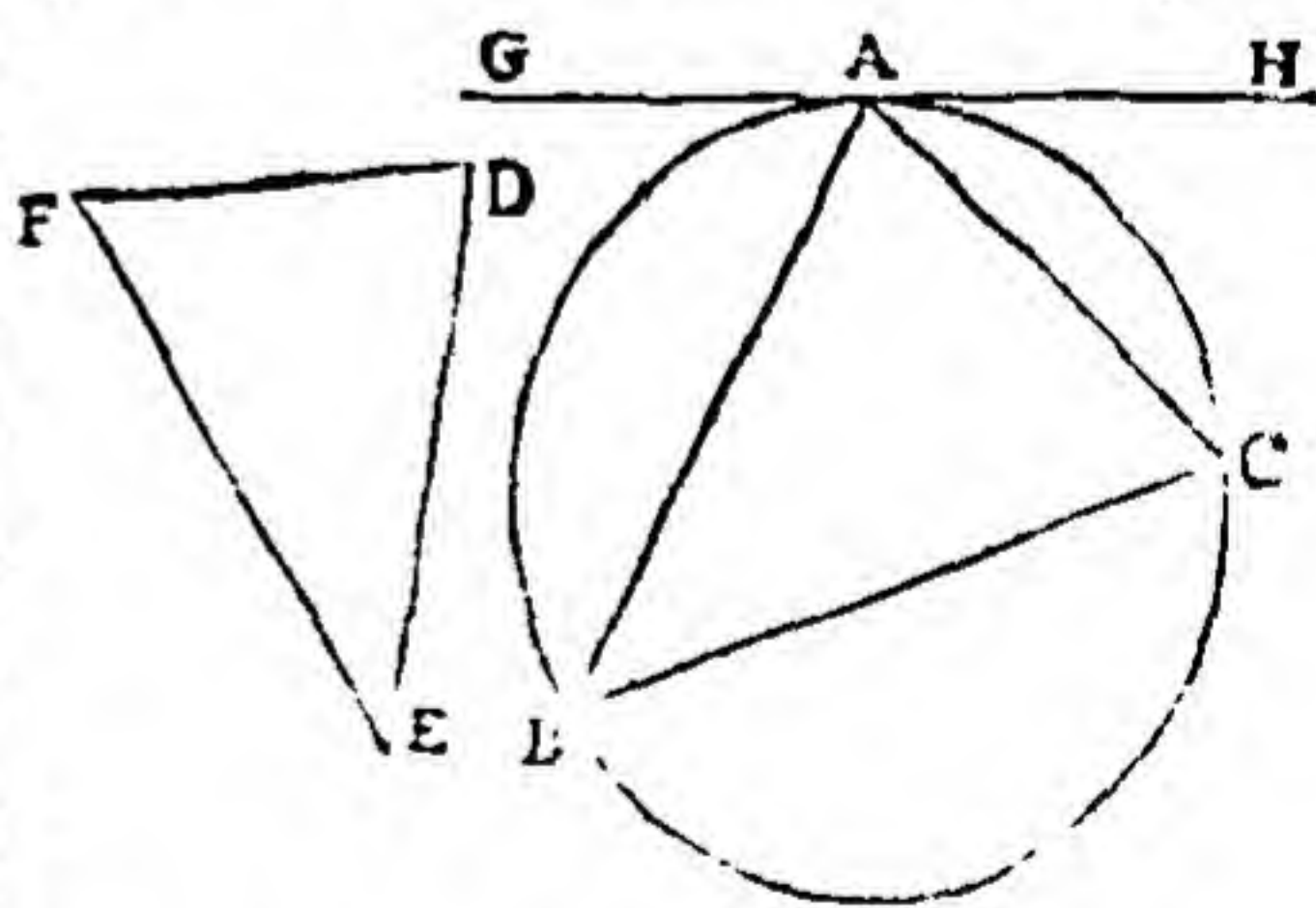
Because the Diameter is the greatest line in a Circle [Prop. 15. El. III.]; therefore this proviso, or limitation, is here made, that the given line, to which an equal line is required to be adapted in the Circle, ought not to be greater than the Diameter.

PROPOSITION II.

IN a given Circle to inscribe a Triangle equiangled to a given Triangle.

Let the given Circle be ABC , and the given Triangle DEF . It is required in the Circle ABC , to inscribe a Triangle equiangled to the Triangle DEF . Let there be drawn a strait line GAH , touching the Circle ABC , in the point A [by Prop. 17. El. III.]. Then to the line AH , and to the point in it A , let the angle HAC be constituted equal to the angle DEF [by Prop. 23. El. I.].

Again, to the line GA , and to the point in it A , let the angle GAB be constituted equal to the angle DFE ; and draw BC . Now forasmuch as a strait line HAG , touches the Circle ABC , and from the Contact is drawn AC ; therefore the angle HAC , is equal to the angle ABC , in the alternate Segment of the Circle [Prop. 32. El. III.]. But the angle HAC is equal to the angle DEF [by Construction]; therefore the angle ABC , is equal to the angle DEF . By the same reason also



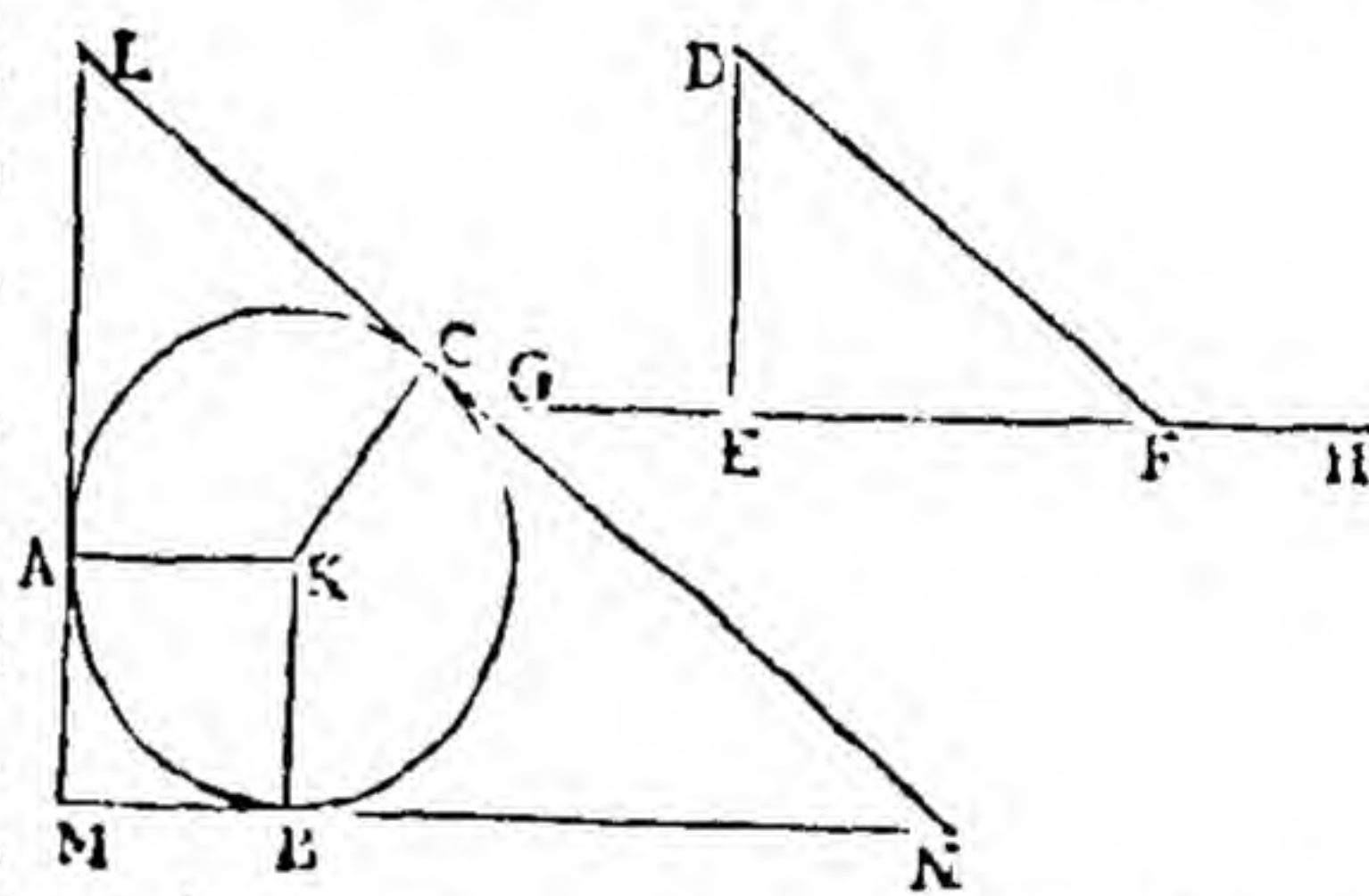
also the angle ACB , is equal to the angle DFE ; therefore the remaining angle BAC is equal to the remaining angle EDF : wherefore the Triangle ABC , is equiangled to the Triangle DEF , and is inscribed in the Circle ABC .

Therefore in a given Circle is inscribed a Triangle equiangled to a given Triangle. Which was to be done.

PROPOSITION III.

A Bout a given Circle to circumscribe a Triangle equiangled to a given Triangle.

Let the given Circle be ABC , and the given Triangle DEF . It is required about the Circle ABC , to circumscribe a Triangle equiangled to the Triangle DEF . Let EF be produced both ways to the points G, H ; and of the Circle ABC let the center K be taken [by Prop. 1. El. III.]; and let a strait line KB be drawn at pleasure. Now to the line KB , and to the point in it K , let there be constituted the angle BKA , equal to the angle DEG [by Prop. 23. El. I.], and also the angle BKC , equal to the angle DFH . Then by the points A, B, C , let there be drawn the strait lines LAM, MBN, NCL , touching the Circle ABC [by Prop. 17. El. III.]. Now forasmuch as LM, MN, NL , touch the Circle ABC , in the points A, B, C ; and from K , the center, to the points A, B, C , are drawn KA, KB, KC ; therefore the angles at the points A, B, C , are right angles [Prop. 18. El. III.]. And because the four angles of the quadrilateral Figure $AMBK$, are equal to four right angles, for that it is divided into two Triangles (by supposing a strait line drawn from K to M , making two Triangles KAM, KBM , each of which have their three angles equal to two right, Prop. 32. El. I.), of which the angles KAM, KBM , are right angles; therefore the remaining angles AKB, AMB , are equal to two right angles. But the angles DEG, DEF , are equal to two right angles [Prop. 13. El. I.]; therefore the angles AKB, AMB , are equal to the angles DEG, DEF , of which AKB is equal to DEG : wherefore the remaining angle AMB , is equal to the remaining angle DEF . In like manner may be demonstrated that the angle LMN , is equal to the angle DFE ; therefore also the remaining angle MLN , is equal to the remaining angle EDF : wherefore the Triangle LMN , is equiangled to the Triangle DEF ; and it is circumscribed about the Circle ABC .



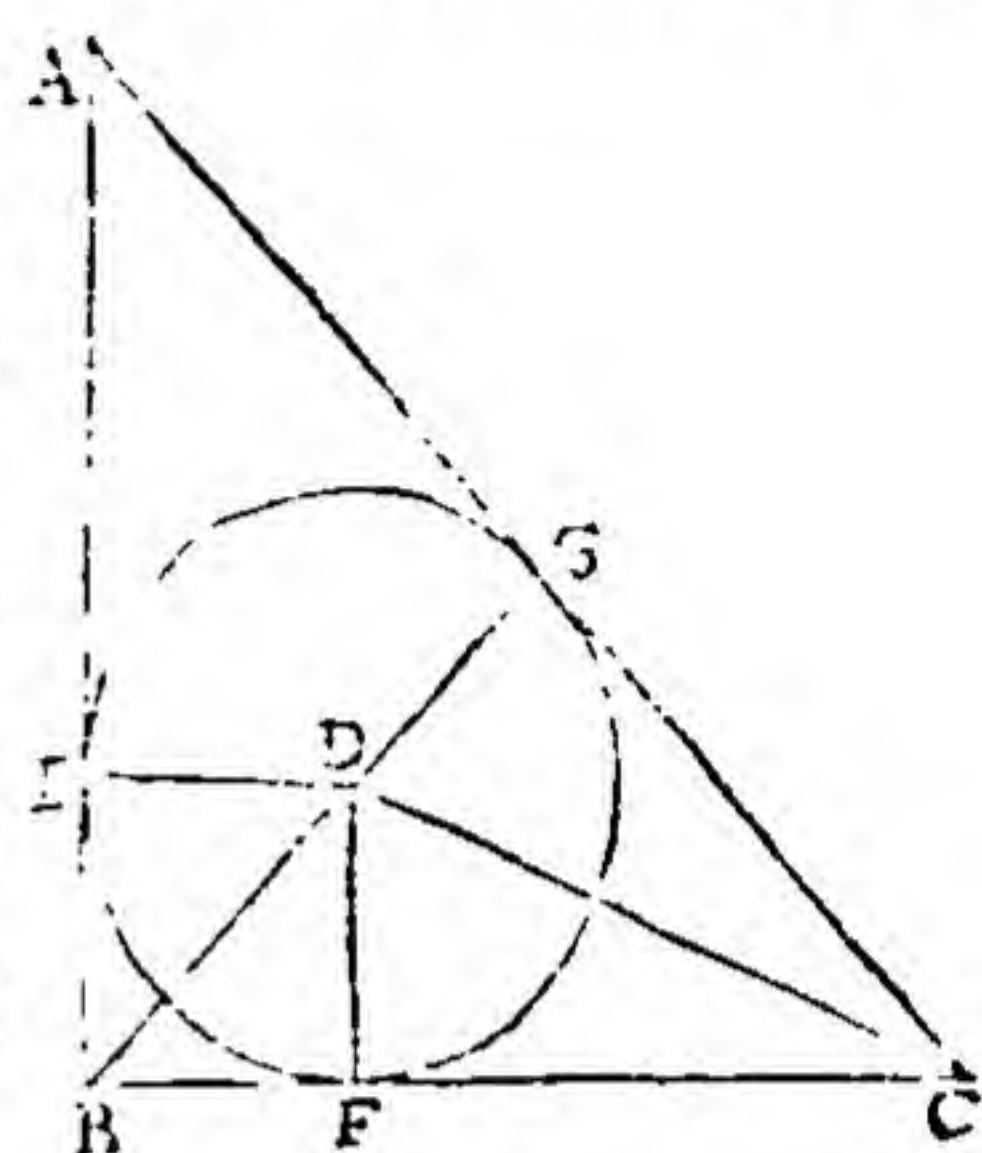
Therefore about a given Circle is circumscribed a Triangle equiangled to a given Triangle. Which was to be done.

THE FOURTH ELEMENT

PROPOSITION IV.

In a given Triangle to inscribe a Circle.

Let the given Triangle be ABC . It is required in the Triangle ABC , to inscribe a Circle. Let the angles ABC, BCA , be cut into halves by the strait lines BD, CD [by Prop. 9. El. I], and let them meet together in the point D ; and from the point D let there be drawn to the lines AB, BC, CA , the perpendiculars DE, DF, DG [by Prop. 12. El. I.]. Now forasmuch as the angle ABD is equal to the angle CBD (for that the angle ABC is cut into halves) and the right angle BED , is equal to the right angle BFD : there are then two Triangles, EBD, FBD , having two angles equal to two angles, and one side equal to one side, namely BD common to both, and subtended under one of the equal angles; therefore they shall have the remaining sides equal to the remaining sides [Prop. 26. El. I.]; wherefore DE shall be equal to DF . By the same reason DG is also equal to DF ; therefore the three lines DE, DF, DG , are equal to one another; wherefore to the center D , and the distance any one of the lines DE, DF, DG , a Circle being described, shall pass through the remaining points, and shall touch the lines AB, BC, CA , because the angles at the points E, F, G , are right. For if the Circle shall cut them, then to the Diameter of a Circle shall, from the extremity, be drawn, at right angles, a strait line falling within the Circle, which is absurd [Prop. 16. El. III.]; therefore to the center D , and distance one of the lines DE, DF, DG , a Circle being described, shall not cut the lines AB, BC, CA : wherefore it shall touch them, and there shall be a Circle inscribed in the Triangle ABC .



Therefore in the given Triangle ABC , is inscribed the Circle EFG . Which was to be done.

PROPOSITION V.

About a given Triangle to circumscribe a Circle.

Let the given Triangle be ABC . It is required about the given Triangle ABC , to circumscribe a Circle. Let AB, AC , be cut into halves in the points D, E [by Prop. 10. El. I.], and from the points D, E , let there be drawn at right angles, to AB, AC , the lines DF, EF [by Prop. 11. El. I.]. Now these lines shall meet either within the Triangle ABC , or in the line BC , or without it. First, let them meet within, at the point F , and let FB, FC, FA , be joyned. Now forasmuch as AD is equal to DB ,



DB ,

DB , and DF common, and at right angles; therefore the base AF , is equal to the base FB [Prop. 4. El. I.]. In like manner we shall demonstrate, that FC is also equal to FA , so that also BF is equal to FC ; therefore these three FA, FB, FC , are equal to one another: wherefore to the center F , and distance, any one of the lines FA, FB, FC , a Circle being described, shall pass also through the remaining points; and there shall be circumscribed a Circle about the Triangle ABC : And let it be described, as ABC .

But again, let DF, EF , meet in the line BC , at the point F , as it is in this Figure, and let AF be joyned.

In like manner we shall demonstrate, that the point F is the center of a Circle circumscribed about the Triangle ABC .

Lastly, let DF, EF , meet without the Triangle ABC , at the point F , as in this last figure; and let FA, FB, FC , be joyn'd.

Now forasmuch as AD is equal to DB , and DF common, and at right angles, therefore the base AF is equal to the base FB . In like manner we shall demonstrate, that FC is equal to FA ; so that also BF is equal to FC : therefore again to the center F , and distance any one of the lines FA, FB, FC , a Circle being described, shall pass also through the remaining points; and shall be circumscribed about the Triangle ABC : And let it be described as ABC .

Therefore about a given Triangle a Circle is circumscribed. Which was to be done.

Corollary.

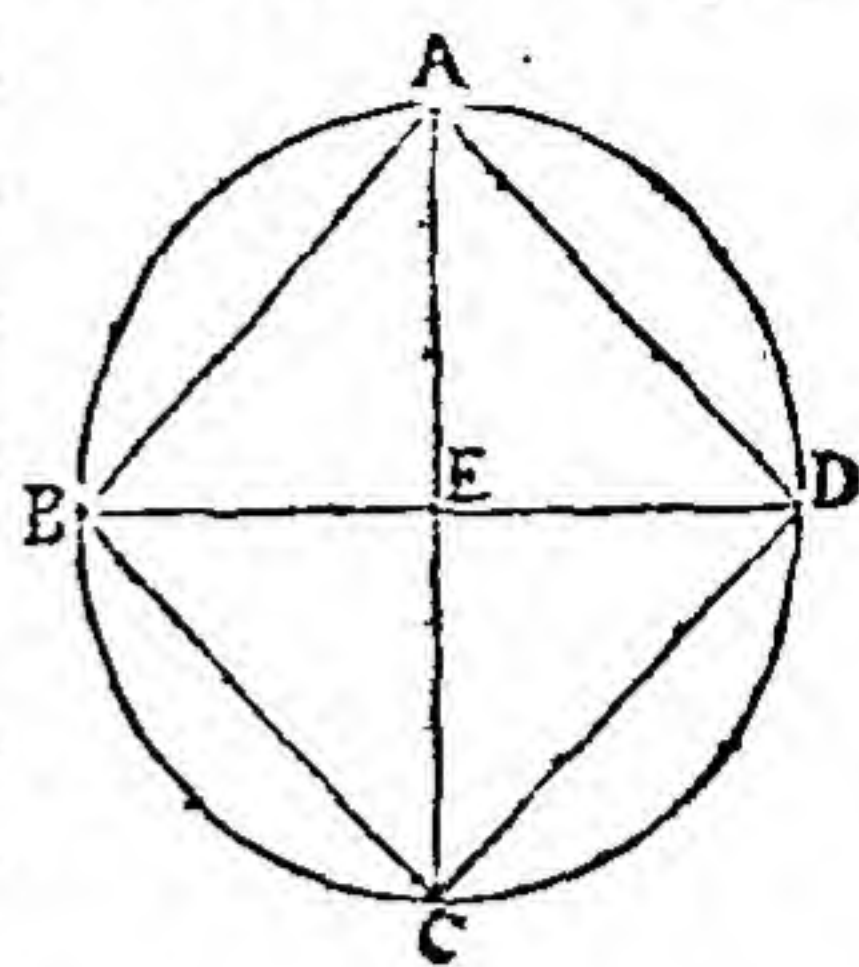
And it is manifest, that when the center of the Circle falls within the Triangle, then the angle BAC , being in a Segment greater than the Semicircle, is less than a right angle [Prop. 31. El. III.]. But when it falls in the line BC , being in the Semicircle, then BAC shall be a right angle. And when the center falls without BC , then the angle BAC , being in a Segment less than the Semicircle, is greater than a right angle. So that when the given angle is less than a right angle, then the lines DF, EF , shall fall within the Triangle. But when it is a right angle, they shall fall in BC : And when greater than a right angle, they shall fall without BC .

PROPOSITION VI.

IN a given Circle to inscribe a square.

Let the given Circle be $ABCD$. It is required in the Circle $ABCD$, to inscribe a square. Of the Circle $ABCD$, let the Diameters AC, BD , be drawn at right angles to one another; and let be joyned AB, BC, CD, DA . Now forasmuch as BE is equal to DE , for the center is E ; and EA is common, and at right angles; therefore the base AB is equal

equal to the base AD . And by the same reason either of the lines BC, CD , is equal to either of the lines AB, AD ; therefore the quadrilateral Figure $ABCD$, is equilateral. I say, that it is also Rectangular. For because the line BD is the Diameter of the Circle $ABCD$, therefore BAD is a Semicircle: wherefore the angle BAD is a right angle [Prop. 31. El. III.]. By the same reason also every one of the angles ABC, BCD, CDA , is a right angle: therefore the quadrilateral Figure $ABCD$ is rectangular. But it has been proved to be equilateral; therefore it is a square, and it is inscribed in the given Circle $ABCD$.

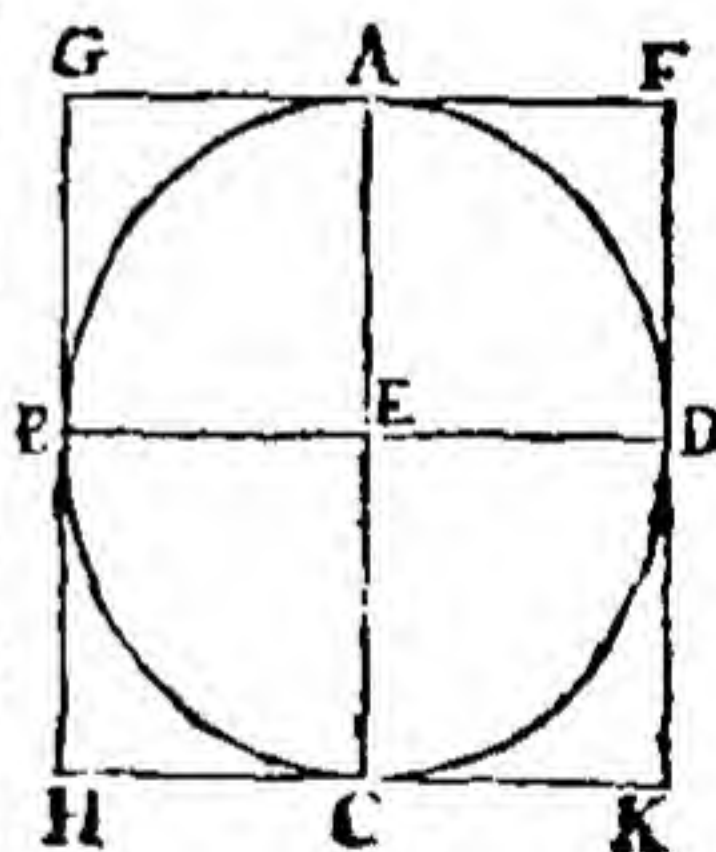


Therefore in the given Circle $ABCD$, is inscribed the square $ABCD$. Which was to be done.

PROPOSITION VII.

About a given Circle to circumscribe a square.

Let the given Circle be $ABCD$. It is required about the Circle $ABCD$, to circumscribe a square. Let two Diameters AC, BD , be drawn at right angles to one another; and by the points A, B, C, D , let there be drawn the lines FG, GH, HK, KF , touching the Circle $ABCD$ [Prop. 17. El. III.]. Now forasmuch as FG touches the Circle $ABCD$, and from E the center, to the Contact at A , is joyned EA ; therefore the angles at A are right angles [Prop. 18. El. III.]; and by the same reason, the angles at the points B, C, D , are also right angles. Now because AEB is a right angle, and that EBG is also a right angle; therefore GH is parallel to AC [Prop. 28. El. I.], and by the same reason AC is also parallel to FK . In like manner we shall demonstrate, that either of the lines GF, HK , is parallel to the line BED : wherefore GK, GC, AK, FB, BK , are Parallelograms; and therefore GF is equal to HK ; as also GH to FK [Prop. 34. El. I.].



“Note, thus far is only proved, that the opposite sides, namely, “ GF is equal to HK , as also GH to FK . Next is to be proved, that all “four are equal to one another.

Now because AC is equal to BD ; but AC is equal to each of the lines GH, FK ; and BD is equal to each of the lines GF, HK ; wherefore also each of the lines GH, FK , is equal to each of the lines GF, HK ; therefore the quadrilateral Figure $FGHK$ is equilateral.

I say, that it is also rectangular. For because $GBEA$ is a Parallelogram, and AEB is a right angle; therefore AGB is also a right angle. In like manner we shall demonstrate, that the angles at the points H, K, F , are right angles; therefore the quadrilateral Figure

$FGHK$

FGHK is rectangular. But it has been proved to be equilateral; therefore it is a square, and it is circumscribed about the Circle ABCD.

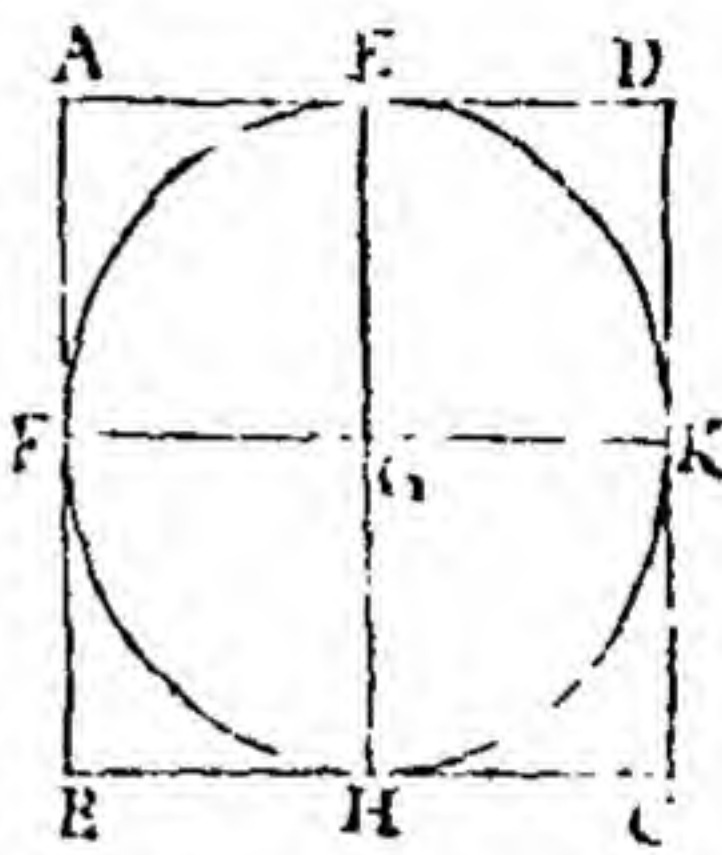
Therefore about a given Circle a square is circumscribed. Which was to be done.

PROPOSITION VIII.

In a given square to inscribe a Circle.

Let the given square be ABCD. It is required in the square ABCD to inscribe a Circle. Let both of the lines AB, AD, be cut into halves in the points E, F, and by E let there be drawn EH parallel to either of the lines AB, DC [by Prop. 31. El. I.]; and by F, let there be drawn FK, parallel to either of the lines AD, BC; therefore every one of the spaces AK, KB, AH, HD, AG, GC, BG, GD, is a Parallelogram; and therefore their opposite sides are equal [Prop. 34. El. I.].

Now forasmuch as AD is equal to AB, and of AD the half is AE, and of AB the half is AF; therefore AE is equal to AF; and also the opposite sides are equal; therefore FG is equal to GE. In like manner we shall demonstrate, that either of the lines GH, GK, is equal to either of the lines FG, GE; therefore the four lines GE, GF, GH, GK, are equal to one another. Wherefore to the center G, and distance any one of the lines GE, GF, GH, GK, a Circle being described, shall also pass by the remaining points, and shall touch the strait lines AB, BC, CD, DA, for that the angles at E, F, H, K, are right angles. For if the Circle cut the lines AB, BC, CD, DA, then to the Diameter of a Circle a strait line being drawn at right angles from the extremity thereof, shall fall within the Circle; which is absurd [Prop. 16. El. III.]; therefore to the center G, and distance any one of the lines GE, GF, GH, GK, a Circle being described does not cut the lines AB, BC, CD, DA; wherefore it shall touch them, and shall be inscribed in the square ABCD.



Therefore in a given square a Circle has been inscribed. Which was to be done.

PROPOSITION IX.

About a given square to circumscribe a Circle.

Let the given square be ABCD. It is required about the square ABCD, to circumscribe a Circle. For the lines AC, BD, being drawn, let them cut one another in F. Now forasmuch as DA is equal to AB, and AC is common; therefore there are the two lines DA, AC, equal to the two lines BA, AC, and the base DC is equal to the base CB; therefore the angle DAC, is equal to the angle BAC; wherefore the

B b

the

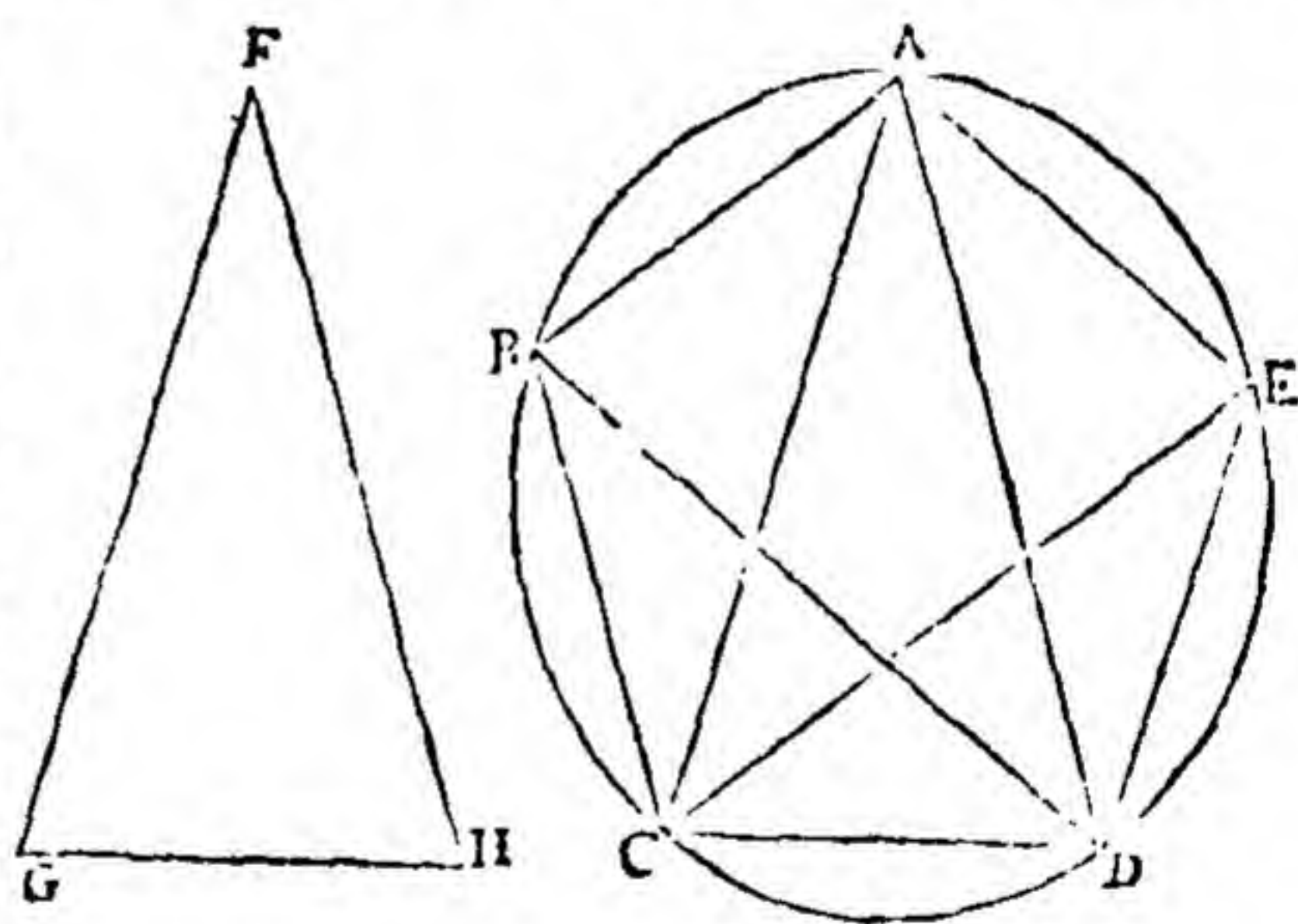
let there be added $\angle CDA$, common to both; therefore the whole $\angle BDA$ is equal to the two angles $\angle CDA, \angle DAC$. But to $\angle CDA, \angle DAC$, is equal the outward angle $\angle BCD$, therefore $\angle BDA$ is equal to $\angle BCD$: but $\angle BDA$ is equal $\angle CBD$, for that the side AD is equal to AB ; that is $\angle DBA$ is equal to $\angle BCD$; therefore the three angles $\angle BDA, \angle DBA, \angle BCD$, are equal to one another. And because the angle $\angle DBC$ is equal to the angle $\angle BCD$; therefore the side BD is equal to the side DC . But BD is put equal to AC ; wherefore also AC is equal to CD ; so that the angle $\angle CDA$ is equal to the angle $\angle DAC$; therefore the angles $\angle CDA, \angle DAC$, are double to the angle $\angle DAC$: but $\angle BCD$ is equal to $\angle CDA, \angle DAC$, and therefore $\angle BCD$ is double to $\angle DAC$. But $\angle BCD$ is equal to each of the angles $\angle BDA, \angle DBA$; wherefore each of the angles $\angle BDA, \angle DBA$, is double to $\angle DAB$.

Therefore there is constituted an equicrural Triangle ADB , having each of the angles at the base double to the remaining angle. Which was to be done.

PROPOSITION XI.

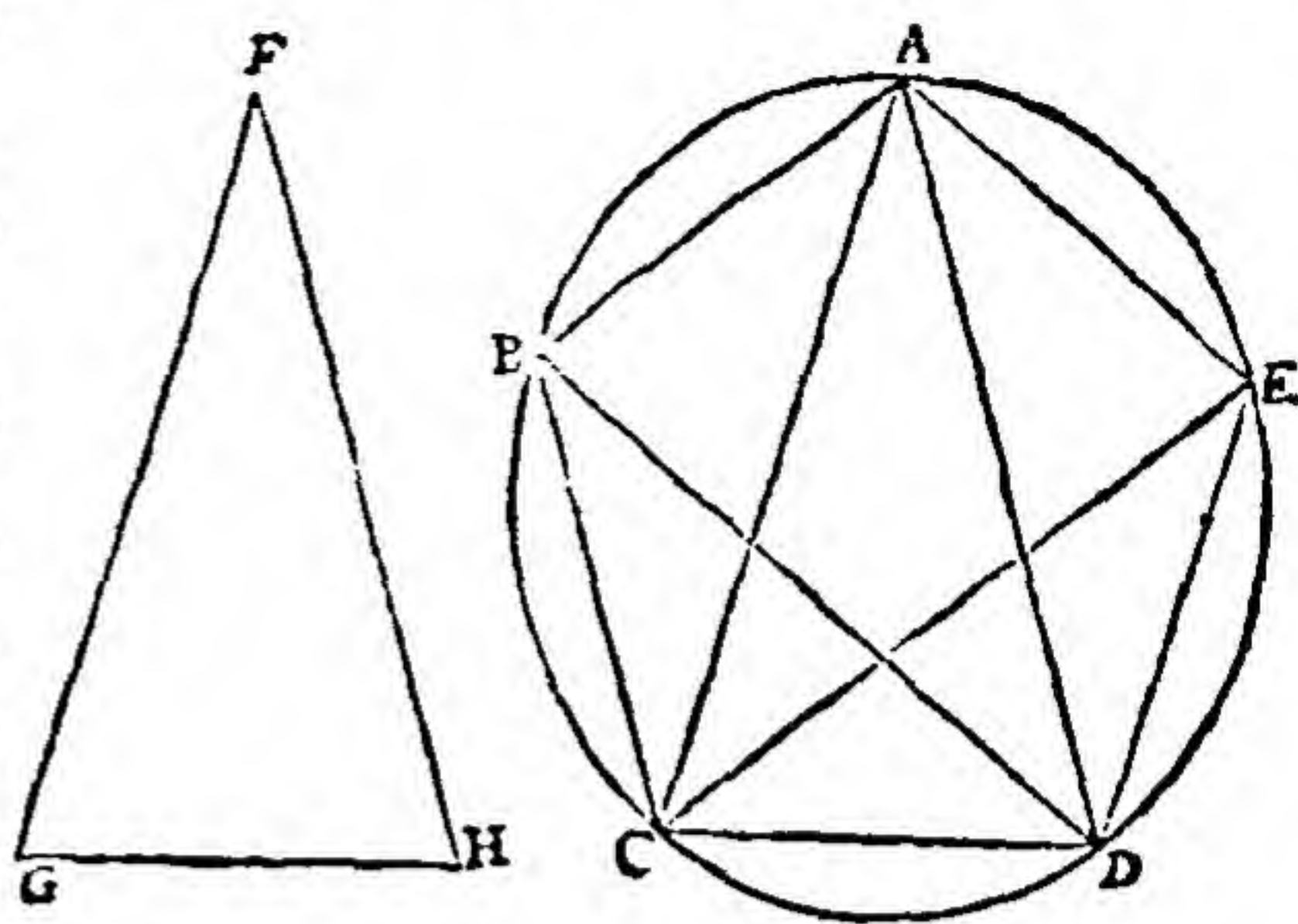
In a given Circle to inscribe an equilateral and equiangular Pentagon.

Let the given Circle be $ABCDE$. It is required in the Circle $ABCDE$, to inscribe an equilateral and equiangular Pentagon. Let there be put an equicrural Triangle FGH , having each of the angles at G, H , double to the angle at F ; and let there be inscribed in the Circle $ABCDE$, the Triangle ACD , equiangular to the Triangle FGH ; so that to the angle at F may be equal the angle $\angle CAD$: And to either of the angles at G, H , be equal either of the angles $\angle ACD, \angle CDA$, and therefore either of the angles $\angle ACD, \angle CDA$, is double to the angle $\angle CAD$. Now let each of the angles $\angle ACD, \angle CDA$, be cut into halves by the straight lines CE, DB [by Prop. 9. El. I.],



and let there be drawn AB, BC, CD, DE, EA . Forasmuch then as each of the angles $\angle ACD, \angle CDA$, is double to $\angle CAD$, and they have been cut into halves by the lines CE, DB ; therefore these five angles $\angle DAC, \angle ACE, \angle ECD, \angle CDB, \angle BDA$, are equal to one another. But equal angles insit on equal circumferences [Prop. 26. El. III.]; therefore the five circumferences AB, BC, CD, DE, EA , are equal to one another. But under equal circumferences are subtended equal straight lines [Prop. 29. El. III.]; wherefore the five straight lines AB, BC, CD, DE, EA , are equal to one another; therefore the Pentagon $ABCDE$ is equilateral. I say, that it is also equiangular. For because the circumference AB , is equal to the circumference DE , let BCD be added in common:

therefore the whole circumference $ABCD$ is equal to the whole circumference $EDCB$. And the angle AED insists on the circumference $ABCD$; as also the angle BAE insists on the circumference $EDCB$; therefore the angle BAE is equal to the angle AED [Prop. 27. El. III.]. By the same reason also every one of the angles ABC , BCD , CDE , is equal to each of the angles BAE , AED : wherefore the Pentagon $ABCDE$ is equiangular. But it has been proved to be equilateral.

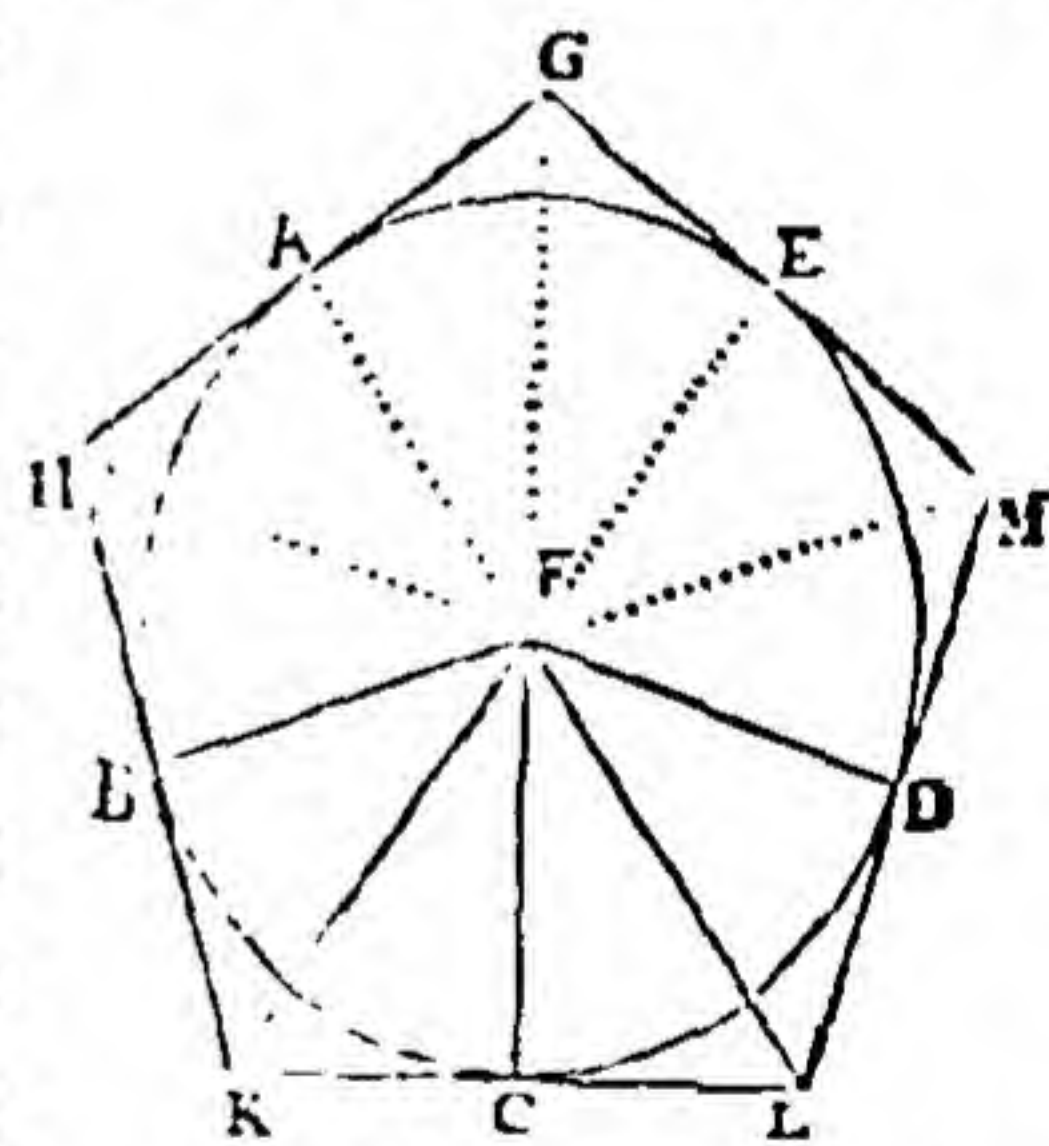


Therefore in a given Circle an equilateral and equiangled Pentagon has been inscribed. Which was to be done.

PROPOSITION XII.

About a given Circle to circumscribe an equilateral and equiangular Pentagon.

Let the given Circle be $ABCDE$. It is required about the Circle $ABCDE$, to circumscribe an equilateral and equiangular Pentagon. Let the points of the angles of the inscribed Pentagon be conceived to be $ABCDE$; so that the circumferences AB , BC , CD , DE , EA , are equal. And by the points A, B, C, D, E , let there be drawn GH, HK, KL, LM, MG , touching the Circle [by Prop. 17. El. III.]: and of the Circle $ABCDE$, let the center F be taken, then let FB, FK, FC, FL, FD , be joined. Now forasmuch as the strait line KL toucheth the Circle $ABCDE$, in the point c , and from the center F to the Contact at c , is drawn FC ; therefore FC is perpendicular to KL [Prop. 18. El. III.]; and each of the angles at c is a right angle: By the same reason also the angles at B, D are right. And because FCK is a right angle, therefore the square of FK is equal to the squares of FC, CK . By the same reason also, the square of FK is equal to the squares of FB, BK ; therefore the squares of FC, CK , are equal to the squares of FB, BK , of which the square of FC is equal to the square of FB ; therefore the remaining square of CK is equal to the remaining square of BK : wherefore BK is equal to CK . And because FB is equal to FC , and FK common, therefore there are the two lines BF, FK , equal to the two lines CF, FK , and the base BK is equal to the base CK ; wherefore the angle BFK is equal to the angle KFC [Prop. 8. El. I.]. And also the angle BKF is equal to the angle FKC ; therefore the angle BFC is double to the angle KFC , and the angle



BK is double to the angle FKC . By the same reason also, the angle CFD is double to the angle CFL , and the angle CLD is double to the angle CLF . And because the circumference BC is equal to the circumference CD ; therefore the angle BFC is equal to the angle CFD [Prop. 27. El. III.]. But the angle BFC is double to the angle FKC , and the angle DFC is double to the angle LFC ; therefore the angle FKC is equal to the angle CFL . Now there are two Triangles FKC , FLC , having two angles equal to two angles, each to each, and one side equal to one side, namely FC , common to both; therefore shall they have the remaining sides equal to the remaining sides, and the remaining angle equal to the remaining angle [Prop. 26. El. I.]; therefore the line KC is equal to the line CL , and the angle FKC to the angle FLC . Now forasmuch as KC is equal to CL , therefore KL is double to KC . By the same reason HK shall be proved double to BK : and now because BK has been proved equal to KC , and that KL is double to KC , as also HK to BK ; therefore HK is equal to KL .

In like manner every one of the lines GH , GM , ML , shall be proved equal to each of the lines HK , KL ; therefore the Pentagon $GHKLM$, is equilateral.

I say, that it is also equiangular. Forasmuch as the angle FKC is equal to the angle FLC , and that the angle HKL , has been proved double to the angle FKC , as also KLM , double to FLC : therefore the angle HKL is equal to the angle KLM .

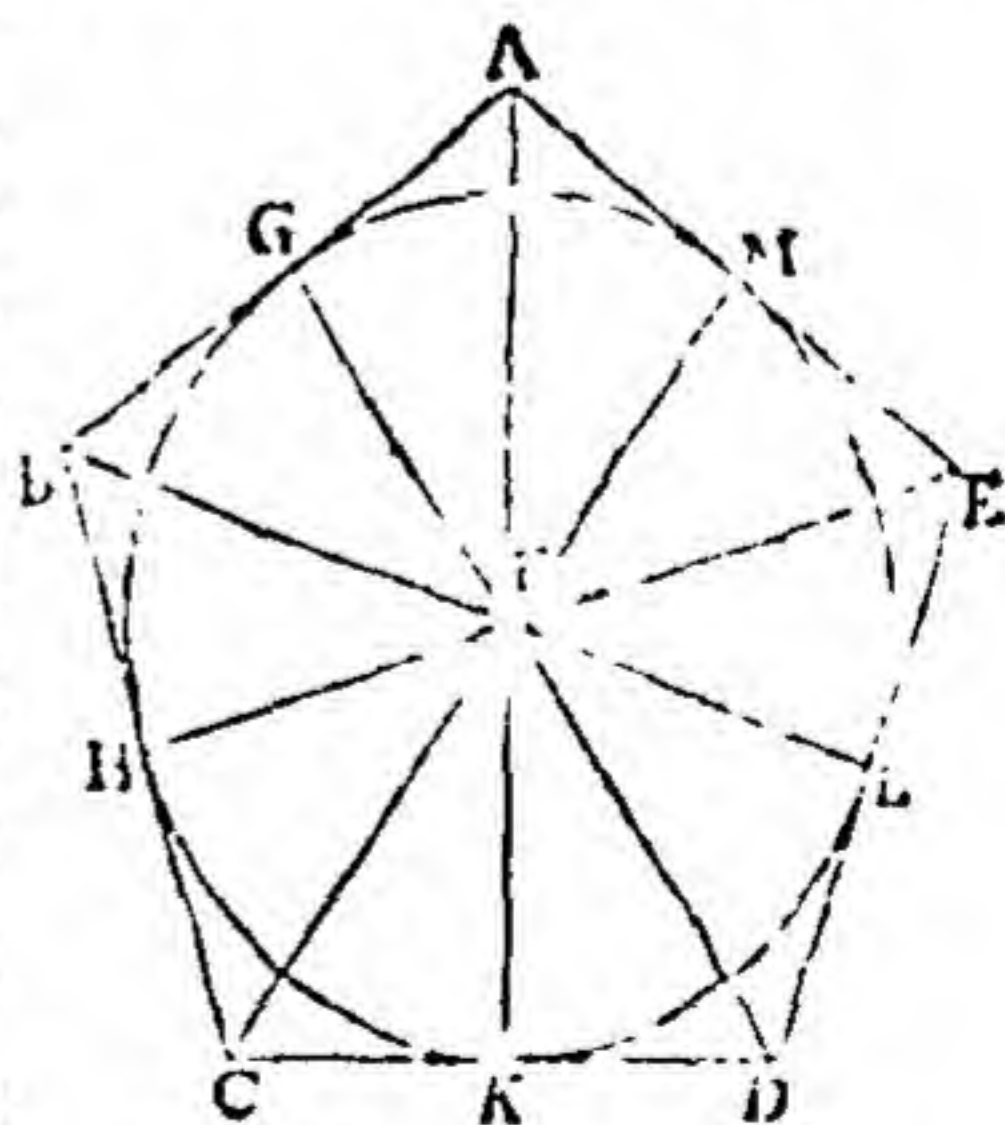
In like manner, every one of the angles KHG , HGM , GML , shall be proved equal to each of the angles HKL , KLM ; wherefore the five angles GHK , HKL , KLM , LMG , MGH , are equal to one another; therefore the Pentagon $GHKLM$, is equiangular. But it has been proved equilateral; and it is circumscribed about the Circle $ABCDE$. Which was to be done.

PROPOSITION XIII.

IN a given Pentagon, which is equilateral and equiangular, to inscribe a Circle.

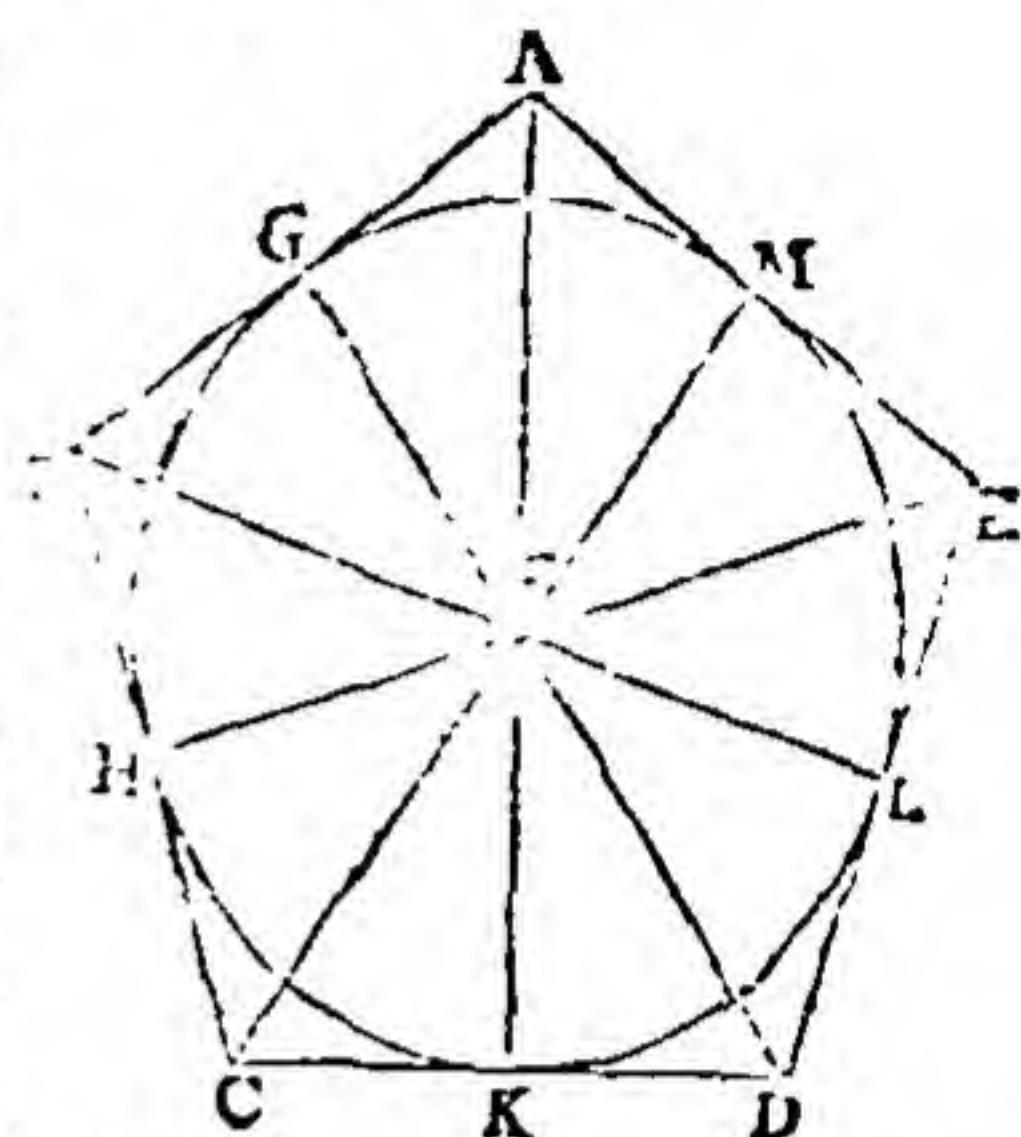
Let the given Pentagon, which is equilateral and equiangular, be $ABCDE$. It is required in the Pentagon $ABCDE$, to inscribe a Circle.

Let each of the angles BCD , CDE , be cut into halves by the lines CF , DF ; and from the point F , wherein the lines CF , DF , do meet, let there be drawn FB , FA , FE . Now forasmuch as BC is equal to CD , and CF common; therefore there are two lines BC , CF , equal to two lines DC , CF , and the angle BCF is equal to the angle DCF ; therefore the base BF is equal to the base DF , and the Triangle BFC is equal to the Triangle DFC , and the remaining angles are equal to the remaining



angles, under which are subtended equal sides [Prop. 4. El. I.]; therefore the angle CBF is equal to the angle CDF . And because the angle CDE is double to the angle CDF ; but the angle CDE is equal to the angle ABC , and CDF , to CBF ; wherefore the angle CBA is double to the angle CBF ; and therefore the angle ABF is equal to the angle FBC : therefore the angle ABC is cut into halves by the line BC . In like manner shall be demonstrated that each of the angles BAE , AED , are cut into halves by the lines FA , FE .

Now from the point F to the lines AB , BC , CD , DE , EA , let be drawn the perpendiculars FG , FH , FK , FL , FM . Now because the angle HCF is equal to the angle KCF , and the right angle FHC , is equal to the right angle FKC ; therefore there are two Triangles FHC , FKC , having two angles equal to two angles, and one side equal to one side, namely FC common to both, and subtended under equal angles; wherefore they shall have the remaining sides equal to the remaining sides [Prop. 26. El. I.]; therefore the perpendicular FH is equal to the perpendicular FK . In like manner shall be demonstrated, that also every one of the lines FL , FM , FG , is equal to either of the lines FH , FK ; therefore the five lines FG , FH , FK , FL , FM , are equal to one another. Wherefore to the center F , and distance any one of the lines FG , FH , FK , FL , FM , a Circle being described, shall pass also through the remaining points, and shall touch the lines AB , BC , CD , DE , EA , because that the angles at the points G , H , K , L , M , are right angles. For if the Circle shall not touch, but cut them, then it shall happen, that to the Diameter of a Circle, a strait line being drawn at right angles from the extremity thereof, does fall within the Circle; which has been proved absurd [Prop. 16. El. III.]; therefore to the center F , and distance any one of the lines FG , FH , FK , FL , FM , a Circle being described, shall not cut the lines AB , BC , CD , DE , EA ; wherefore it shall touch them. Let it be described, as $GHLKM$.



Therefore in a given Pentagon, which is equilateral, and equi-angled, a Circle is inscribed. Which was to be done.

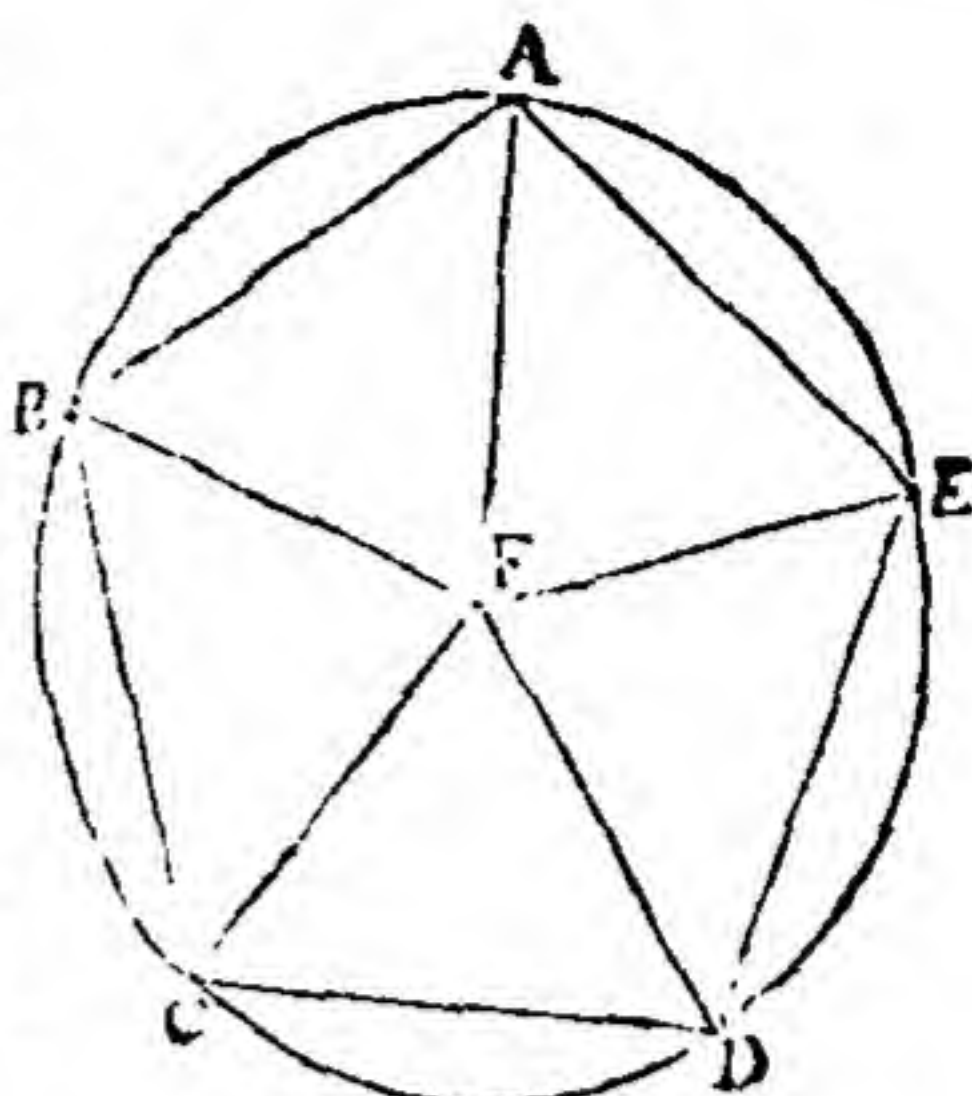
PROPOSITION XIV.

A Bout a given Pentagon, which is equilateral, and equiangled, to circumscribe a Circle.

Let the given Pentagon, which is equilateral, and equiangled, be $ABCDE$. It is required about the Pentagon $ABCDE$, to circumscribe a Circle. Let each of the angles BCD , CDE , be cut into halves by the lines CF , DF ; and from the point F , wherein the lines CF , DF , do

do meet, let there to the points B, A, E , be joyned the lines FB, FA, FE . As it was in the foregoing Proposition so may it here be demonstrated, that every one of the angles CBA, BAE, AED , is cut into halves by the lines FB, FA, FE .

Now forasmuch as the angle BCD is equal to the angle CDE , and of the angle BCD , the half is FCD , and of CDE , the half is CDF , therefore the angle FCD is equal to the angle FDC , so that the side FC is equal to the side FD .



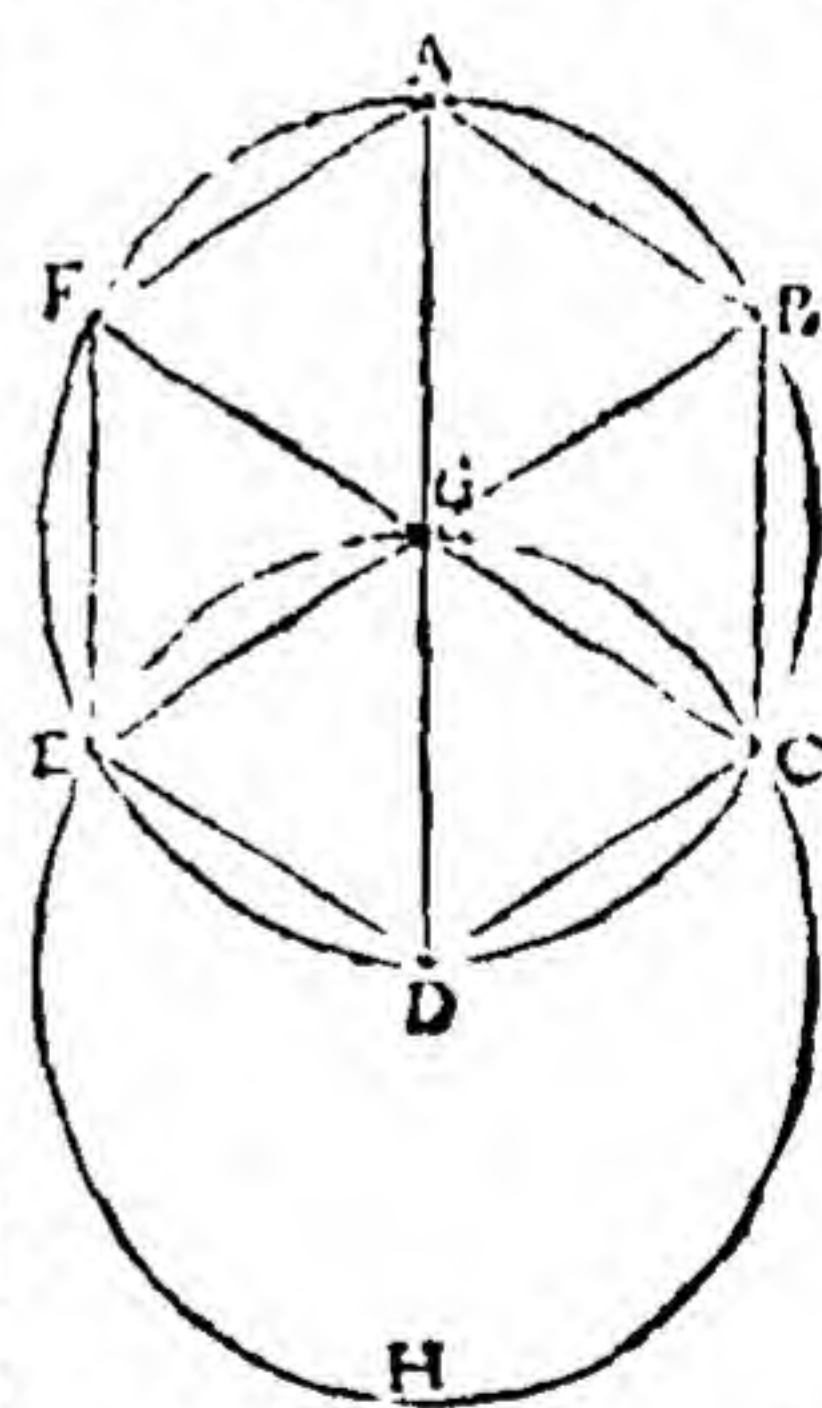
In like manner shall be demonstrated, that every one of the sides FB, FA, FE , is equal to each of the sides FC, FD . Wherefore the five lines FA, FB, FC, FD, FE , are equal to one another; therefore to the center F , and distance, any one of the lines FA, FB, FC, FD, FE , a Circle being described, shall pass also through the remaining points, and shall be circumscribed about the Pentagon $ABCDE$, which is equilateral and equiangular. Let it be circumscribed, and be the Circle $ABCDE$.

Therefore about a given Pentagon, which is equilateral and equiangular, a Circle is circumscribed. Which was to be done.

PROPOSITION XV.

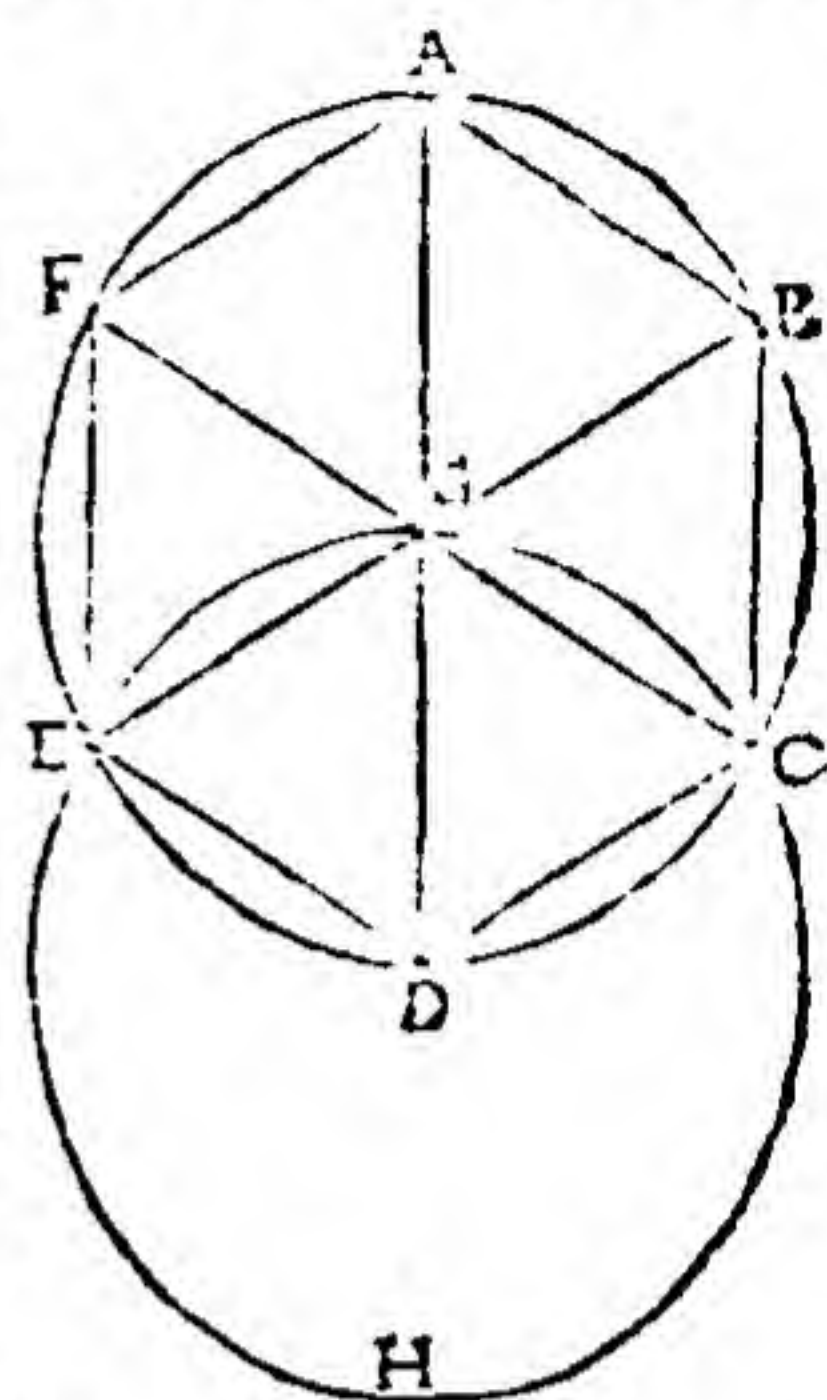
IN a given Circle to inscribe an equilateral, and equiangular Hexagon.

Let the given Circle be $ABCDEF$. It is required in the Circle $ABCDEF$, to inscribe an equilateral, and equiangular Hexagon. Of the Circle $ABCDEF$, let the Diameter AD be drawn, and the center G be taken. Now to the center D , and distance DG , let the Circle $EGCH$ be described, and EG, CG being joyned, let them be produced to the points B, F , and let AB, BC, CD, DE, EF, FA , be joyned. I say, that the Hexagon $ABCDEF$, is equilateral and equiangular. Now forasmuch as the point G is the center of the Circle $ABCDEF$; therefore GE is equal to GD . Again, because D is the center of the Circle $EGCH$, therefore DE is equal to DG . But GE has been proved equal to GD ; wherefore GE is equal to ED ; therefore the Triangle EGD is equilateral, and the three angles EGD, GDE, DEG , are equal to one another. Now because in equicrural Triangles, the angles of the base are equal to one another [Prop. 5. El. I.], and the three angles of a Triangle are equal to two right [Prop. 32. El. I.]; therefore the angle EGD is a third part of two right angles. In like manner the angle DGC shall be proved a third of two right angles; and because the



the line CG , standing upon the line EB , makes the collateral angles EGC , CGB , equal to two right angles [Prop. 13. El. I.]; therefore the remaining angle CGB is also a third of two right angles: wherefore the angles EGD , DGC , CGB , are equal to one another: And also the vertical angles BGA , AGF , FGE , are equal to EGD , DGC , CGB [Prop. 15. El. I.]; wherefore the six angles EGD , DGC , CGB , BGA , AGF , FGE , are equal to one another: But equal angles insit upon equal circumferences [Prop. 20. El. III.]; therefore the six circumferences AB , BC , CD , DE , EF , FA , are equal to one another. But under equal circumferences are subtended equal strait lines [Prop. 29. El. III.]; therefore the six strait lines are equal to one another: wherefore the Hexagon $ABCDEF$ is equilateral.

I say, that it is also equiangled. For because the circumference AF is equal to the circumference ED , let there be added the circumference $ABCD$ common; therefore the whole circumference $FABCD$, is equal to the whole circumference $EDCBA$. But the angle FED insits upon the circumference $FABCD$, and the angle AFE upon the circumference $EDCBA$; therefore the angle AFE is equal to the angle FED [Prop. 27. El. III.]. In like manner shall be demonstrated, that the remaining angles of the Hexagon $ABCDEF$, are every one equal to either of the angles AFE , FED ; therefore the Hexagon $ABCDEF$, is equiangled. But it has been proved also equilateral; and it is inscribed in the Circle $ABCDEF$.



Therefore in a given Circle, an equilateral, and equiangled Hexagon is inscribed. Which was to be done.

Corollary.

From hence it is manifest, that the side of an inscribed Hexagon, is equal to the Radius of the Circle.

And if by the points A, B, C, D, E, F, we draw Tangents to the Circle, there shall be circumscribed about the Circle an equilateral and equiangled Hexagon, according to what hath been said of the Pentagon. And moreover, by the like as hath been said of the Pentagon, we shall in a given Hexagon inscribe a Circle, and also circumscribe.

Corollary 2. added.

Because the circumference of a Circle is greater than the Perimeter of any Polygon inscribed in it; and every side of an inscribed Hexagon is equal to the Radius; therefore the circumference of the Circle, being greater than the six sides of the inscribed Hexagon, is also greater than the six Radii, that is, than three Diameters of the Circle.

It being therefore manifest from this Proposition of *Euclide*, that the circumference is more than triple of the Diameter, Geometricians have in all Ages enquired how much more it is.

The great *Archimedes* has brought it within the easiest limits, and the best for common use.

Therefore if we suppose the Diameter to be 7, and so consequently divided into

into seven parts, then the circumference shall be more than thrice seven, that is; more than 21 by almost $\frac{1}{7}$ part of the Diameter.

And hereupon the circumference, compared to the Diameter, is generally taken to be as 22 to 7: this proportion (tho' somewhat too great) being near enough the truth for any common use. To which only end this Rule was given by *Archimedes*, who could otherwise have proceeded nearer and nearer, to any approximation desirable; but because after all, the nature of the subject admits not of a Geometrical exactness, and equality, That great Master of Geometry rightly judg'd it most convenient, to state the mensuration between the Circumference and Diameter in the easiest and readiest Terms, as 22 to 7 almost. So that if the Diameter be supposed to be 7, as for instance 7 Inches, then shall the circumference of the Circle be almost 22 Inches. And the like in Feet, Cubits, Miles, or any other measure.

Advertisment.

We are lastly to observe in this place, that the Equilateral Triangle, the Square, the Equilateral Pentagon and Hexagon, are the four simple and primitive Figures, from which all other Regular Polygons do arise, that are mutually with a Circle, or with one another Inscriptible and Circumscribable: therefore *Euclide* has most accurately, in these four Figures, set forth a general method sufficient for this kind of Subject, and applicable to all other regular Polygons. But of them in particular he makes here no mention; because these four are only requisite to the consideration of the five regular *Platonick* Bodies, wherewith *Euclide* concludes his Elements. And besides, the rest, as they are infinite in multitude, so are they divided from these after one and the same manner of Construction; and their Demonstration is agreeable every way to what is here already set forth in these primitive Figures.

First then, from an inscribed Square is constituted an inscribed Octagon, by the bisection of a Quadrantal Arch [Prop. 30. El. III.], and by drawing strait lines from the angular points of the inscribed Square, to the points of bisection.

Again, if there be drawn by the angular points of the inscribed Octagon, strait lines touching the Circle, then shall be constituted a circumscribed Octagon, like as before in the Circumscription of the Pentagon [Prop. 12. El. IV.]. Now again, if the Octagonal Arch be bisected, there may, in like manner, be inscribed and circumscribed, a regular Polygon of 16 sides; and so forwards of 32, of 64, &c. infinitely.

Secondly, from the bisection of the Pentagonal Arch, may in like manner be inscribed and circumscribed a Decagon: And from the bisection of a Decagonal Arch may be inscribed and circumscribed a regular Polygon of 20 sides, and so forward of 40, of 80, &c. infinitely.

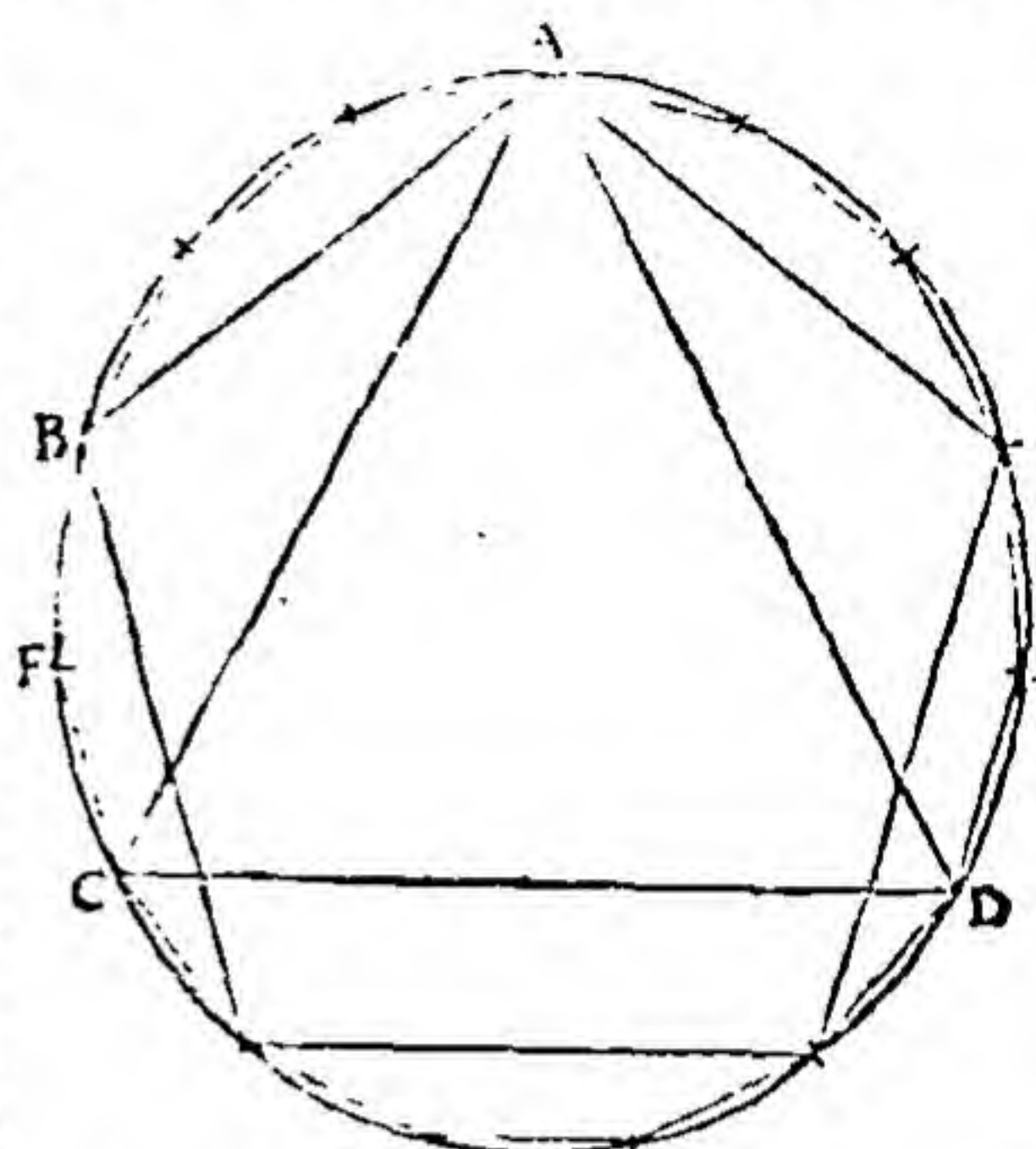
Lastly, from the bisection of an Hexagonal Arch, may be inscribed and circumscribed a Duodecagon: And from the bisection of a Duodecagonal Arch, may be inscribed and circumscribed a regular Polygon of 24 sides, and so forward of 48, of 96, &c. infinitely.

In this method therefore, by the bisection of a given Arch, there are from a Square, Pentagon, and Hexagon, constructed all Polygons of this kind: Only in the following Proposition is constituted a Polygon of 15 sides; which, although it be effected by the bisection of an Arch also; yet it is in a peculiar and different manner from the forementioned Polygons. For which reason it is subjoyn'd by *Euclide* to the precedent Propositions, to compleat this Element.

PROPOSITION XVI.

IN a given Circle to inscribe an equilateral and equiangled Quindecagon, or a Figure of fifteen sides.

Let the given Circle be $ABCD$. It is required in the given Circle $ABCD$, to inscribe an equilateral and equiangled Quindecagon. Let AC be the side of an equilateral Triangle inscribed in the Circle $ABCD$: and of an equilateral Pentagon let the side be AB ; therefore of what equal parts the Circle $ABCD$ is fifteen, of such the circumference ABC , being the third part of the Circle, shall be five: and the circumference AB , being the fifth part of the Circle, shall be three; therefore the remaining Arch BC is two of those equal parts. Let BC be cut into halves in the point E [by Prop. 30. El. III.]; wherefore each of the circumferences BE, EC , is the fifteenth part of the Circle $ABCD$. If therefore drawing the strait lines BE, EC , we adapt continually, in the Circle $ABCD$, strait lines equal to them, there shall be inscribed in the same an equilateral and equiangled Quindecagon. Which was to be done.



In like manner, as before in the Pentagon, if by the divisions of the Circle, we draw Tangents to the Circle, there shall be circumscribed an equilateral and equiangled Quindecagon. And moreover, by the like as before said in the Pentagon, we shall in a given Quindecagon, Equilateral, and Equiangled, inscribe a Circle; and also circumscribe.

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The *Quindecagon* is the only derivative *Polygon* that *Euclide* thought necessary to be consider'd, after the four *Primitive Figures*, namely, a Triangle, a Square, a Pentagon, and an Hexagon: because of its peculiar manner of Construction, from the inscription of an *Equilateral Triangle*, and *Pentagon* compared together. Yet it may be said, that a *Polygon* of 24 sides might also have been constructed in the self same manner, from the inscription of a *Square* and *Hexagon* compared together. But we are to know, that this *Polygon* of 24 sides arises more naturally from the bisection of an *Hexagon* and then of a *Duodecagon*; like as others, from the bisection of a *Square*, or of a *Pentagon*, as is observ'd in the foregoing Advertisement. And therefore *Euclide* judged it inartificial to take notice of it in this place, as he hath done of a *Quindecagon*, which admits of no other way of Construction.

But of the *Heptagon* and *Nonagon*, *Euclide* makes no mention, because, as before, for the inscription of the *Pentagon*, there was first to be inscribed an *Equicrural Triangle*, having the angles at the base double to the angle at the *Vertex*; and then those angles were to be bisected: So, for the inscription of an *Heptagon*, it is first requisite to inscribe an *Equicrural Triangle*, having the angles at the base triple to the angle at the *Vertex*; and then to divide those angles into three equal angles.

Again, for the inscription of a *Nonagon*, it is first necessary to inscribe an *Equilateral Triangle*, and then to divide every one of its angles into three equal angles,

whereby to set forth the Arch of a *Nonagon*, for that thrice 3 is 9; wherefore these *Polygons* depend upon the Trisection of an angle. But HOW TO TRISECT any Angle, or Arch, as *Euclide* hath demonstrated HOW TO BISECT, in Prop. 9. El. I. and Prop. 30. El. III. falls not within the power of the *Euclidéan plain Geometry*, whose instruments are only a *strait Line*, and a *Circle*, according to the three simple *Postulata*, laid down at our entrance into the Elements. For every Angle cannot, by the help of a strait line and a Circle only, be divided into three equal Angles. Yet, notwithstanding the incapability of such a Trisection, *Orontius* and many others, not having a true and full insight into the nature of this matter, that is, not understanding what belongs to Plain, and what to Solid Geometry; what belongs to Magnitudes of two Dimensions, and what to Magnitudes of three: They have after much toyl, lost their labour and reputation, therein vexing themselves with impossibilities.

THE FIFTH ELEMENT.

THIS Element depends upon none of the foregoing, but stands alone as an *universal Mathesis*. It is like Metaphysics to Natural Philosophy: a Transcendent Element of pure, and prime Mathematics, and so much abstracted not only from Matter in any Subject, but also from every particular kind of Subjects, so as to be equally applicable to all the Species of Quantity; to the Sciences, Geometry, and Arithmetic; and besides, universally to all other things, which are capable of comparison, such as Force, and Power in Agents; Intention, and Remission in Qualities; Velocity, and Tardity in Motions; Gravity, and Levity in Ponderations; Modulation in sounds; Value, and Estimation in Things; and whatsoever else may admit of any Gradation.

But *Euclide* in a Geometrical method pursues his course, and does accordingly apply this Element to Magnitudes: yet in such an artificial and subtil Form of Demonstration, that it might in general be made use of wheresoever in the nature of things, the reason of Man can compare one thing with an other.

This Doctrine of Proportions cannot be well explained without the use of Numbers; and therefore whoever intends rightly to understand this Element, must come furnished with a moderate skill in Arithmetic. We have therefore apply'd Numbers to the Definitions and Propositions, for illustration sake to the younger Students.

I should farther advise that with the Study of this Element, also *Euclid's* Elements of Numbers were together perused, especially those Propositions where Proportions are concern'd. For the Doctrine of Proportions is chiefly, or rather only explicable by Numbers: and what here is apply'd to Magnitudes, was secretly derived from those Elements, which do much further a right understanding of this. It will be at first sufficient for Beginners only to read the Propositions of those Elements, and carefully to observe the *Expositions*; which may instruct them enough for their present use in this Element, without giving themselves the trouble of being convinced by Demonstrations.

DEFINITIONS.

Of Part and Multiple.

DEFINITION I.

A Part is a magnitude of a magnitude, a less of a greater, when the less measures the greater.

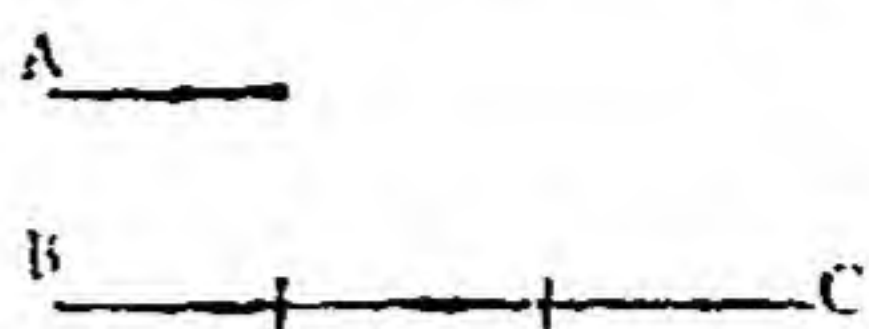
DEFINITION II.

A Multiple is a greater of a less, when the greater is measured by the less.

A Part

A Part and Multiple are each a magnitude of a magnitude; a Part a less magnitude of a greater; a Multiple a greater magnitude of a less: both combin'd in a mutual respect to one another.

As A is said to be a part of B C, when repeated some certain times, as here 'tis thrice, It doth exactly measure, and compleat the magnitude B C.



For a Part is here to be understood in a peculiar sense; and not as when it is said that the whole is greater than its part: where a part is taken indifferently for any portion of the whole, as a less quantity contained in a greater.

But now in this place, by a Part is meant such a portion of the whole, which repeated measures the whole precisely. And again, in this respect the whole is call'd a Multiple of that part, either Duple, Triple, Quadruple, &c. because it contains the same just so many times, as twice, thrice, four times, &c. and is noted thus, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c. And the Part which so many times repeated, measures the whole, is accordingly said to be one half, or one third, or one fourth part, &c. of that whole or Multiple; and is noted thus, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c.

Euclide begins with Part and Multiple as a proper foundation of the Doctrine of Proportions, because these are in the nature of an Unite and a Number, by which only, the Measure, Value, and Proportion of one thing to another can be expressed. For as an Unite is a part and measure of every Number, and every Number is a Multiple of an Unite; so in magnitudes a part is as an Unite, the measure of its Multiple; and every magnitude may as a Multiple be divided by equal partitions into measuring parts, as Number into measuring Unites. Besides, a Part and Multiple, not only answer to Unity and Number, but also to Numbers themselves. For any Number may be a part of some other Numbers, and these again be Multiples of the same. For instance, 2 is a part of 12, because 2 taken 6 times measures, or makes 12, and is therefore a sixth part of 12. In like manner 3 is a fourth part, 4 a third, 6 an half of 12. And again 12 is a Multiple of each of these: Sextuple of 2, Quadruple of 3, Triple of 4, Duple of 6. So that 2, 3, 4, and 6; tho' each be a Number, yet in respect of 12, each being a part of 12, is as an Unite. For 2 is $\frac{1}{6}$, 3 is $\frac{1}{4}$, 4 is $\frac{1}{3}$, 6 is $\frac{1}{2}$ of 12.

But again, 5 is not in this sense a part of 12, because 5 being twice taken makes but 10, and thrice taken makes 15, and so added to it self doth not measure 12, but is either under, or over it. For the same reason neither 7, 8, 9, 10, 11, are said to be a part of 12. But in this case such a portion of any Number, or Magnitude is called Parts, for that it contains some certain and measuring parts of the whole, but is not it self a measure of that whole: As 8 measures not 12; yet because it contains 4, a measure and part of 12 some certain times, therefore 8 is said to be parts of 12, namely two thirds, or two third parts of 12. Likewise 5 measures not 12, yet because it doth certain times contain 1, the Monade or Unite, which is the common part and measure of all Numbers, therefore 5 is properly said to be parts of 12, as being five Unites of 12 the whole.

And in general *Euclide* hath demonstrated in Prop. 4. El. VII. That every less Number is of every Greater either a Part, or Parts.

Such like Quantities both in Number, and Magnitude, are distinguished by the names of Quotal, and Quantal Parts, usually called *Pars Aliquota*, and *Pars Aliquanta*.

A Quotal part measures the whole: which is then called a Multiple of that part.

A Quantal part measures not the whole: but repeated is either less or greater than it.

From hence we may perceive that a Quotal part is either an Unite, or if a Number, yet used as an Unite in the mensuration of the whole. And that a Quantal part is an Aggregate of Quotal parts, which together are not a measure of the whole, that is, make not any Quotal part thereof.

Of Equimultiples.

Moreover in Number and Magnitude, when two or more Magnitudes are equally Multiple of other Magnitudes, that is do equally, or equal times (*ισίαις* says *Euclide*) contain other Magnitudes, they are then call'd *Equimultiples* of their respective Magnitudes, or Quotal parts. As 20 and 12 are Equimultiples of 5 and 3, for that 20 contains 5 four times, as likewise 12 does 3, so that how many *fives* are in 20, so many *threes* are in 12: the multitude or number of the Quotal parts being in both Multiples equal, *viz.* four in each. And therefore all Equimultiples as they equally contain their Quotal parts, so are they again equally divisible into the same number of Quotal parts: As 20 into 4 fives, and 12 into 4 threes, which kind of Division is most frequently used in the Demonstrations of this Element.

Of Proportions, and Proportionals.

DEFINITION III.

Proportion is an habitude of two Homogeneous Magnitudes unto one another, according to Quantity.

Proportion is by *Euclide* called *λόγος*, *Logos*, a word among the *Greeks* of various significations, and commonly Translated *Ratio*, as ambiguous a word as the *Greek*. *Cicero* therefore calls it *Proportio*, a name properly used where the consideration is what *Portion* one thing is of an other.

Proportion therefore in general is an Habitude, Relation, or Comparison of two things to one another, as of A compared to B, according to something, which is common to them both, or of which they both partake, each in some degree of comparison toward the other. A the first of the two is in the ordinary way of speaking the *Antecedent*, likewise B the second is called the *Consequent*, unto which the Antecedent is compared. The Antecedent and the Consequent, are said to be the Terms of the Proportion, for that in them the Proportion between Antecedent and Consequent, is bounded and terminated.

In this place Proportion is only considered between two Magnitudes; and therefore, as all other things comparable to one another, so these are also to be Homogeneous, that is, of the same kind: as a Line to a Line, a Superficies to a Superficies, a Solid to a Solid, is to be compared according to the Dimensions that each do in *sua specie* partake of: and therefore the comparison is to be made according to Quantity, that is, as far as appertains to Quantity: not in respect to any Quality, Power, Weight, Motion, Price (as Lead or Gold), or any other Estimation whatsoever.

Neither again is Quantity here taken *absolutely*, or in a Predicamental Notion, as a *Genus* to Continual, and Discrete Quantity, to Magnitude, and Multitude. But it is to be understood *relatively*, in order to such a *quantitative* Valuation of Magnitude, as where the Quantity of one Magnitude is comparatively to be estimated by an other.

In this sense is *Euclide* to be understood by *κατὰ πηλικότητα*, according to Quantity. For the *Greeks* make a just distinction between *ποσότης*, and *πηλικότης*, between Quantity absolute, as considered in its own nature, and Quantity relative, in a respect to Mensuration and Estimation. The *Latines* use only *Quantitas* for both, as *Rectius* for *ἐξ ἑαυτῆς* and *ἐκ τῆς*; but where proper words are wanting, the sense must make out the proper meaning of ambiguous words.

For the better understanding this present matter, review the Annotations at Def. 1. El. I. concerning the application of Number to Magnitude; where the whole business about the Quantity of Magnitudes is fully explained. And further observe, that the Quantity of every number is shewn by the name of the number, as Ten signifies so many Unites collected into one number under that name: Unity,

Unity, or a Monade being the only prime constituent part, and therefore the common measure also of all numbers, giving the Quantity of every number by a known name in all Languages.

But the Quantity of Magnitudes doth not appear after such an open manner, because every Magnitude, be it never so great, or so little, is in it self only *unum Integrum*, one Integral thing, and tho' divisible into infinite parts; yet hath no prime constituent part for a common measure of Magnitudes, like an Unite in numbers. When therefore it is asked *Quantum*, or *quam Magnum*, how much or how great a thing is? The question tacitely relates to some arbitrary measure commonly received amongst us, by which Magnitudes are usually estimated: as an Inch, Foot, Cubit, or the like. And when a Magnitude is said to contain a certain number of such, or such a measure; That then is reputed to be the quantity of the same Magnitude: but in reality it is its proportion to 1. that is, to its *Geometrical Unite*, and measure. If an Inch be put for a general measure of Magnitudes, then the *Quantity of a Yard* is said to be 36 Inches, because it contains the stated measure so many times. In like manner a Foot is said to be 12 Inches. But again, if a Yard as a certain length be compared to a Foot as another length, then a Yard shall be found to be triple of a Foot: and this is called the *Proportion of a Yard* to a Foot. Likewise in general, the quantity of any number is ever according to its name so many Unites, because an Unite is the natural constituent of all numbers. As the *Quantity* of 12 is always 12 Unites: but in particular comparisons of number to number, the *Proportion* of 12 compared to 4, is said to be triple of 4, or in a triple proportion; and compared to 3 is quadruple. The value of the Antecedent in Proportion being changeable according to the change of the Consequent; because in such particular comparisons the Consequent is as a measure, by which the Antecedent ought to be estimated. For in every Proportion is considered how much the Antecedent contains of the Consequent; the *πηλίκως*, or *Quantuplum*, what Quantuple the Antecedent is of the Consequent. For be the Antecedent either equal, greater or less than the Consequent, it is always the *Quantum* of the Consequent contained in the Antecedent, which gives the proportion of Antecedent to Consequent. And as the Antecedent contains more, or less of its Consequent, so 'tis proportionally valued in a respect to that Consequent. As to give a familiar instance, if a Penny be made the measure of Mony; then the *Quantity* of a Shilling shall always be accounted 12 Pence. But the *Proportion* of a Shilling compared to a Groat, or to a Crown, or to a Pound, is in these divers comparisons of a different value; triple of a Groat, a fifth part of a Crown, a twentieth part of a Pound, as containing so much of each Consequent. And in this sense Proportion is said to be an Habitude according to Quantity.

The Division of Proportions.

Proportion is either of Equality, when the Antecedent is equal to the Consequent; or of Inequality, when greater or less.

If the Antecedent be greater, then it is called *Proportion of the greater Inequality*, for that the comparison is of the greater to the less.

If the Antecedent be the less, it is called *Proportion of the less Inequality*, because the less is compared to the greater.

Moreover, because there are many Homogeneous Magnitudes which are incommensurable to one another (as the side of a Square and its Diameter); so that their mutual Proportions, or how much one contains of the other, cannot be set forth by any common measure, nor be expressed by any number whatsoever; therefore in magnitudes, Proportion is again divided into Effable and Ineffable, Expressible and Inexpressible by number: and commonly called proportion Rational and Irrational. This fifth Element is framed with such an artifice as indifferently to comprehend both.

Proportion of Equality is always Rational (tho' the Terms be sometimes Irrational); for that every thing may have its equal, and be to an other in a Rational account,

account, as 1 to 1. It is also the ground from whence all other Proportions do arise; and a principal Subject of the preceding Elements, tho' not under the name of Proportion: As that *Vertical Angles are equal to one another, the three Angles of a Triangle are equal to two Right, &c.* Besides infinite other such like Propositions throughout all Geometry. It is also of a most general use in Algebra, and the Doctrine of Equations.

Proportions of Inequality which are Rational, are distinguished into five kinds of the greater Inequality, and into as many of the less.

The Varieties of Rational Proportion.

Of Multiple Proportion, and Submultiple.

The most simple Proportions of Inequality, are founded in the first and second Definitions of Part and Multiple.

If in comparison of the one to the other, the Multiple be Antecedent and the Part be Consequent, then it is called *Multiple Proportion*. If the Part be Antecedent and the Multiple be Consequent, then it is called *Submultiple Proportion*. As 12 compared to 4 is Multiple Proportion, and named triple: And 4 to 12 is Submultiple Proportion, and named Subtriple. The like appellation is used in all Multiple and Submultiple Proportions; as Quadruple, Subquadruple; Quintuple, Subquintuple, &c. The other Rational proportions of Inequality, are made by the various Compositions of Part and Multiple, as followeth.

Of Multiple Superparticular, and Submultiple Superparticular.

First, if above the exact Multiple of the Consequent, there remains in the Antecedent any *Quotal* part of the Consequent, as an half, a third, a fourth, or a tenth part of the Consequent, (or otherwise thus named, a Sesquialteral, a Sesquitertrial, a Sesquiquartal, a Sesquidecimal part, &c.) then the proportion is called *Multiple Superparticular*, because the overplus besides the exact Multiple is a particular and measuring part of the Consequent. As 13 to 4 is in Multiple Superparticular proportion, which is known to be so, by dividing the greater by the less; where then the Quotient $3\frac{1}{4}$ shews 13 to contain 4 thrice, and one fourth part of the Consequent 4; wherefore this proportion is named triple Sesquiquartal, and is noted thus $3\frac{1}{4}$. So 10 to 4 is in Multiple superparticular proportion duple Sesquialteral $2\frac{1}{2}$, that is $2\frac{1}{2}$: For where the Numerator is a part, that is, a measure of the Denominator, dividing the Denominator by the Numerator, and this by it self, it will be brought to an Unite, and the proportion plainly appear to be superparticular, as here $\frac{1}{2}$ is reducible to $\frac{1}{2}$. So in all Superparticulars the Numerator is, or may ever be reduced to an Unite: As 40 to 12 is $3\frac{1}{3}$ or $3\frac{1}{3}$, Triple Sesquitertrial. Again, upon transversion of the Terms, the less is compared to the greater, and called Submultiple Superparticular: as 13 to 4, inverted, is 4 to 13 viz. Subtriple sesquiquartal, and is noted thus $\frac{1}{3}$, which signifies that 4 the Antecedent, contains four parts of the Consequent, consisting of 13 such equal parts.

Of Multiple Superpartient, and Submultiple Superpartient.

But now, if above the exact Multiple of the Consequent, the Surplusage be a *Quotal* part of the Consequent, then the proportion is called *Multiple Superpartient*, for that the Overplus is not any quotal part of the Consequent, but some quotal parts taken together, which make a quotal part, that measures not the Consequent: As 8 to 3 is in proportion Multiple Superpartient: for dividing the Antecedent 8 by 3 the Consequent, the Quotient $2\frac{2}{3}$ shews 8 to contain 3 twice, and two thirds of the Consequent 3: therefore this proportion is named Duple Superbitertrial, and according to the Quotient is noted $2\frac{2}{3}$. So 22 to 8 is $2\frac{5}{4}$ or $2\frac{1}{2}$, Duple Supertriquartal. For in all Superpartients where the Numerator and Denominator

minator happen to have a common part and measure, as here $\frac{8}{3}$ have 2 to each in common; then are they by Division to be reduced to other numbers, which are Prime to one another, that is, have no number for a measure common to them both: as $\frac{8}{3}$ is reduced to $\frac{3}{2}$. And again, by transversion comparing the less to the greater, the proportion is called Submultiple Superpartient, as 8 to 3, inverted, is 3 to 8, Subduple, and is noted thus $\frac{3}{8}$, which signifies, that 3 the Antecedent, contains three parts of the Consequent, consisting of eight such equal parts.

Superparticular and Subsuperparticular.

Now if the Antecedent be not in any manner a Multiple of the Consequent, but contains the Consequent *Only once*, and moreover a particular *Quotal part* of the Consequent, then the proportion is called by the single name of *Superparticular*: as 3 to 2 is in proportion Superparticular, according to the Quotient $1\frac{1}{2}$, which sheweth 3 the Antecedent to contain 2 the Consequent once, and one half of two; and is named proportion Sefquialteral: so 15 to 12 is $1\frac{1}{4}$, or $1\frac{1}{4}$. For the Numerator 3 being a quotal part of the Denominator 12, $\frac{3}{12}$ is by Division reduced to $\frac{1}{4}$, and the proportion shewn to be Sefquiquartal $1\frac{1}{4}$. So in all Superparticular proportions the Quotient is always an Unite with a fraction, whose Numerator is likewise an Unite, or reducible to an Unite. Again, by transversion comparing the less to the greater, as 2 to 3, the proportion is Subsuperparticular, and named Subsefquialteral, which is thus noted $\frac{2}{3}$: shewing that the Antecedent 2 contains two parts of 3 the Consequent.

Superpartient and Subsuperpartient.

Lastly, if as before, the Antecedent be not any ways a Multiple of the Consequent, but contains its Consequent *Only once*, and moreover *some parts* (which together measure not the Consequent) then the proportion is thereupon called *Superpartient*; as 8 to 5 is by the Quotient $1\frac{3}{5}$ shewn to be in proportion Superpartient, and particularly Supertriquintal; that is, the Antecedent 8 contains the Consequent 5 once, and moreover three parts of the Consequent 5. So 14 to 10 is $1\frac{2}{5}$, or $1\frac{2}{5}$, by diving 4 and 10 by their common measure 2, and this is named proportion Superbiquintal. So in all Superpartient proportions the Quotient is always an Unite with a fraction, whose Numerator is ever a number: and by this it is distinguished from the Quotient of a Superparticular proportion, where the Numerator of the fraction is ever to be an Unite.

For further, note that in this matter of fractions, whensoever the Numerator can measure the Denominator, the same may divide it self, and the Denominator; and then shall that Numerator be brought to an Unite, and the fraction be Superparticular in its least Terms.

And when both Numerator and Denominator can be measured by an other number, then each of them being divided by that common measure, the fraction will be Superpartient and exposed in its smallest Terms.

Also in all fractions, whether of Magnitudes or Numbers, the Denominator is ever a supposed *Totum*, which consists of so many parts, as the Number of that Denominator signifies.

Again, to finish all the Varieties of Rational proportions; the Superpartient is likewise by transversion of its Terms in comparing the less to the greater, called Subsuperpartient, as 5 to 8, or $\frac{5}{8}$ is Subsupertriquintal: and 10 to 14, or $\frac{5}{7}$ is Subsuperbiquintal.

These are the five kinds of Rational proportion of the greater Inequality, *Multiple*, *Superparticular*, *Superpartient*, *Multiple Superparticular*, *Multiple Superpartient*. To which answer as many of the less Inequality, arising from the transversion of the same Terms, and distinguished by adding *Sub* to the other Appellations.

Now here from the Quotients you may observe, that all these kinds of Rational proportions arise from Unity, Part, and Multiple. For one compared to one

makes proportion of Equality: One with a part to one makes Superparticular proportion: One with parts to one makes Superpartient. Again, many to one makes Multiple proportion: Many with a part to one makes Multiple superparticular: Many with parts to one makes Multiple superpartient, as these Quotients represent: which are also the Denominators or Exponents of Proportions in the manner following.

The Species. Superparticular. Superpartient. Multiple.

Exponents. $1\frac{1}{2}$ $1\frac{1}{3}$ $1\frac{1}{4}$ $1\frac{1}{5}$ $1\frac{2}{3}$ $1\frac{2}{4}$ $1\frac{2}{5}$ $1\frac{2}{6}$ $1\frac{3}{4}$ $1\frac{3}{5}$ $1\frac{3}{6}$ $1\frac{4}{5}$

Least Terms. $\frac{2}{1}$ $\frac{3}{1}$ $\frac{4}{1}$ $\frac{5}{1}$ $\frac{3}{2}$ $\frac{4}{2}$ $\frac{5}{2}$ $\frac{6}{2}$ $\frac{4}{3}$ $\frac{5}{3}$ $\frac{6}{3}$ $\frac{7}{3}$

The Species. Multiple Superparticular. Multiple Superpartient.

Exponents, $2\frac{1}{2}$ $2\frac{1}{3}$ $3\frac{1}{4}$ $4\frac{1}{5}$ $5\frac{1}{6}$ $6\frac{1}{7}$ $2\frac{2}{3}$ $2\frac{2}{4}$ $3\frac{2}{5}$ $4\frac{2}{6}$ $5\frac{2}{7}$

Least Terms. $\frac{3}{2}$ $\frac{4}{3}$ $\frac{5}{4}$ $\frac{6}{5}$ $\frac{7}{6}$ $\frac{8}{7}$ $\frac{4}{3}$ $\frac{5}{4}$ $\frac{6}{5}$ $\frac{7}{6}$ $\frac{8}{7}$

Of the Exponent of Rational Proportion.

Now to discover unto which of these kinds the proportion between any two proposed Terms (as between 45 and 40) should be referred, we are to reduce those Terms unto two such others, which shall be one and the same Character and Exponent, common to all possible Terms in that proportion, and which therefore must necessarily be the least and prime Terms of the same.

How then to find out the least and prime Terms of any proportion, we are to divide the greater by the less; and then the Quotient gives the sole and proper Terms of that proportion. For *as the Divisor is to the Dividend, so is an Unite to the Quotient: and as the Dividend to the Divisor, so the Quotient to an Unite.* Here therefore the Antecedent or Consequent being brought to an Unite, the least of Terms, the Quotient is manifestly the only common Exponent, and the certain standard of any proportion that can be raised from Unity. Which also may be the same proportion in an infinite variety of several Terms. As 9 to 3, 12 to 4, 18 to 6, the Quotient 3, that is, 3 to 1, or $\frac{3}{1}$ is the common Exponent of them all. So 45 to 40, 27 to 24, 18 to 16, 9 to 8. Here between these several Terms the Exponent of their proportions is the common Quotient $1\frac{1}{2}$, which shews the proportion to be in every one Sesquioctaval. And the like infinitely in this and in all the other kinds of Rational proportions, the Quotient expounds and specifies the proportion. Thus the Quotient is the Exponent of every proportion in its proper Quantity, Species, and Name, which therefore was by the Ancients called *ποσὺν τε λεγέσθαι*, *Proportionis Fundum*, the Fundamental proportion, or the proportion in its Fundamental Terms. Whereas then the Quotient is the Exponent of a Proportion, therefore the Notation of the proportion between any two Magnitudes, as A and B, is in Species thus properly signified $\frac{A}{B}$; that is, A divided by B: which being thus noted $\frac{A}{B}$, signifies the Quotient, or Exponent of the proportion between A and B. So $\frac{C}{D}$ is C divided by D, and notes the proportion of C to D. And when the proportions are equal, it is thus represented by their Quotients, $\frac{A}{B} = \frac{C}{D}$: when unequal, the greater thus, $\frac{A}{B} > \frac{C}{D}$, and the less thus, $\frac{A}{B} < \frac{C}{D}$.

Of Arithmetical Proportion.

Lastly, there is an other kind of Habitude between two Magnitudes or Numbers *κατὰ ὑπερβολήν*, according to the *Hpyeroche*, or Excess of one above the other: that is, according to the difference in majority, or minority between two Magnitudes or Numbers. As in comparing 6 to 2, or 2 to 6 is considered, not as before how much 6 the Antecedent contains of 2 the Consequent, or two the Antecedent contains of 6 the Consequent; but how much 6 the greater exceeds 2 the less, or 2 the less is exceeded by 6 the greater: that is, what is the difference in majority or minority between 6 and 2, or 2 and 6. In both comparisons either

of 6 the greater to 2 the less, or of 2 the less to 6 the greater, the excess, or difference is the same, namely 4. But now the proportion of 6 to 2 is triple, three to one; and of 2 to 6 is subtriple, one to three: And this only kind of Habitude was taken by the Ancients to be *Λίγος*, Proportion.

But the Modern Mathematicians call the Habitude according to Excess, or difference by the name of Arithmetical Proportion; and the other defined by *Euclide*, is by them for distinction sake called Geometrical Proportion: altho' both be applicable indifferently to Magnitudes and Numbers: Arithmetical as well to Magnitudes, as to Numbers, and Geometrical as well to Numbers, as to Magnitudes.

Arithmetical Proportion was likely so called, because Numbers in their natural order of 1, 2, 3, &c. are not otherwise distinguished than from their Excess, or Difference by an Unite: therefore where the Excess, or Difference between two Numbers, or Magnitudes is the thing considered, there that Habitude is said to be an Arithmetical Proportion. But it might have been more properly called Proportion of Majority and Minority.

Now in Geometrical Proportion because there is considered how much one Term contains of the other, therefore the proportion between the two Terms is shewn in the Quotient, by dividing the greater by the less.

But in Arithmetical Proportion because there is considered the difference between the two Terms, therefore this Proportion of majority and minority, appears in the Remainder, by subtracting the less from the greater: Division being made use of in Geometrical, and Subtraction in Arithmetical proportions, as the different nature of these two kinds of Proportion do so require.

DEFINITION IV.

Magnitudes are said to have proportion to one another, which being multiplied can exceed each the other.

We have noted before, that upon comparison of one thing to an other, the same must be made in a matter capable of augmentation and diminution, and of which the things compared together do in common participate. Now in Magnitudes compared according to quantity, this Definition discovers when Magnitudes have a participation of one anothers quantities, by this signal property, that upon the multiplication of themselves they can mutually exceed each other: As the side of a Square doubled exceeds the Diameter: the Diameter of a Circle quadrupled exceeds the Circumference, and these again multiplied can exceed the others. Now by this reciprocal Excess, and Comprehension of one another, it manifestly appears that they fully communicate in each others Magnitudes, and therefore are capable of mutual comparison according to Quantity. Whereas a finite strait line multiplied never so often cannot exceed the Magnitude of an Infinite: The angle of Contact in Circles, tho' infinitely augmented, can never exceed the least strait-lined angle. These therefore not being to be made comprehensive of each others Magnitudes by any possible multiplication, cannot be said to have proportion to one another. Thus *Euclide* determines in this matter. But some farther imagine that he expounds in this Definition what Magnitudes should be accounted Homogeneous. His words import nothing towards such a sense; but rather suppose that there are Homogeneous Magnitudes incapable of mutual proportion; and therefore gives us here a Touchstone, by which to know what Homogeneous Magnitudes can be said to have proportion to one another. Moreover in proportion the first consideration is, that the compared Magnitudes be Homogeneous (a word here taken in that sense as vulgarly understood): and to avoid Philosophical disputes (no ways suitable to an Elementary doctrine in Geometry) about angles, or other Magnitudes, what are Homogeneous, and what not, *Euclide* passes by this, (as elsewhere the like Controversies) and without such a consideration, or determination upon this point, only shews in general how to discover when any kind of quantities are between themselves capable of proportion; tho' the same be un-

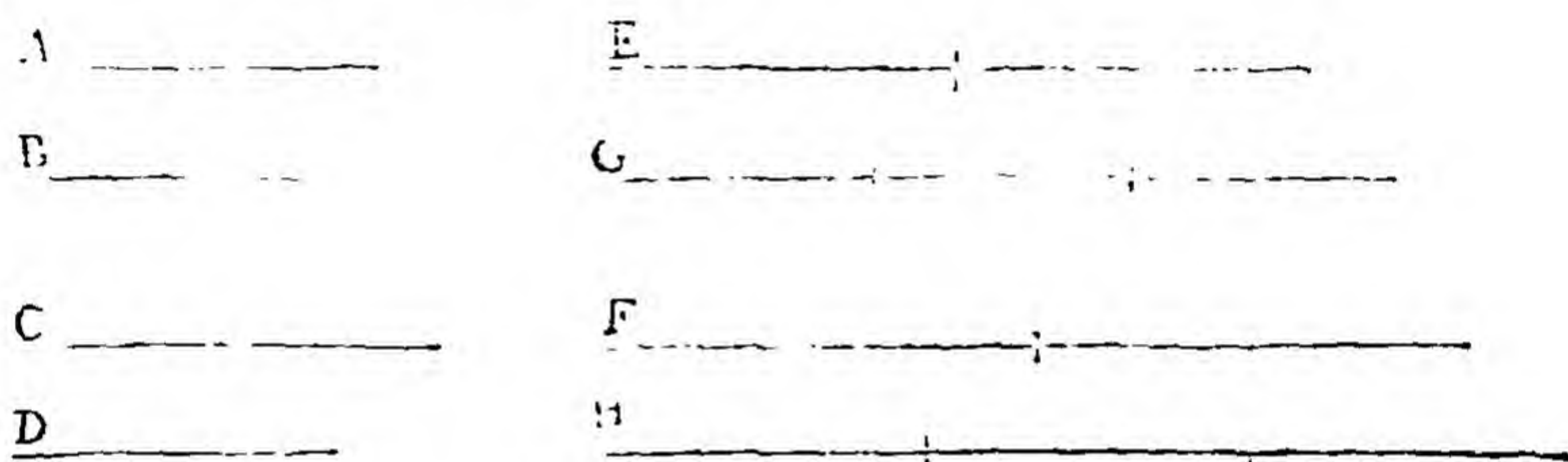
known: And of such, on both sides, capable and incapable, we have now given some clear instances.

It is besides very incongruous, and against this whole Elementary method to make a Definition to be preposterously an interpretation of a word used before.

DEFINITION V.

M*agnitudes are said to be in the same proportion, the first to the second, and the third to the fourth, when according to any multiplication whatsoever, the Equimultiples of the first, and third compared to the Equimultiples of the second and fourth, are either together Deficient, or together Equal, or together Exceeding each the other.*

Let there be four Magnitudes A, B, C, D; where A is compared to B, and C to D. Then let be taken any Equimultiples whatsoever of A the first and of C the third, the two Antecedents. As let E be of A, and F of C any Equimultiples, each of each, either duple, triple, quadruple, decuple, centuple, &c. as here they are duple. And again, let be taken any Equimultiples of B the second, and of D the fourth, the two Consequents, namely, let G be of B, and H of D any Equimultiples, either the same as before of the Antecedents, or any other Equimultiples whatsoever, as here they are triple.



Now when ever it is demonstrated that according to any Multiplication whatsoever, the Equimultiples E and F of the Antecedents A and C, compared to the Equimultiples G and H of the Consequents B and D, each to each, E to G, and and F to H; when these Equimultiples I say are proved to be either together less, or equal, or Greater, E than G, and F than H; then these Magnitudes A, B, C, D, are said to be in the same proportion, A to B, as C to D. And therefore when in any one particular instance the contrary shall be demonstrated, that the Equimultiple of one Antecedent exceeds the Equimultiple of its Consequent, and the other exceeds not, but is either equal, or less; then those exposed Magnitudes are not in the same proportion, the first to the second as the third to the fourth; because the agreement of the Equimultiples in a joint Defect, Equality, or Excess, ought to hold in any multiplication whatsoever.

This is a general TEST, and an infallible Character of Proportional Magnitudes, Commensurable, or Incommensurable. But note, that Incommensurable Magnitudes can never have their Equimultiples equal; for then they would prove to be Commensurable to one another; therefore Incommensurables are shewn to be proportional only from the joint Excess, or Defect of their Equimultiples. Whereas Commensurable Magnitudes are capable both of a joint equality, and inequality, of their Equimultiples: And in respect only to Commensurable Magnitudes the equality of the Equimultiples was here added, that the Definition might comprehend all kind of Magnitudes; and the three sorts of Equimultiples according to Defect, Equality, or Excess, might answer to all possible multiplications indifferently, without considering whether the Magnitudes be either Commensurable, or Incommensurable.

The Notation of Proportionals is in *Species*, or Symbols thus commodiously signified $A : B :: C : D$; That is, as A to B, so C to D. Or thus, $\frac{A}{B} = \frac{C}{D}$: that is, the Exponent $\frac{A}{B}$ is equal to the Exponent $\frac{C}{D}$, and shews the proportions to be the same.

DEFINITION VI.

Magnitudes that have the same proportion, let them be called ANALOGALS, or Proportionals.

After the Definition, which had now declared what Magnitudes are said to be in the same proportion; there is next a name given to them. Let them be called *Ἀνάλογον*, *Analogon*, says *Euclide*. This word is used adverbially, and literally translated is *Equiproportionally*: So that it is indifferent to say Magnitudes to be in the same proportion, or in one word to be *Equiproportionally*, or *Analogally*, the first to the second, as the third to the fourth. For *Ἀνά* is a Preposition of equality, and *Ἀνάλογον* implies *ἀνά τὸν λόγον*: In, or of equal proportion, or the same proportion repeated. From *Analogon* is the abstract *Analogia*, that is equality of proportions, which is commonly rendred *Analogy*, or *Proportionality*.

Annotations on the Fifth Definition.

This Definition all the *Greeks* received without exception, and *Archimedes* himself makes frequent use of it, as the only general and infallible sign of proportional Magnitudes. But many of the Modern Geometricians stumble at it; for want it seems of a through inspection into the bottom of this matter, which the Ancients better understood.

First, Therefore we shall examine upon what motives Men are commonly induced to except against it. Secondly we shall explain the nature of this Definition, and what was *Euclid's* intent, and meaning, in defining proportional Magnitudes after such a manner. Thirdly, we shall prove that this Definition is most suitable to the nature of Magnitude, and fully satisfactory when rightly understood.

First then the joint, or Simultaneous Defect, Equality, or Excess in the Equimultiples of Antecedents, and Consequents, which affection *Euclide* puts for the determining of proportional Magnitudes, seems to have no affinity, or conjunction with the usual, and natural notion that Men have of things, which they account, and name Proportionals: and therefore they are startled at such a strange, and unexpected Definition, so much different from the common Idea they have of this matter. For in all affairs wherein Men are conversant, whether concerning Quantities, Qualities, Powers, Actions, Motions, Value, Commerce, or other Negotiations, they do naturally judge by *Number*, and *Measure*, when things have between themselves the same proportion, and when not. Where then between things compared together they find an agreement according to *Number*, and *Measure*, there such are properly accounted Proportional to one another. As when A is as much of B, as C is of D, *Tantum, Quantum*, the one of the other, then A to B, and C to D, are commonly reputed to be in the same proportion. So that in Proportionals the Antecedents are conceived to be always EQUIQUANTUPLES of their Consequents: and this Equality to be set forth in number, and measure.

Wherefore Men thus rationally conceiving aforehand the condition, and state of things they call proportional, and being thus already acquainted both with the thing, and name, they are thereupon prepossessed in their judgements, and their very prænotion, or natural præcognition of things Proportional, necessarily creates in them a prejudice against *Euclid's* Definition: wherein there appears no relation to number, or measure, nor any coherence with their own Conception of Proportionals.

This seems to have been the chief ground of the Exceptions, that are made
D d 3 against

against *Euclid's* Definition. For the 11th. Definition of the Third Element, where *like Segments of Circles* are defin'd by the *Equality of Angles which they receive*, is as lyable to be excepted against as this Definition of Proportional Magnitudes; because the similitude of Circular Segments appears not from the equality of those Angles: but does properly consist in this, that they are the same, or equal portions of their own Circles. But then how to set forth, why they are the same, that is, to define *like Segments of Circles* Essentially, from the immediate, and formal Cause of their likeness lyes under the same difficulties with the Definition of *like Proportions of other Magnitudes*. Wherefore in both these Cases *Euclide* is constrained to take a sign of the thing, instead of the thing it self, that is, to give a Secondary, or Artificial Definition instead of the Prime, and Natural. For in Mathematics Definitions are not always Philosophical, but Specially framed in order to their Geometrical uses; as here Def. 5. El. V. is on purpose contrived for the Demonstrations of Proportional Magnitudes. Yet it is to be observed that Def. 11. El. III. (tho' of the same nature with this) is received without dispute, because Men being less conversant with the true nature of *like Segments of Circles*, than of *like Proportions*, come not prepossessed with any other notions of their own, but do acquiesce in what *Euclide* gives them to understand by *like Segments of Circles*. Whereas they quarrel at this Definition of Proportional Magnitudes, finding themselves disappointed in their own thoughts and præconception of Proportionals. We must acknowledge the vulgar notion among Men to be most agreeable to the nature of Proportionality, and of all things, that can be esteemed Proportionals. For number alone it is, by which in common Commerce the Measure and Value of things is most naturally made, and signified: but where Number, and Measure cannot be apply'd as fit instruments for a Mathematical proof in all such things, that are really Proportionals, there for Demonstration sake some other certain, and definitive mark is to be taken: as *Euclide* hath done in his Definition of Proportional Magnitudes.

But where the Subject would bear it, *Euclide* clears this matter, and gives an other Definition of Proportional Numbers, tho' not of Proportion it self; for that his former Definition of proportion given in this 5th. Element answers as well to Numbers, as to Magnitudes, only changing the names, and putting *Numbers* in the place of *Homogeneous Magnitudes*. For in both Numbers and Magnitudes the Habitude is κατὰ μέγεθος, according to Quantity.

Now the 20th. Definition of the 7th. Element concerning proportional Numbers is very plain, and conformable to the common conception that Men have of things proportional, in these perspicuous words.

The Definition of Proportional Numbers.

Numbers are Proportional, when the first is of the second, and the third of the fourth, equally Multiple; or the same Part; or the same Parts.

This Definition is most natural, and plainly shews Numbers to be proportional from the immediate causes, which make them so. For in *equally Multiple*, or the *same Part*, or the *same Parts* is expressly laid down wherein the Antecedents do equally contain their Consequents, that is, are each Equiquantuples of their Consequents. Only 'tis to be noted, that *Multiple* is here to be interpreted in a more general sense than usual, as not strictly signifying precisely many times, or more than once, for one number to contain an other number, as one to be exactly Double or Triple, &c. of the other; but numbers are here understood to be *Equally Multiple* of others, when the Antecedents do equally, or alike contain the Quantity, that is, the numerical parts of their Consequents, one as much, or as often as the other does in any manner. As either simply once in Proportion of *Equality*: or once with a Part, or Parts in proportion *Superparticular*, or *Superpartient*: or else more than once with a Part, or Parts in proportion *Multiple Superparticular*, or *Superpartient*. Here therefore for a farther illustration of this matter, take these

examples: In which we have according to the Form of this Definition given the Varieties of four Rational Proportionals, both particularly in Numbers, and generally in *Species*.

Multiple.

$$6 . 2 :: 9 . 3 .$$

$$3 A . A :: 3 B . B .$$

Multiple Superparticular.

$$7 . 2 :: 21 . 6 .$$

$$3\frac{1}{2} A . A :: 3\frac{1}{2} B . B .$$

• *Superparticular.*

$$5 . 4 :: 15 . 12 .$$

$$1\frac{1}{4} A . A :: 1\frac{1}{4} B . B .$$

Submultiple.

$$3 . 12 :: 5 . 20 .$$

$$\frac{1}{3} A . A :: \frac{1}{5} B . B .$$

Multiple Superpartient.

$$11 . 4 :: 33 . 12 .$$

$$2\frac{3}{4} A . A :: 2\frac{3}{4} B . B .$$

Superpartient.

$$7 . 4 :: 21 . 12 .$$

$$1\frac{1}{2} A . A :: 1\frac{1}{2} B . B .$$

Subsuperparticular.

$$9 . 12 :: 15 . 20 .$$

$$\frac{3}{4} A . A :: \frac{3}{5} B . B . \&c.$$

Thus in *Multiple*, *Part*, or *Parts*, taken sometimes *singly*, and sometimes *jointly*, are comprehended all kinds of Rational Proportions, and Proportionals.

Whereas then this Definition evidently sets forth the nature of Proportional Numbers, which also very much corresponds with the common notion of Men; wherein they deem things to be proportional; it may be reasonably demanded, why some such like Definition might not have been accommodated to proportional Magnitudes; seeing that the Definition of Proportion answers both to Magnitudes, and Numbers? But the reasons why so clear, and natural a Definition could not be apply'd to Magnitudes, are irresistibly cogent, and manifest to a Geometrician, tho' not falling within the cognizance of the vulgar.

And to explain this matter, we are to recollect 1st. That Proportion is an Habitude according to Quantity. 2^{dy}. That the Quantity of a Magnitude, or Number is only to be shewn in its reference, or proportion to some *certain*, and *known measure*. 3^{ly}. That every Measure is some one simple thing: In Magnitude it is an Inch, or Foot, or Cubit, or any other *stated* measure: In Numbers it is an Unite; of which it *naturally* consists; therefore as afore said, every single number carries in its very name the quantity of it self; that is, its proportion to an Unite, the common Constituent, and natural measure of all Numbers: As the quantity of 9 is nine Unites; that is, nine times one, or 9 to 1: The quantity of 10 is so many Unites, or 10 to 1. And again, in comparison of number to number, 9 compared to 10, is nine parts of 10, or the proportion of 9 to 10. The quantity of 5 to 10 is an half part of 10, or 5 to 10; Every less number compared to a greater being apparently a part, or some parts of the greater.

But Magnitude (tho' infinitely divisible into parts) is not, as Number, an Aggregate of certain parts. It hath no such fundamental beginning, nor any *Original Part*; No *Minimum*, or *Geometrical Unite* from which to demonstrate in an Elementary method the equality, and inequality of Proportions; as *Euclide* hath clearly delivered the whole matter in his Elements of Numbers.

What of this kind Magnitude hath, or can have, is only arbitrary, and variable after the customs, and usages of several Nations, and People, which therefore cannot give a natural foundation to the Geometrician, whereon to build a general doctrine of proportional Magnitudes upon the same grounds, or such a kind of Definition from Multiple, Part, and Parts, on which that of numbers is settled, and Demonstrations framed accordingly.

And altho' there be but one True, and the same essential notion of Proportionality both in Magnitude, and Number, which consists in *this*, that the *Antecedents* are *Equiquantuples* of their *Consequents*; which in numbers is plainly, and demonstratively to be set forth from an Unite, the *natural Minimum* of all numbers: yet in Magnitudes, because there is not a *natural Minimum*, the same, or like method cannot possibly be used.

Again,

Again, besides this main impediment, founded in the very nature of Magnitude, there is another more invincible obstacle of Incommensurability among Magnitudes, which renders Incommensurable Magnitudes utterly incapable of any imaginary common Part, or that one can ever be made a Multiple of the other; and therefore that plain, and natural Definition of proportional Numbers can be no ways applicable to all Magnitudes, whereby to demonstrate that they are proportional to one another.

Euclide therefore is necessitated to seek for a more general Character of Proportionals, which may serve for all Magnitudes, Commensurable, and Incommensurable. And tho' he is in Magnitudes Incommensurable to one another deprived wholly of any helps from a common Part, or that one can ever be made a Multiple of the other; which are the fundamental instruments of demonstration in the Doctrine of Numbers: yet because all Magnitudes Commensurable, and Incommensurable, may have any Multiples of themselves, as duples, triples, quadruples, &c. and so by consequence any Equimultiples; for that any two Magnitudes may each be alike doubled, tripled, quadrupled, &c. Therefore *Euclide* has recourse to such a Multiplication, as the only, or at least the most commodious remedy left in this matter, and searches how far an Equimultiplication of the Antecedents, and an Equimultiplication of the Consequents might contribute toward a sure and certain discovery what Magnitudes (whether Commensurable or Incommensurable to one another) are proportional, and what not, when they are compared together; which proved very useful to his purpose.

§. 1. For *Euclide* found demonstratively in any four proportional numbers (and in no other but proportional numbers), that if the Antecedents be equally multiplied according to any multiplication whatsoever, and again the Consequents be equally multiplied according to any multiplication whatsoever; then shall the Equimultiples of the Antecedents be proportional to the Equimultiples of their Consequents in some one, or other proportion for ever. So those that are once Proportionals shall in this manner ever be some Proportionals.

This singular Power, and Property of four Proportionals, always to make from such a multiplication other four Proportionals, is the foundation of *Euclid's* Definition of proportional Magnitudes; as will clearly appear in the following Paragraphs.

- §. 2. For because all proportions are either of equality, or of the greater, or lesser inequality; therefore in any four Proportionals if one of the Antecedents be either equal to its Consequent, or greater, or less; the other Antecedent also must accordingly be either equal, greater, or less than its Consequent. For if one Antecedent should be equal to its Consequent, and the other unequal, or the one be greater, and the other less, it is an immediate contradiction, and impossibility to suppose equals to be to one another in the same proportion with unequals; or the greater to the less to be in the same proportion with the less to the greater.
- §. 3. Seeing then that of four Proportionals the Equimultiples of the Antecedents, and Consequents taken according to any multiplication whatsoever, are found to be always Proportionals (*as it is noted in the first Paragraph*): therefore the Equimultiples of the Antecedents are together equal, or together greater, or together less than the Equimultiples of the Consequents (*by the foregoing Paragraph*). When therefore it shall be demonstrated that according to any multiplication whatsoever the Equimultiples of the Antecedents [*viz.* of the first, and third Terms] are together equal, greater, or less than the Equimultiples of the Consequents [*viz.* of the second, and fourth Terms], then are the Antecedents themselves in the same proportion to their Consequents, the first to the second, as the third to the fourth, according to the 5th. Definition.

Thus we see that *Euclid's* Characteristic property of Proportional Magnitudes is but one single remove from the immediate, and apparent Cause of the same, as it hath been laid down in the first Paragraph. For four Magnitudes are said by *Euclide* to be Proportionals, when the Equimultiples of the Antecedents are together either equal, greater, or less, than the Equimultiples of their Consequents, according to any multiplication whatsoever; Because according to any multiplication whatsoever the Equimultiples of the Antecedents are always proportional to the Equimultiples of their Consequents. But this intermedial Proposition was concealed by *Euclide*, and in silence passed over: as Geometricians use to do, who are not obliged to give the reason of their Definitions. They are only to be privately assured of the truth of that property, which is attributed to the thing: and it lyes wholly on our part either to accept it, or to make manifest some unaptness, or insufficiency in the use of it. Now for the matter in hand, *Euclid's* Definition of Proportional Magnitudes hath stood the tryal of about two thousand years unshaken: and altho' the Antecedents, and Consequents multiplied according to any multiplication whatsoever, seem in words to make this thing perplex, and troublesome (as *Tacquet* unjustly calls it a Labyrinth) yet in use, and practice it is most plain and easie, as we shall find in the first, and last Propositions of the 6th. Element.

If it be farther demanded why *Euclide* did not define Magnitudes to be proportional, when the Equimultiples of the Antecedents were porportional to the Equimultiples of the Consequents; but rather when the Equimultiples of the Antecedents were together either equal, greater, or less than the Equimultiples of the Consequents, seeing that this is only a Result of that; the reason is apparent. For to define four Magnitudes to be proportional, when the Equimultiples of the Antecedents are proportional to the Equimultiples of the Consequents, is to define the same thing by it self, *Ignotum per æquè Ignotum*; because we are still as much to seek how to know that these Equimultiples are proportional, which was the gross oversight of *Campanus*, and *Orontius* in this matter. But how to know when the Equimultiples of the Antecedents are together either equal, greater, or less than the Equimultiples of the Consequents, each Equimultiples being taken according to any multiplication whatsoever, can be readily made out, and be at once demonstrated by some known Geometrical Proposition: As may be found in all the Ancient Greek Geometricians, with what facility it is apply'd upon any occasion of demonstrating Magnitudes to be Proportionals: Therefore *Euclide* gave this general Concomitant of any four Proportionals for an infallible Indicant, and the definitive Character of Proportional Magnitudes.

And for a farther Explanation of this matter, we shall here give a demonstration of the Fundamental Proposition mentioned in the first Paragraph, upon which *Euclid's* Definition of Proportional Magnitudes is grounded, and from whence it doth most naturally arise.

Of four Proportionals the Equimultiples of the Antecedents are proportional to the Equimultiples of the Consequents, according to any Multiplication whatsoever.

Let the four Proportionals be $A . E :: B . C$. and let N denote any number multiplying the Antecedents A, B , and let n denote any number multiplying the Consequents E, C . I say, that

$$\S A \text{ in } N . E \text{ in } n :: B \text{ in } N . C \text{ in } n.$$

For let it be $A . E :: B . C$. and then Alternately. Prop. 13. El. VII.

$$\S A . B :: E . C . \text{ Therefore}$$

$$\begin{array}{l} A . B :: A \text{ in } N . B \text{ in } N . \text{ and } \\ E . C :: E \text{ in } n . C \text{ in } n . \end{array} \} * \text{ Prop. 17. El. VII.}$$

$$\S \text{ But } A . B :: E . C . \text{ Therefore}$$

$$A \text{ in } N . B \text{ in } N :: E \text{ in } n . C \text{ in } n . \text{ and Alternately}$$

$$\S A \text{ in } N . E \text{ in } n :: B \text{ in } N . C \text{ in } n . \text{ Which was to be demonstrated.}$$

¶ c

* If

* *If a Number multiply two Numbers the Products are in the same proportion with the given Numbers.* This is next to a common notion, for who naturally sees not that as *three to two*, so *4 threes to 4 twos*? And upon this obvious Principle *Euclide* multiplied the Antecedents by any one number whatsoever, and again the Consequents by any one number whatsoever; from whence by one, or two easie steps he found (as we see) his Definition of Proportional Magnitudes.

Thus from the very natural Power of Proportionality to make of any four Proportionals, other four Proportionals for ever; whose Antecedents therefore shall also ever be together either equal, greater, or less than their Consequents, *Euclide* puts this necessary Adjunct for the general, and next immediate Indicant of all Proportionals.

For whether the proportions be Rational, or Irrational; the Magnitudes Commensurable, or Incommensurable, there is this agreement in all things conceived to be Proportionals, that the Antecedents do equally, or alike contain certain quantities of their Consequents; altho' this Equality, or Similitude cannot be declared, and signalized by Equimultiples, or the same Part, or the same Parts of the Consequent, as it is in Numbers; yet in regard that Proportionality is every where of the same nature, therefore one common Property is by *Euclide* justly applied to all for a standing Indicant of any four Proportionals.

For the side of one Square is the same portion of its Diameter, that the side of an other is of its Diameter, as truly and really, as if it were explicable by Part, or Parts, by Number, or some common Measure: So that in Magnitudes Incommensurable, there is something in nature, which is Equivalent, and *Tantamount*. For the Square of the side is demonstrated to be equal to half the Square of the Diameter; and therefore the sides of all Squares are manifestly Equiquantuples, and the same portions of their own Diameters, and so in the same proportion to them: the nature of Proportionality, or Equiquantipleness between Antecedents, and Consequents, being no ways altered in having the proportions either effable, or ineffable by number, and measure. For the Affections, and Properties of Proportionals, are the same in Numbers, and Magnitudes, only they cannot be demonstrated after the same manner.

And therefore to require a Definition of Proportional Magnitudes (answerable to that of Numbers) from which to demonstrate by some certain measures in what manner the Antecedents do equally contain their Consequents, in the making of equal proportions, where between the Antecedents, and Consequents there are *in nature no such measures*, is an impossibility unreasonably required of *Euclide* to make possible. But what is possible, he performs: and gives a Definition, or rather a Definitive mark, which certainly shews when Magnitudes are Proportional, tho' not *How*, and in what particulars they are Proportional: and in this Case to demand of him a demonstration of his own Definition is *Iniquissimum mandatum*, repugnant to the doctrinal methods of all Arts, and Sciences. It had been a business proper enough for a *Scholiast*, or Commentator to illustrate, and as a Philosopher, to set forth the causes of the different Definitions given by *Euclide* of proportional Numbers, and of proportional Magnitudes, from the different Natures and Constitutions of Number, and Magnitude: but this had been improper for the Elementator himself to submit unto.

In short therefore, to sum up this whole and much controverted matter. Forasmuch as it hath been shewn, that Magnitudes are incapable of being made known, how they are in the same proportion to one another from *any Equimultiples, or the same Part, or the same Parts of the Consequents*, according to the Definition of proportional Numbers; therefore instead of such a Primary, and Essential Definition, *Euclide* laid hold of the next immediate Property that he found to spring from the natural Power of proportionability, and might produce an infallible Character of all Proportionals: which is thus apparently deduced.

Of all Proportionals, and only of Proportionals any whatsoever Equimultiples of the Antecedents, shall ever be Proportional to any whatsoever Equimultiples of the Consequents.

Wherefore of all Proportionals, and only of Proportionals any whatsoever Equimultiples of the Antecedents ever are together either equal, greater, or less, than any whatsoever Equimultiples of the Consequents.

And therefore where any whatsoever Equimultiples of the Antecedents are demonstrated to be together either equal, greater, or less, than any whatsoever Equimultiples of the Consequents, there the first exposed Magnitudes were Proportionals: The Antecedents Equimultiples of their Consequents, each of each Equimultiple in any proportion whatsoever, Rational or Irrational.

Thus now this general Test, and property of Proportionals, that by some of our Moderns hath been calumniated for being so remote and intricate, is in brief shewn to be nearly conjoyn'd with the nature of whatsoever Quantities are, or can be esteemed Proportionals, and laid open in few words.

If any thing yet remains of dissatisfaction, it lyes unavoidably in the nature of Magnitude it self, which hath no *Minimum*, and therefore no natural, or standing Measure; where also incommensurability between Magnitudes is not capable of any imaginable common Measure; or of any better evidence than what *Euclide* hath given in his Definition of Proportional Magnitudes: and not in any deficiency either in Geometry, or the Geometrician.

DEFINITION VII.

When of Equimultiples the Multiple of the first shall exceed the Multiple of the second; and the Multiple of the third shall not exceed the Multiple of the fourth; then the first is said to have to the second a greater proportion, than the third hath to the fourth.

In Def. 5. Magnitudes were determined to be in the same proportion, when the Equimultiples of the Antecedents should together be either equal, greater, or less than the Equimultiples of their Consequents, according to any multiplication whatsoever; and therewithal 'twas necessarily implied, that when it should be found otherwise in any one multiplication, then those Magnitudes were not in the same proportion to one another.

Now this Definition farther declares which Antecedent hath to its Consequent the greater proportion, and which the less.

As for instance, let A 4, be compared to B 3; and C 6, to B 5: then let be taken Equimultiples of the Antecedents A 4, and C 6; namely *quadruples* E 16, and F 24. And again, take of the Consequents B 3, and D 5; other Equimultiples; namely *quintuples* G 15, and H 25. Here 16 the *quadruple* of the Antecedent 4 exceeds 15 the *quintuple* of the Consequent 3: but 24 the *quadruple* of the Antecedent 6, exceeds not 25 the *quintuple* of the Consequent 5. Therefore the Antecedent A 4, is said to have to the Consequent B 3, a greater proportion, than the Antecedent C 6, has to the Consequent D 5: and accordingly the Quotients shew the same inequality of these proportions. For 4 to 3 is $1\frac{1}{3}$, or *sesquitertial*: and 6 to 5 is but $1\frac{1}{5}$, or *sesquiquintal*.

When therefore in Magnitudes, or Numbers such a disagreement is demonstrated to be between the Equimultiples of the Antecedents, and of the Consequents in

equality, excess, or defect upon any one multiplication; then as they are not in the same proportion by Def. 5; so this Definition expounds which Antecedent hath to its Consequent the greater proportion, and which the less.

But again it is to be observed, that Magnitudes, or Numbers may also have an agreement between the Equimultiples of the Antecedents and Consequents in equality, excess, or defect, according to some multiplications; yet not according to all, as it is required in this matter of Proportionals.

For in the former instance take of the Antecedents 4 and 6, the quadruples 16 and 24; and of the Consequents 3 and 5, the triples 9 and 15. Here of the Antecedents 4 and 6, their Equimultiples 16 and 24, exceed 9 and 15 the Equimultiples of the Consequents 3 and 5, each exceeding each, 16, 9, and 24, 15: yet the proportion of 4 to 3 and of 6 to 5, is not the same in both, as was now before shewn.

Wherefore tho' in some multiplications of Disproportionals, the Equimultiples of the Antecedents and Consequents may happen to agree in a joint equality, excess, or defect: yet if any one multiplication shall be demonstrated to disagree, then by Def. 5. these Equimultiples did not arise from four Proportionals; for the reason before given that Proportionals shall ever make the Equimultiples of the Antecedents proportional to the Equimultiples of the Consequents; and be therewithal either together equal, or greater, or less according to any multiplication whatsoever: which is the given Signal of Proportionals.

DEFINITION VIII.

A *Nalogy, or Proportionality is a similitude of Proportions.*

Magnitudes, which have the same proportion were before in Def. 6. named *Analogals*, or Proportionals. Now again in the Abstract is here defined Analogy. And what in the comparison of Magnitudes was called *Equality*, that in the comparison of proportions this Definition calls *Similitude*: So that to have *the same* proportion or *like*, or *equal*, signifies all one thing. Yet *Euclide* never uses to say Magnitudes in *like*, or *equal* proportion; but always Magnitudes in the same proportion, or having the same proportion.

This Definition therefore ought farther to be examined, which is more suitable to the speculations of a Philosopher, than of any use to a Geometrician. For first, in Def. 3. *Euclide* tells us what is proportion; next in Def. 4. what Magnitudes are capable of proportion to one another; then in Def. 5. what Magnitudes are in the same proportion, and therewithal what are not. Again in Def. 6. to Magnitudes in the same proportion he gives a name; and lastly Def. 7. shews when Magnitudes are not proportional, where the proportion is the greater, and where the less. And thus he having finished his Explanations of Proportions, Proportionals, and Disproportionals; it is manifestly to no purpose to define again Analogy to be a similitude of Proportions: which bare Definition neither shews wherein it consists, nor by any sign to know when there is between two proportions a similitude, that some use might be made thereof in the following Demonstrations. But for this end a Sign and Character of Proportionals was given before in Def. 5. Plainly then this is a superfluous Definition, a remark of some Scholiast shuffled into the Text without any order, and to no use. The next that follows is likewise a By-note to as little purpose, and both evidently interrupt the Coherence of *Euclid's* genuine Definitions.

DEFINITION IX.

A *Nalogy is in three Terms at the fewest.*

This is so far from being one of *Euclid's* Definitions, that it is most clearly no
3 Definition

Definition at all. Besides in strictly speaking it is not true. For Analogy is always in four Terms, two and two in the same proportion, Antecedent to Consequent, and Antecedent to Consequent: tho' it may sometimes be only between three Magnitudes, or Numbers expressed and exposed: As let there be three Magnitudes A, B, C, in the same proportion A to B, as B to C; that is, in whatsoever proportion A is to B, in the same again is the self same B to C. Here are in reality only three Magnitudes, but between them two proportions, and therefore two Antecedents, and two Consequents: so of necessity there are always four Terms in every Analogy, the first to the second, and the third to the fourth; and therefore when the Magnitudes are only three, the middle is necessarily understood to supply the second, and third Term, and the like in Numbers. Now forasmuch as two equal proportions may sometimes be in three, sometimes in four Magnitudes, or Numbers, therefore equal proportions are distinguished into *Continual*, and *Discrete*.

Of Continual Proportion.

When Magnitudes, or Numbers in any proportion whatsoever are continued in the same proportion, by an immediate Coherence of Terms with one another, each intermedial being twice taken, first as a Consequent to the preceding, next again as an Antecedent to the following, then this is called Continual proportion: as A to B, so B to C, so C to D, and is thus noted, A, B, C, D —. And the Magnitudes, or Numbers are said to be *ἐξῆς ἀνάλογον*, *Deinceps proportionales*, continually Proportionals.

Of Discrete Proportion.

Again, when the intermedial Magnitudes, or Numbers are not continued, or twice taken, as a Consequent, and an Antecedent, but the proportions are as A to B, so C to D (noted thus A . B :: C . D); yet so is not B to C; the intermedials B and C being discontinued, and separated from each other in that proportion, then this is called Discrete proportion.

So that a Proportion is said to be continual, when the Proportions (whether two, or three, or more) are the same, and exposed in continued Terms. As A to B, so B to C, so C to D: Or in numbers, as 16 to 8, 8 to 4, 4 to 2.

And Discrete Proportion is, when the Proportions are likewise the same, but exposed in discontinued Terms. As A to B, so C to D, so E to F: Or in numbers, as 12 to 8, 6 to 4, 3 to 2.

Of Concatenate Proportions.

There are also Concatenate Proportions, *ἐξῆς λόγος*, *Proportiones deinceps*; when the proportions are different (not all the same, as in Continual, and Discrete proportions) yet are exposed in continued Terms, *ἐν ἐξῆς ὅροις*. As 15 to 10, 10 to 6, 6 to 3, are to be called *Concatenate Proportions*.

Therefore we are carefully to distinguish between Continual Proportion, and Proportions in continued Terms. By continual Proportion is always meant the same, or equal Proportions repeated orderly in continued Terms. By the other is to be understood various, and different proportions, which likewise follow one another in continued Terms. As 4 to 3, 3 to 2 are Concatenate proportions: But 8 to 4, 4 to 2, are to one another in Continual proportion: as 8 to 4, 6 to 3, are in Discrete proportion.

It is farther worthy of observation that in Discrete proportion, because the intermedials break off one from the other, therefore an Analogy may be here between Magnitudes of different kinds, so that a Line may be to a Line, as a Superficies to a Superficies, or a Solid to a Solid. But in Continual proportion, because all the Terms from the first to the last are continued in the same proportion to one another, therefore they must all be Homogeneous, for that Heterogeneousals can have no proportion one to an other: therefore in Continual proportion the

Terms are either all Lines, or all Planes, or all Solids. Likewise in Concatenate proportions where the proportions are put various, and different; yet are in *Continued Terms* Concatenated, as A to B in one proportion, B to C in an other, C to D in some other, there the Magnitudes must also for the same reason be all Homogeneous, like as they were in Continual proportion.

DEFINITION X.

When three Magnitudes are Proportional, the first is said to have to the third a Duplicate Proportion of the first to the second.

Let A, B, C, be three proportional Magnitudes, as A the first to B the second, so B the second to C the third. From which continuation of proportions the proportion of A the first to C the third is understood to be compounded of the two intermedial proportions, *viz.* of the proportion of A the first to B the second, and of the proportion of B the second to C the third: and therefore A the first is said to have to C the third a Duplicate proportion of A the first to B the second; because the proportion of A to B is the same with that of B to C, and so is doubled, or twice continued between the proportion of A the first to C the third. But if the two intermedial proportions be not the same, and one of them be different from the other, as 4 to 3 in proportion Sesquitercial, and 3 to 2 in proportion Sesquialteral, then 4 the first cannot be said to have to 2 the third a Duplicate proportion of 4 the first to 3 the second, as in equal proportions; but yet in reference to this Definition, it is said by *Euclide* at Prop. 23. El. VI. to have a proportion compounded of 4 to 3, and of 3 to 2, by which the extent of this tenth Definition, and *Euclid's* meaning is plainly declared. So that whether the intermedial proportions be equal, or unequal, the proportion of the extremes is in general understood by *Euclide* to be compounded of the intermedial proportions: And when the intermedial proportions are put the same, then the proportion of the first to the third, is in Special called by *Euclide* a Duplicate proportion of the first to the second.

Now to illustrate this farther in Numbers, let the three Proportionals be 9, 3, 1, The proportion of 9 to 1 is said to be a duplicate proportion of 9 to 3, because between 9, and 1 the proportion of 9 to 3, or triple proportion is twice continued. For 9 is triple of 3, and 3 of 1: so that the Noncuple proportion of 9 to 1 is said to be triple proportion duplicated; that is, to be compounded of two triple proportions, which continuedly intervene between 9, and 1.

DEFINITION XI.

When four Magnitudes are proportional, the first is said to have to the fourth a triplicate proportion of the first to the second: And so forward always more by one, as long as the Analogy shall be continued.

Let A, B, C, D, be four proportional Magnitudes as A to B, so B to C, so C to D, in continual proportion, then A the first is said to have to D the fourth, a triplicate proportion of A the first to B the second; because between the Term A, and D, the same proportion of A to B is thrice iterated.

Likewise if there be a Series of five Terms A, B, C, D, E, the proportion of A the first to E the fifth, is said to have a quadruplicate proportion of A the first to B the second, and so forward in a continue Chain the proportion of the extremes has its name from the number of the intermedial proportions; Duplicate from two, Triplicate from three, &c. which proportion of the extremes is also said to be compounded of those intermedial proportions.

When

When equal proportions are continued in many Terms, it is usually called *PROGRESSION*, or Geometrical Progression.

DEFINITION XII.

Homologal Magnitudes are Antecedents to Antecedents, and Consequents to Consequents.

When four Magnitudes as A, E, B, C, are said to be Proportionals, it is farther requisite to set forth distinctly the *Analogism*, or order of these proportional Magnitudes in their proper Terms; *that is*, to declare which are the Antecedents, and which the Consequents, and besides which Antecedent belongs to each Consequent: whether A to E as B to C, or otherwise. If it be put, as A to E so B to C, then A, and B the Antecedents are called Terms *Homologal*: and E and C the Consequents are likewise Terms *Homologal*; that is, *Comproportional*, or Terms of the same condition as *Conjugate* or *Consociate Pairs* in the order, and disposition of four Proportionals. The Antecedents being the Terms compared, and the Consequents the Terms to which is made the comparison.

Now when four Magnitudes are proportional, and the *Analogism*; *that is*, the *Homologal* Terms are orderly set forth, as A . E :: B . C, *Euclide* shews in the following Definitions what changes, and besides what alterations the four primary, or first proposed Proportionals are capable of; so that the changed, or altered Terms may still be proportional to one another. These variations are made five manner of ways.

DEFINITION XIII.

Alternate Proportion is a Sumption of the Antecedent to the Antecedent, and of the Consequent to the Consequent.

Here is made only a change in position of the primary Proportionals, without any alteration of the same Terms, and the Sumption, or Comparison of one Antecedent to the other as a Consequent; and of one Consequent as an Antecedent to the other Consequent is called Alternate proportion: As let the primary Proportionals be A to E, as B to C, and thus set forth.

$$\begin{array}{l} \text{Primary Proportionals, } \left\{ \begin{array}{l} 15 . 5 :: 12 . 4. \\ A . E :: B . C. \end{array} \right. \\ \text{then Alternately, } \left\{ \begin{array}{l} A . B :: E . C. \\ 15 . 12 :: 5 . 4. \end{array} \right. \end{array}$$

The Analogy between the Terms of Alternate proportion is demonstrated in Prop. 16.

DEFINITION XIV.

Inverse Proportion is a Sumption of the Consequent as Antecedent, to the Antecedent as Consequent.

Here the Proportional Terms are otherwise changed in position only, the Consequents into Antecedents, and the Antecedents into Consequents: The inward Terms into the outward, and the outward into the inward: As let there be again

$$\begin{array}{l} \text{Primary Proportionals, } \left\{ \begin{array}{l} 15 . 5 :: 12 . 4. \\ A . E :: B . C. \end{array} \right. \\ \text{then Inversely, } \left\{ \begin{array}{l} E . A :: C . B. \\ 5 . 15 :: 4 . 12. \end{array} \right. \end{array}$$

We may observe, that in Alternate proportion only one Antecedent is changed

changed into one Consequent, and one Consequent into one Antecedent. But in inverse proportion, both the Consequents are made Antecedents; and both the Antecedents are turned into Consequents, whereby proportions of the greater, and less inequality are transmuted into one another.

It is likewise Inverse proportion when the Terms are taken backward, or contrarily, as if $A . E :: B . C$. It is likewise between the Terms inversely $C . B :: E . A$.

The Analogy of inverted proportion is demonstrated by a Corollary of Prop. 4.

DEFINITION XV.

C*omposition of proportion is a Sumption of the Antecedent together with the Consequent as one, to the same Consequent.*

Here now is made an alteration of the primary, or first proposed Terms, by adding each Antecedent, and Consequent, together as one Term, and so compared to its correspondent Consequent. For let here be as before,

$$\begin{array}{l} \text{Primary Proportionals} \left\{ \begin{array}{l} 15 . 5 :: 12 . 4. \\ A . E :: B . C. \end{array} \right. \\ \text{then by Composition,} \left\{ \begin{array}{l} A + E . E :: B + C . C. \\ 15 + 5 (=20) . 5 :: 12 + 4 (=16) . 4. \end{array} \right. \end{array}$$

The Analogy between the Terms of Compound proportion is demonstrated in Prop. 18.

DEFINITION XVI.

D*ivision of proportion is a Sumption of the excess, wherein the Antecedent exceeds the Consequent, to the same Consequent.*

Here again is made another alteration of the primary Terms, by subtracting each Consequent from its Antecedent, and then each excess is assumed as an Antecedent, and compared to its proper Consequent: As let there be

$$\begin{array}{l} \text{Primary Proportionals} \left\{ \begin{array}{l} 15 . 5 :: 12 . 4. \\ A . E :: B . C. \end{array} \right. \\ \text{then by Division,} \left\{ \begin{array}{l} A - E . E :: B - C . C. \\ 15 - 5 (=10) . 5 :: 12 - 4 (=8) . 4. \end{array} \right. \end{array}$$

The Analogy between the Terms of Divided proportion is demonstrated in Prop. 17; and 'tis manifest, that divided proportion must be of the greater inequality, for that the Consequent is to be subtracted from the Antecedent.

What is commonly translated Composition, and Division of proportion, and called by *Euclide Synthesis*, and *Diæresis*, is only an addition, and subtraction of Proportional Terms: an addition of each Consequent to its Antecedent, and a subtraction of each Consequent from its Antecedent. But Composition, of which we have made some mention in the Annotations on the 10th and 11th foregoing Definitions, is taken in a far different sense, and called *σύνθεσις*, *Synkeisis*, a Composition, or Commixture of Proportions one with another, and not a *Synthesis*, or addition of the Terms to one another: As shall be farther explained at the 5th Definition of the 6th Element.

DEFINITION XVII.

C*onversion, or ANASTROPHE of proportion is a Sumption of the Antecedent to the excess, wherein the Antecedent exceeds the Consequent.*

As let it be $\left\{ \begin{array}{l} 15.5 :: 12.4. \\ \dots A.E :: B.C. \end{array} \right.$
 Then by *Conversion*, $\left\{ \begin{array}{l} A.A-E :: B.B-C. \\ 15.15-5 (=10) :: 12.12-4 (=8). \end{array} \right.$

The Analogy between the Terms of *Converted* Proportion is demonstrated by the Corollary of Prop. 19th.

Again, the Consequent also might have been compared to the same excess, as $E.A-E :: C.B-C. 5.10 :: 4.8$. But this is only proportion of *Division inverted*, and inversion of proportion having been laid down before; it were artificial to repeat the same again as a new kind of change. For it is to be noted that the simple, or primary Proportional Terms being either compounded, divided, or converted, may in like manner be again changed by Alternation, and Inversion. As

by Composition, and Alternation, $A+E.B+C :: E.C.$
 by Division, and Alternation, $A-E.B-C :: E.C. \&c.$

Again, if the Terms be only three in continual proportion, as $A, M, E :: 8, 4, 2$, they may in like manner be changed, and altered, for that M the middle Term is both a Consequent, and Antecedent. As

$8.4 :: 4.2.$
 $A.M :: M.E.$ and
 by Composition, and Alternation, $A+M.M+E :: M.E.$
 by Division, and Alternation, $A-M.M-E :: M.E. \&c.$

All these changes, and alterations of four Proportionals may in a view be thus set forth.

The primary Proportional Terms.

| | | |
|--------------|-------------------|---|
| | $A.E :: B.C.$ | |
| Alternation, | $A.B :: E.C.$ | } The primary Terms changed only in position. |
| Inversion, | $E.A :: C.B.$ | |
| Composition, | $A+E.E :: B+C.C.$ | } The primary Terms altered by Addition, and Subtraction. |
| Division, | $A-E.E :: B-C.C.$ | |
| Conversion, | $A.A-E :: B.B-C.$ | |

Here are first put four primary Terms, two, and two in the same proportion, as A to E , so B to C , from which are made by five ways of variation four other proportional Terms, two, and two in some one, and the same proportion to one another.

But now besides these, there may be put after another manner more Terms than four; as five, or six, or seven, &c. in two distinct orders, with an equal number of Terms in each order, two, and two in the same proportion, from which are also made four other Terms proportional to one another, as this following Definition declares.

DEFINITION XVIII.

Proportion of Equidistance (commonly called proportion of Equality) is a Sumption of the extremes by Subtraction of the inter-medial Terms; when, there being many magnitudes in one Rank, and others as many in an other Rank, taken two, and two in the same proportion, the first shall be assumed to the last in the preceding Rank, and likewise the first to the last in the following Rank.

Here is put a double Series of Magnitudes, as $\left\{ \begin{array}{l} 18, 3, 6, 9, 12, \\ A, B, C, D, E, \\ F, G, H, I, K, \\ 24, 4, 8, 12, 16, \end{array} \right.$ F f

Two,

Two, and two of each Rank being in the same proportion to one another, A to B, as F to G: and again, B to C as G to H, &c. Then subtracting, or passing over the intermedials in each Rank, if A the first be assumed to E the last, in one Rank; and F the first to K the last, in the other Rank, this *Sumption*, or comparison of these extremes, A to E, and F to K, is called *proportion of Equidistance*, or of *Equality*.

Here I have adventured to give a *new Name* to this Definition from *Euclid's* very words $\Delta\iota\sigma\tau\epsilon\ \lambda\acute{o}\gamma\epsilon\varsigma$: It is commonly translated *ex aequalitate Ratio*, or *ex aquo Ratio*, Proportion of *Equality*, or even proportion: but without any reason given of this interpretation to inform us what it plainly means: $\Delta\iota\sigma\tau\epsilon\ \lambda\acute{o}\gamma\epsilon\varsigma$, strictly translated, is *Ratio ex aquo*. But *ex quo*? Of what? Certainly *ex aquo Intervallo*. For what can more suitably be understood by $\Delta\iota\sigma\tau\epsilon$ than in taking it for $\delta\iota\ \iota\sigma\epsilon\ \delta\iota\alpha\sigma\eta\mu\alpha\tau\epsilon\varsigma$, *ex aquo Intervallo*, for an equal interval, or *even distance* of the extreme Terms from one another. The use of this Definition makes this interpretation manifest; for that the compared extremes are always taken at the same distance from one another, by subtracting an equal number of the intermedials in each Series. And indifferent it is, whether the other extreme from the first, be either the third, the fourth, the fifth, the sixth, or any further Term, only, for instance, if A the first be assumed to D the fourth in one Rank, then also F the first is assumed to I the fourth in the other, and the like in the rest: For that the Equidistance of the extremes is constantly to be observed in both the Ranks; as *Euclid's* words imply, when in the Sumption of the extremes he puts the Magnitudes in each Rank to be equal in multitude, and so the assumed extremes, the first to the last shall ever be equidistant from one another in each of the Ranks.

The Analogy between the extreme Terms of equidistant proportion is demonstrated in the 22^d. and 23^d. Propositions of this fifth Element.

Besides the Physical reason why these extremes are Analogals, or Proportionals, is very apparent, in that the proportion of the two extremes in each Rank, *viz.* of A to E, and of F to K, is compounded of the same intermedial proportions, and therefore in common sense those extremes must be in the same proportion to one another, A to E as F to K. For upon a full consideration of this matter, we shall find, that proportion *ex aquo* is nothing else, but a comparison of two proportions to one another, which are each compounded of like or equal proportions, that intervene between the extreme Terms of both Ranks: which extremes are therefore easily perceived to be Proportionals, as also is hereafter demonstrated.

And farther, this Definition of proportion *ex aquo* (a proportion of admirable use) is derived from the 10th. and 11th. Definitions of Duplicate, and Triplicate proportion, and agrees with them in a like Sumption of the extremes to one another: and also in that the proportion of the extremes is upon the same ground said to be compounded of the intermedial proportions, exposed in continued Terms, $\epsilon\tau\ \tau\iota\varsigma\ \epsilon\tau\tau\epsilon\varsigma\ \epsilon\iota\varsigma\ \epsilon\iota\varsigma$, as are those proportions which *Euclide* calls compounded. And the only difference between them is, that in a Series of Duplicate, and Triplicate proportion the intermediate proportions are all throughout one and the same: whereas in proportion *ex aquo* each Series may be of various proportions; which yet in their order are the same with one another, the first proportion of one Rank the same with the first of the other; the second the same with the second, &c. So that the proportions of the extremes in this 18th. Definition, just as before in the 10th. and 11th. Definitions, are alike said to be compounded of equal intermedial proportions. The agreement between these Definitions in this point is of special remark, and whoever shall studiously pierce into the depth, and subtil Contrivances of the 5th, 10th, and 18th. Definitions, wherein *Euclide* hath shewn a wonderful sagacity in setting forth Proportional Magnitudes, and the composition of proportions, will upon a just consideration find, that nothing in all these Elements ought to be more worthily admired, or received with greater esteem; notwithstanding the Cavils of some modern Geometricians.

$\Delta\iota\sigma\tau\epsilon$ is a manner of brief expression usual with the *Greeks*, and like to *Diatestafon*, and *Diapafon* in Musical Notes.

In

In this proportion *ex æquo*, or of Equidistance, there are two Cases. For the Analogal intermedial Terms of both Ranks, two, and two in the same proportion, may proceed either in order, one Term after the other, or else some of the Terms may be taken out of order, as the two following Definitions of Ordinate, and Perturbate Analogy do declare.

DEFINITION XIX.

Ordinate Analogy is, when it shall be as Antecedent to Consequent so Antecedent to Consequent: and again as Consequent to some other Magnitude, so Consequent to some other Magnitude.

Let the double Series $\left\{ \begin{array}{l} 18, 3, 6, \\ A, B, C, \\ \text{be as before,} \quad \left\{ \begin{array}{l} F, G, H, \\ 24, 4, 8, \end{array} \right. \end{array} \right.$

And the Analogy be as A the Antecedent to B the Consequent, so F the Antecedent to G the Consequent: And again as B the Consequent to C an other Magnitude, so G the Consequent to H an other Magnitude in a direct order.

This orderly progress in all the Analogal Terms of both Ranks successively one after the other, is called Ordinate Analogy, wherein the extremes taken *ex æquo* A to C, and F to H, are demonstrated to be proportional to one another in Prop. 22^d.

And the proportions of the extremes are each said to be compounded of the intermedial proportions taken in an Ordinate Analogy to one another.

DEFINITION XX.

Perturbate Analogy is, when in the precedent Rank it shall be as Antecedent to Consequent, so in the following, Antecedent to Consequent: and again in the Precedent, as the Consequent to some other Magnitude, so in the following, some other Magnitude to the Antecedent.

Let there be again a double Series, as $\left\{ \begin{array}{l} 12, 4, 2, \\ \dots A, B, C, \\ X, F, G, \\ 18, 9, 3, \end{array} \right\} \left\{ \begin{array}{l} 12 \cdot 8 \cdot 6 \cdot \\ \dots A \cdot B \cdot C \cdot \\ X \cdot F \cdot G \cdot \\ 12 \cdot 9 \cdot 6 \cdot \end{array} \right.$

And the Analogy be as A the Antecedent to B the Consequent, so F the Antecedent to G the Consequent: And again, as B the Consequent to C an other Magnitude in a direct order, so X an other Magnitude to F the Antecedent in a disturbed order. From the preposterous Sumption of two Terms in one of the Ranks, as namely here of X to F, making X the Term last put to become the first extreme, this *ἡσυχία* is called Perturbate Analogy: Wherein the extreme Terms tho' taken *ex æquo* A to C, and X to G, are demonstrated in Prop. 23. to be proportional to one another, as well as the extremes in Ordinate Analogy.

And the proportions of the extremes are each said to be compounded of the intermedial proportions taken in a Perturbate Analogy to one another.

For the intermedial proportions of both Ranks, either in Ordinate, or Perturbate Analogy, are alike the same (tho' not taken in a like order); and therefore the extremes being alike compounded of the same intermedial proportions, are easily conceived to be in the same proportion to one another. And this is the Physical, or Philosophical reason of this matter, which is Geometrically demonstrated in Prop. 22. and 23. El. V.

In the Definitions of Ordinate, and Perturbate Analogy, *Euclid's* words imply only three Magnitudes in each Series, whereas they might have been farther continued at pleasure, if the matter had so required. But he having before fully defined Proportion *ex æquo* in general, let the Terms be never so many; three Terms now sufficed to shew in what order, or disorder the extremes might be *ex æquo* taken, and assumed to one another, and still be either way found proportional to one another. For this reason *Euclide* puts only three Terms as most readily serving to set forth the two Cases of Ordinate, and Perturbate Analogy, which may happen in this proportion of Equidistance, or Equal Interval of Terms.

DEFINITION XXI.

Harmonical Analogy is when there being three Magnitudes, or Numbers, it shall be as the first to the third, so the difference between the first and second, to the difference between the second and third.

Otherwise.

Harmonical Analogy is a Sumption of the extremes to the differences between the extremes, and the middle Term.

Let there be three Magnitudes, or Numbers $\overset{6}{A}, \overset{4}{M}, \overset{3}{E}$. If A be to E, as the difference between A and M to the difference between M and E, that is, if A be to E, as $A - M$ to $M - E$, then this is called Harmonical Analogy, and signified after this manner, if A, M, E, be put as Terms of Harmonical Proportion.

$$\begin{array}{l} A : E :: A - M \quad . \quad M - E . \\ 6 : 3 :: 6 - 4 (=2) . \quad 4 - 3 (=1) . \end{array}$$

This Analogy is called Harmonical, because many Musical Consonancies suit often with the Terms of this Analogy, as 6 to 4, or proportion Sefquialteral is *Diapente*, 4 to 3, or Sefquitertial is *Diateffaron*, 6 to 3, or Duple proportion is *Diapafon*.

There is likewise Harmonical Analogy in four Terms, as A, M, N, E: 30, 12, 8, 5: when as the first is to the fourth, so is the difference of the first, and second to the difference of the third, and fourth.

$$\begin{array}{l} A : E :: A - M \quad . \quad N - E \\ 30 : 5 :: 30 - 12 (=18) . \quad 8 - 5 (=3) \end{array}$$

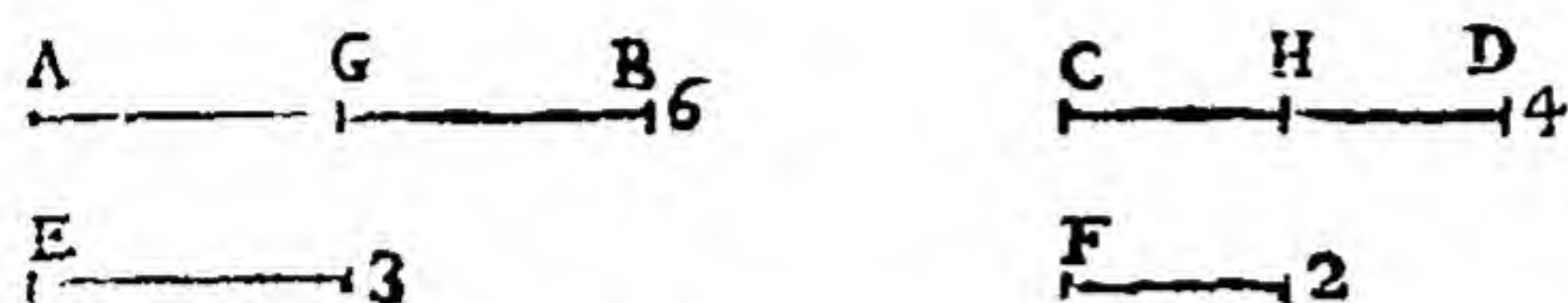
I have added this Definition to the rest of *Euclid's*, because most Commentators make mention of Harmonical Proportion as worthy of Consideration; and for that it hath some affinity with proportion *ex æquo*.

Clavius in his Comments hath abundantly set forth, and explained the various Properties, and mutual Correspondencies of Proportions Geometrical, Arithmetical, and Harmonical, whom the Studios may consult with much profit and delight.

PROPOSITION I.

If there be Magnitudes how many soever Equimultiples of as many other Magnitudes, each of each; Quotuple one of the Magnitudes is of one, Totuple shall all be of all.

Let there be magnitudes how many soever AB , CD , Equimultiples of as many other magnitudes E , F , each of each. I say, that Quotuple AB is of E , Totuple shall AB , CD together, be of E , F together.



If $AB \cdot E :: CD \cdot F$. Then $AB \cdot E :: AB + CD \cdot E + F$.

$6 \cdot 3 :: 4 \cdot 2$. Then $6 \cdot 3 :: 6 + 4 \cdot 3 + 2$.

For because AB is equimultiple of E , as CD is of F ; therefore how many magnitudes are in AB equal to E , so many are in CD equal to F .

Let AB be divided into the magnitudes equal to E , namely into AG , GB ; and CD into the magnitudes equal to F , namely into CH , HD .

Now then the multitude of these magnitudes CH , HD , contained in CD , shall be equal to the multitude of those magnitudes AG , GB , contained in AB [*for that* AB , CD *are by supposition Equimultiples of* E , F .]

And forasmuch as AG is equal E , and CH to F ; therefore AG , CH together, are equal to E , F together [Ax. 2. If equals be added to equals, the wholes are equal.].

By the same reason GB is equal to E , and HD to F ; therefore also GB , HD together, are equal to E , F together [by Ax. 2.].

Wherefore how many magnitudes are in AB equal to E , so many are in AB , CD together, equal to E , F together.

Therefore Quotuple AB is of E , Totuple shall AB , CD together, be of E , F together.

If therefore there be magnitudes how many soever Equimultiples of as many other Magnitudes, each of each, &c. Which was to be demonstrated.

ANNOTATIONS.

To this Proposition answer in Numbers the 5th. and 6th. Propositions of the seventh Element, which are thus comprised.

PROP. V, VI. EL. VII.

If a number be a Part, or Parts of a number, and another be the same Part, or Parts of an other number, also both together, shall be the same Part, or Parts of both together, that one is of one.

3

Ff 3

This

This is differently expressed from our present Proposition, but yet to the same purpose. For what *Euclide* demonstrates in Magnitudes from Multiples, the same is in Numbers more perspicuously demonstrated from Submultiples; because Numbers have a common Part, *à Mores*, The Monade, a natural Measure of all Numbers: from whence may be taken a certain and standing beginning for the framing of demonstrations much more clearly, than what can be made out in Magnitudes, which are wholly destitute of such a natural foundation; therefore *Euclide* is forced to make use of Multiples, for that (as we have formerly noted) every Magnitude may have any Multiple of it self, and all Incommensurables may be equally multiplied. From which Equimultiplication *Euclide* most ingeniously, and with admirable Artifice manageth this Element of Proportional Magnitudes.

Forasmuch therefore that equals added to equals make the wholes equal, He from thence proves that equal Proportions added to equal Proportions make in the sum the same equal Proportions.

And the meaning of this Proposition is, that if there be magnitudes never so many, Equimultiples of as many others, that is, in the same Multiple proportion to as many other Magnitudes, then as one of the Antecedents is to one of the Consequents, so all the Antecedents shall be to all the Consequents.

$$2B. \quad 2D. \quad 2F.$$

$$\text{As if } A.B :: C.D :: E.F. \&c. \quad \text{Then } A.B :: A + C + E.B + D + F.$$

$$6.3 :: 4.2 :: 10.5. \quad 6.3 :: 6 + 4 + 10.3 + 2 + 5.$$

Euclide here in his demonstration gives an instance only in duple proportion between two and two magnitudes for brevity sake: because we may in like manner easily proceed to any further Multiple proportion whatsoever, triple, quadruple, &c. And also to any number of Magnitudes infinitely; which are to one another in the same Multiple proportion.

But *Euclide* expresseth Magnitudes that are to one another in the same Multiple proportion by Equimultiples, because the word Equimultiples does more immediately denote the Antecedents to contain an equal number of their Consequents, into which they may be accordingly divided: And of such a division *Euclide* foresaw he was to make a special use in the demonstration of this Proposition; as likewise elsewhere.

Now what is here set forth only in Multiple proportion, the same is afterwards universally demonstrated at Prop. 12th. in any other kind of Proportion, both Rational, and Irrational. But this was premised to prove other Propositions, which were necessarily required towards the demonstration of that 12th. Proposition. Like as the 16th. Proposition El. I. proves the outward angle of a Triangle to be greater than either of the inward and opposite angles, to make way for the proof of some other Propositions, which were requisite to demonstrate that the same outward angle was equal to the two inward and opposite angles, Prop. 32. El. I.

PROPOSITION II.

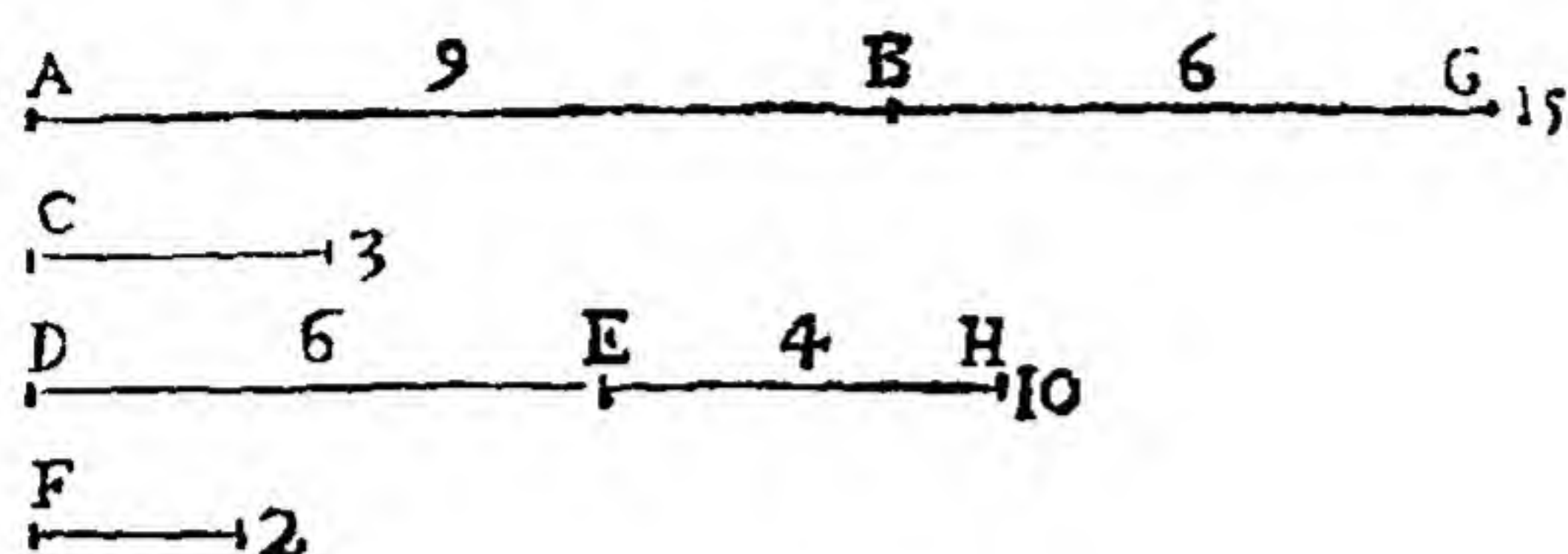
I*F the first be Equimultiple of the second, as the third is of the fourth: and there be a fifth Equimultiple of the second, as a sixth is of the fourth: then the first and fifth taken together shall be Equimultiple of the second, as the third and sixth are of the fourth.*

Let AB the first be Equimultiple of C the second, as DE the third is of F the fourth: and let there be BG a fifth Equimultiple of C the second, as EH a sixth is of F the fourth.

I say,

I say, that AG the first and fifth together taken, shall be Equimultiple of C the second, as DH the third and sixth together taken, is of F the fourth.

For because AB is equimultiple of C , as DE is of F ; therefore how many magnitudes are in AB equal to C , so many are in DE equal to F .



$3C \quad 3F \quad 2C \quad 2F \quad 3C + 2C \quad 3F + 2F$
 $AB.C::DE.F$ and $BG.C::EH.F$. Then $AB + BG.C::DE + EH.F$.
 $9.3::6.2$ and $6.3::4.2$. Then $9 + 6.3::6 + 4.2$.

By the same reason how many are in BG equal to C , so many are in EH equal to F .

Wherefore how many are in the whole AG equal to C , so many also are in the whole DH equal to F .

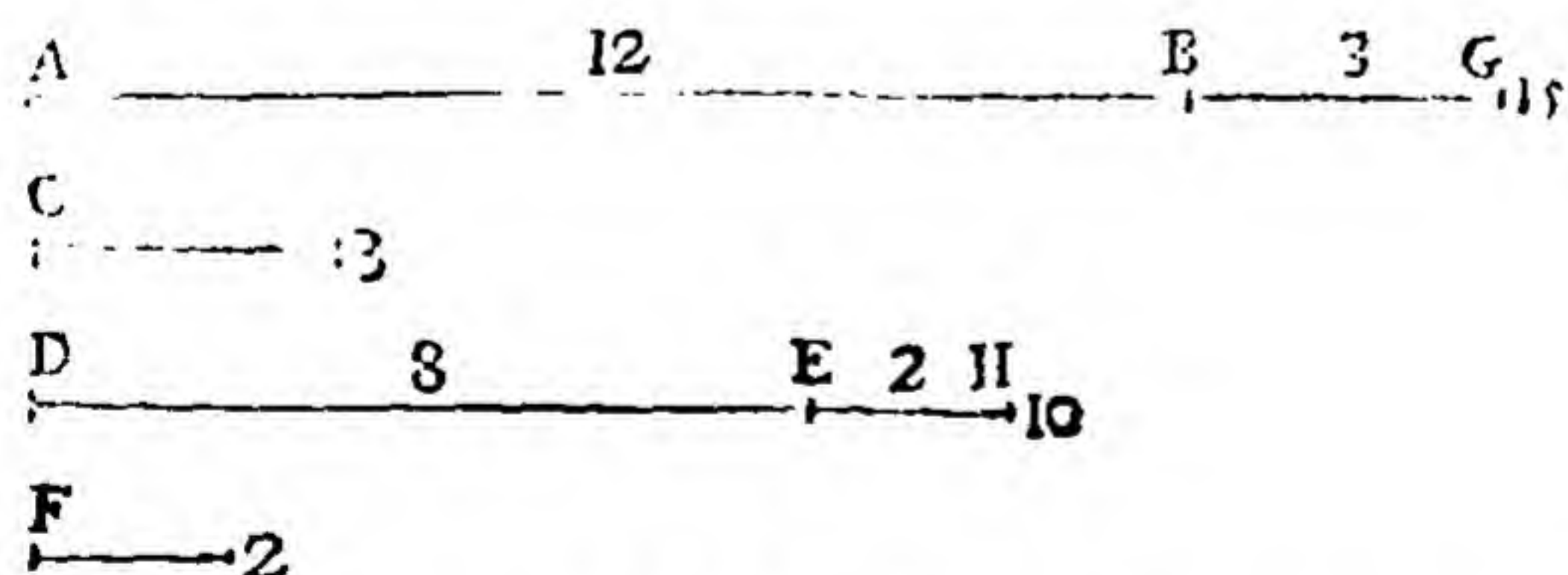
Therefore Quotuple AG is of C , Totuple DH is of F .

And therefore AG the first and fifth together taken, shall be equimultiple of C the second, as DH the third and sixth together taken, is of F the fourth.

If therefore the first be equimultiple of the second, as the third is of the fourth, &c. Which was to be demonstrated.

ANNOTATIONS.

This Proposition puts the fifth and sixth Magnitudes to be equimultiples of the second, and fourth: but the proportion would hold the same, if the fifth were only equal to the second, and the sixth to the fourth. For in this Case also the first, and fifth together shall be equimultiple of the second, as the third, and sixth together is of the fourth.



For let AB the first be 12, and BG the fifth be 3, equal to the Consequent C , 3. Also let DE the third be 8, and EH the sixth be 2, equal to the Consequent F , 2.

Then $AB + BG.C::DE + EH.F$. That is,

$$AB + C.C::DE + F.F.$$

$$12 + 3.3::8 + 2.2.$$

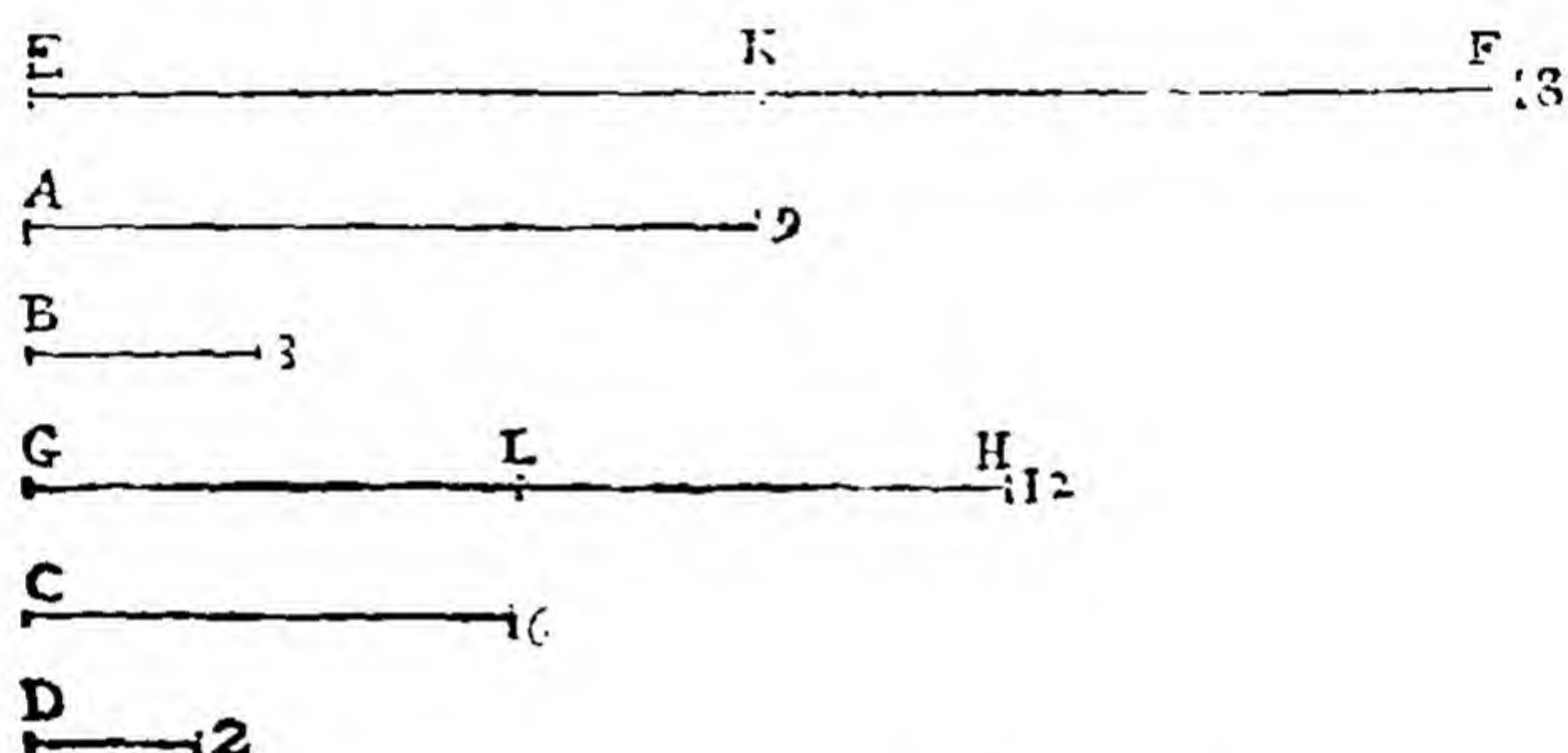
But here now 'tis to be observed, that when the fifth and sixth Terms, BG, EH ,
5
are

are put equal to C, F, the second and fourth, only the Consequents are added to the first and third, AB, DE, the Antecedents; and then this will be the very same with that, which *Euclide* calls Composition of Proportion in Def. 14. El. V. For the Consequents C, and F are plainly added to the Antecedents, AB, DE, in that they are equal to BG, EH, the fifth and sixth Terms. And besides, this Composition of Proportion is particularly demonstrated hereafter at Prop. 18. and made a distinct notion, and of a different use from this Proposition, which is applied after a more general manner, and not confined to the addition of the Consequents to the Antecedents; but admits of any other Terms whatsoever: and not only in equimultiple Proportion as in this place, but also universally in any kind of Proportion, as is demonstrated at Prop. 24th.

PROPOSITION III.

If the first be equimultiple of the second, as the third is of the fourth: and there be taken equimultiples of the first and third, then by Equidistance each of the taken equimultiples shall be equimultiple of each, one of the second, the other of the fourth.

Let A the first be equimultiple of B the second, as c the third is D the fourth: and of A and c let be taken equimultiples EF, GH. I say, that *ex æquo*, by equidistance [Def. 18.] EF is equimultiple of B the second, as GH is of D the fourth.



$$\begin{array}{l} \text{EF} \quad \text{GH} \\ A : B :: C : D. \text{ Then } \textit{ex æquo} \quad 2A : B :: 2C : D \\ 9 : 3 :: 6 : 2. \text{ Then } \textit{ex æquo} \quad 18 : 3 :: 12 : 2. \end{array}$$

Forasmuch as EF is equimultiple of A, as GH is of c; therefore how many magnitudes are in EF equal to A, so many are in GH equal to c.

Let EF be divided into the magnitudes equal to A, namely into EK, KF; and GH into GL, LH equal to c.

Now then the multitude of the magnitudes EK, KF shall be equal to the multitude of the magnitudes GL, LH.

And forasmuch as [by Supposition] A is equimultiple of B, as c is of D; and EK is equal to A, and GL to c; therefore EK is equimultiple of B, as GL is of D.

By the same reason KF is equimultiple of B, as LH is of D.

§

Because

Because therefore EK the first, is equimultiple of B the second, as GL the third is of D the fourth: and a fifth KF is equimultiple of B the second, as a sixth LH of D the fourth; therefore EF the first and fifth together taken, is equimultiple of B the second, as GH the third and sixth together taken, is of D the fourth [by Prop. 2. El. V.].

If therefore the first be equimultiple of the second, as the third is of the fourth: and of the first and third there be taken equimultiples, then *ex æquo* each, &c. Which was to be demonstrated.

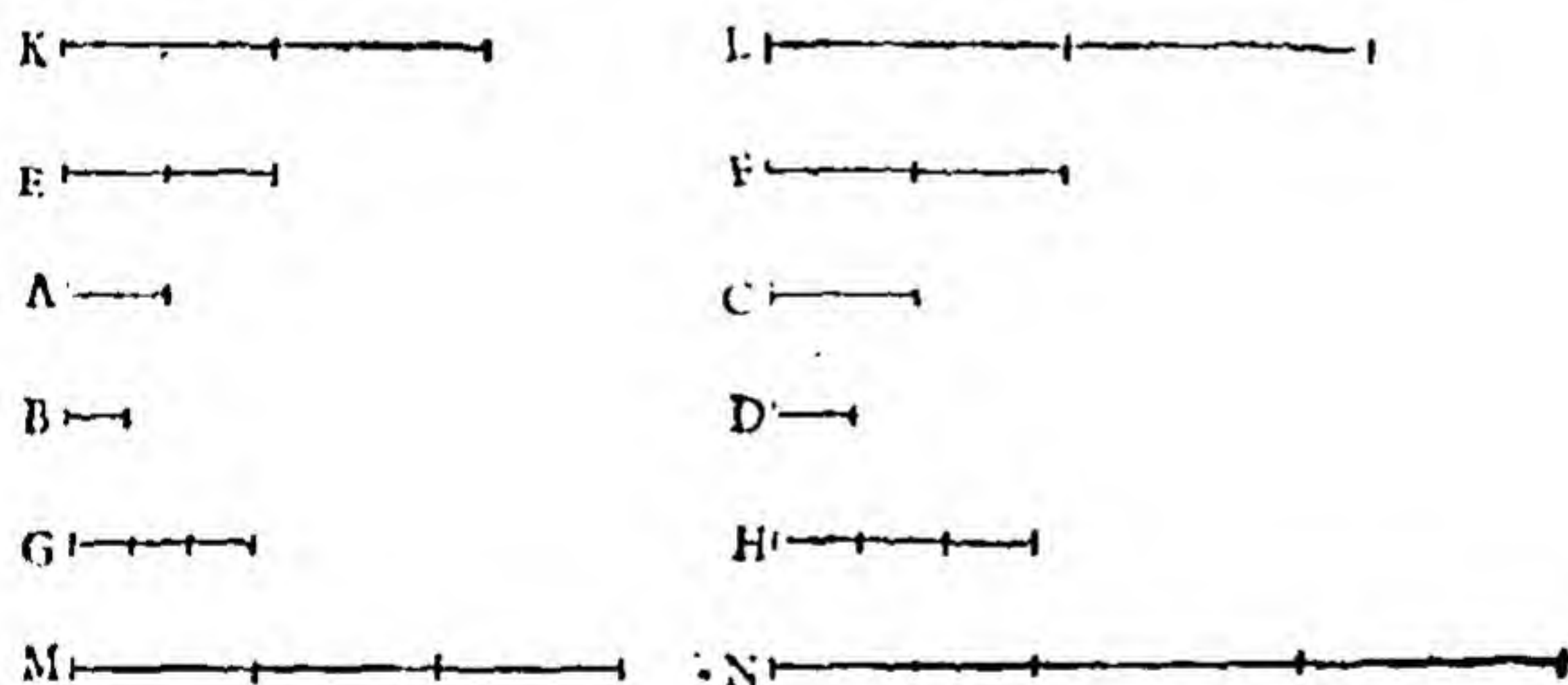
This is universally demonstrated at Prop. 22. and at present proposed only in multiple proportion to be made use of for the demonstration of the following Proposition.

PROPOSITION IV.

If the first hath to the second the same proportion, that the third hath to the fourth, then according to any multiplication whatsoever, the equimultiples of the first and third shall have to the equimultiples of the second and fourth the same proportion, being compared to one another.

Let A the first have to B the second the same proportion, that C the third hath to D the fourth: and let there be taken of A and C the first and third any equimultiples E and F ; and of B and D the second and fourth any equimultiples G and H .

I say, that as E is to G , so is F to H . For again, of E and F let be taken any equimultiples K and L ; and of G and H any equimultiples M and N .



$A : B :: C : D$. Then

$E : G :: F : H$.

$K : M :: L : N$.

Now because E is equimultiple of A , as F is of C ; and of E and F hath been taken equimultiples K and L : Therefore [*ex æquo* by Prop. 3.] K is equimultiple of A , as L is of C .

Likewise because G is equimultiple of B , as H is of D ; and of G and H hath been taken equimultiples M and N : Therefore *ex æquo*, M is equimultiple of B , as N is of D .

Forasmuch then that [by Supposition] as A the first is to B the second,

G

second,

second, so c the third to d the fourth; and of a and c hath been taken any equimultiples k and l; and of b and d any equimultiples m and n: therefore [* by the fifth Definition] if k exceeds m, l exceeds n, if equal, 'tis equal, if less, less.

But k, l are equimultiples of e, f; and m, n of g, h according to any multiplication whatsoever: Therefore [* by Def. 5.] as e is to g, so is f to h.

If therefore the first hath to the second the same proportion that the third hath to the fourth, then &c. Which was to be demonstrated.

Corollary.

Of Inverse Proportion.

From hence 'tis manifest, that if four Magnitudes be proportional, they shall also be *Inversely* proportional.

For if e and f equimultiples of a and c, the first and third be greater than g and h the equimultiples of c and d, the second and fourth; then on the contrary g and h shall be less than e and f: and if e and f be less, then shall g and h be greater; if equal, equal: Therefore *Inversely*, b shall be to a, as d to c: the second to the first, as the fourth to the third.

A Declaration of this fourth Proposition in Numbers,

Shewing the equimultiples of the Antecedents to be proportional to the equimultiples of the Consequents in some kind of proportion, either of equality, or of the greater, or less inequality.

$$\begin{array}{l} 6. \quad 3 :: 4. 2 \quad 6. \quad 3 :: 4. 2. \quad 6. \quad 3 :: 4. 2. \\ A. \quad B :: C. D \quad A. \quad B :: C. D. \quad A. \quad B :: C. D. \\ 2A. 3B :: 2C. 3D. 2A. 4B :: 2C. 4D. 2A. 5B :: 2C. 5D. \\ 12. 9 :: 8. 6. 12. 12 :: 8. 8. 12. 15 :: 8. 10. \end{array}$$

| Proportion Scsquiterial. | Proportion of Equality. | Proportion Subscsquiquartal. |
|---|---|---|
| The Antecedents A, C
Multiplied by 3, and
the Consequents B, D
by 3. | The Antecedents A, C
Multiplied by 2, and
the Consequents B, D
by 4. | The Antecedents A, C
Multiplied by 2, and
the Consequents B, D
by 5. |

ANNOTATIONS.

[* By Def. 5.] The turning of this Definition, in which lyes the chief force of *Euclid's* demonstration may seem to the younger Students somewhat perplext. But we ought to consider, that in all Definitions the Subject, and Prædicate are simply convertible, and do mutually put one another. So here there are first four Magnitudes proposed to be proportional; and thereupon their equimultiples, taken according to any multiplication whatsoever, are supposed to be together either equal, greater, or less, as the Definition requires. Again, on the contrary, because the equimultiples of four Magnitudes are found to be so affected, therefore those four Magnitudes are concluded to be proportional. Thus here the Definition of proportional Magnitudes, and the *Converse*, or *Definitum* are immediately turned upon one another in the demonstration of this Fundamental Proposition, the Cause putting the Effect, and the Effect the Cause.

Adver-

Advertisement.

This Proposition ought to be of a more special Remark, than is commonly taken notice of. It is the principal and most immediate property of four Proportionals, and the sole ground from whence *Euclide* raised his Definition of proportional Magnitudes; as hath been explained in the Annotations on the 5th. Definition. Where it was also made manifest, that the Primary, and Essential Conception of Proportionals consists in this general notion, that the Antecedents do equally contain their Consequents: How much one is of one, so much the other is of the other. As to instance clearly and distinctly in this main matter. If the Antecedents be *Equimultiples* of their Consequents, each of each, as if A be triple of B, and C triple of D, then in common sense A the first is to B the second, in the same *Multiple* proportion, that C the third is to D the fourth. And so again, if the Antecedents be any otherwise *Equiquantuples* of their Consequents, each of each, then the first is to the second in the same proportion that the third is to the fourth: Let the Proportion be any whatsoever possibly can be. This *Equiquantupleness* in Numbers *Euclide* expressly sets forth in Def. 20. El. VII. But now in Magnitudes it cannot be expressed from their own natural Constitution (as in Numbers) wherein the Antecedents are equimultiples of their Consequents; for that Magnitudes have no Original measure, or Geometrical Unite, whereby to express their quantities, but only what is made by consent among our selves. And then again there are numberless Magnitudes which can have no common measure at all to be made between them, whereby to express their mutual proportions. Wherefore seeing that the Essential and Physical Definition of proportional Magnitudes cannot be made use of in Geometrical demonstrations (as it is in Numerical) unless we could set forth wherein the Antecedents do equally contain their Consequents: therefore recourse must be had to some other property, which does flow from this *Equiquantupleness*, that is, from the Essence of Proportionality. Now in this 4th. Proposition such an immediate property of four Proportionals is demonstrated of them: namely, that the equimultiples of the Antecedents shall be proportional to the equimultiples of their Consequents, according to any multiplication whatsoever, when the four Magnitudes, whereof they are equimultiples are proportional, let the Magnitudes be Commensurable, or Incommensurable, the proportions Rational, or Irrational. But how again, and by what deductions from this same property *Euclide* gave us an other infallible sign of all kinds of proportional Magnitudes in the place of a Formal, or Essential Definition of Proportionals, the Annotations on the 5th. Definition have before fully declared.

PROPOSITION V.

If a Magnitude be equimultiple of a Magnitude, as a part detracted is of a part detracted, then shall the Remainder be equimultiple of the Remainder, as the whole is of the whole.

Let the Magnitude AB be equimultiple of CD, as the part detracted AE is of the part detracted CF.

I say, that the Remainder EB shall be equimultiple of the Remainder FD, as the whole AB is of the whole CD.

For Quotuple AE is of CF, Totuple let EB be made of an other Magnitude CG.

And because AE is equimultiple of CF, as EB is of CG, therefore AE, EB together, are equimultiple of CF, CG together, that is, the whole AB is equimultiple of the whole GF, as AE is of CF [by Prop. I.

G g 2

All

All equimultiple of all, as one of one.] But AE is also supposed equimultiple of CF , as AB is of CD .

Wherefore AB is equimultiple of each GF , and CD , and therefore GF is equal to CD [by Ax. 7.]

Let CF common to both be taken away, therefore the Remainder GC , is equal to the Remainder FD .



$$\begin{array}{ccccccc} 3CD & 3CF & 3FD & 3CD \\ AB.CD :: AE.CF. & \text{Then} & EB.FD :: AB.CD. \\ 18.6 :: 12.4. & \text{Then} & 6.2 :: 18.6. \end{array}$$

Forasmuch then that AE is equimultiple of CF , as EB is of GC , and GC is equal to FD ; therefore AE shall be equimultiple of CF , as EB of FD .

But AE is supposed equimultiple of CF , as AB of CD ; therefore the Remainder EB is equimultiple of the Remainder FD , as the whole AB is of the whole CD .

If therefore a Magnitude be equimultiple of a Magnitude, as a part detracted is of a part detracted, &c. Which was to be demonstrated.

This here in multiple proportion is universally demonstrated at Prop. 19.

To this Proposition answer in Numbers the 7th. and 8th. Propositions of the seventh Element, only what is here demonstrated of Multiple proportion, the like is there demonstrated *e contra* of Submultiple proportion; for the reasons before noted upon Prop. I. of this Element.

PROP. VII, and VIII. El. VII.

If a number be a Part, or Parts of a number, such as a number detracted is of a number detracted; so also the Remainder shall be the same Part, or Parts of the Remainder, which the whole is of the whole.

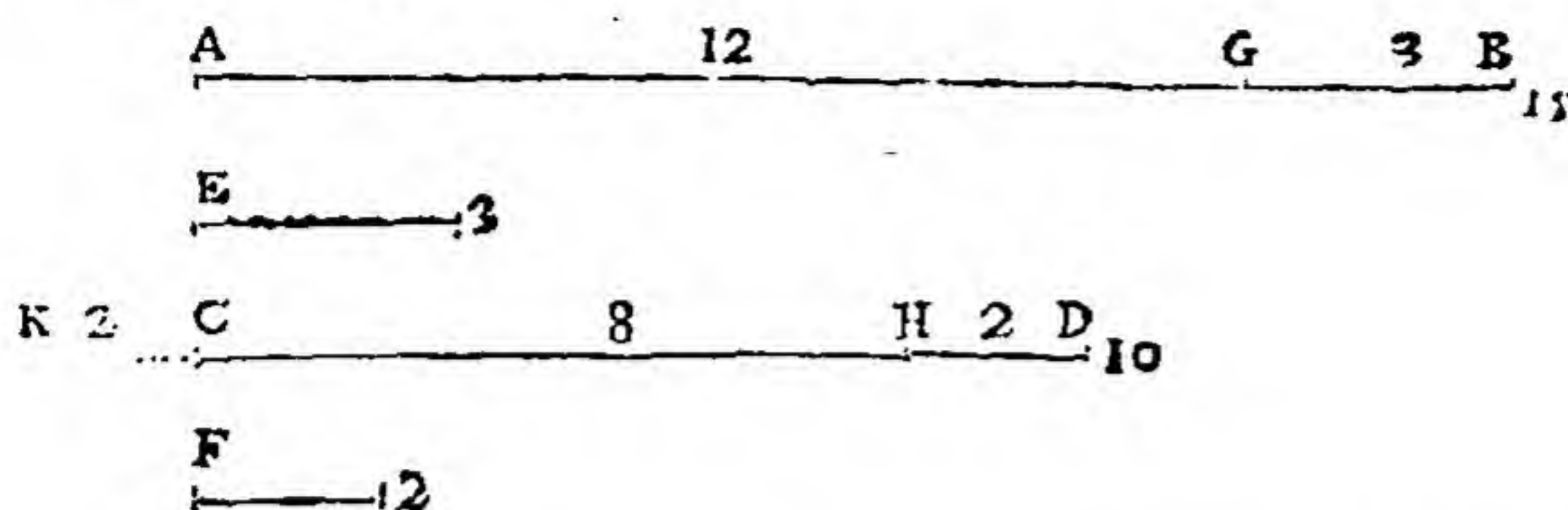
PROPOSITION VI.

If two Magnitudes be equimultiple of two Magnitudes, and some parts of them detracted be equimultiple of the same Magnitudes, then the Remainders also shall be either equal to the same Magnitudes, or equimultiples of them.

Let the two Magnitudes AB, CD be equimultiples of the two Magnitudes E, F , and the parts detracted AG, CH , be some equimultiples of the same E, F .

I say, that the Remainders GB , HD , are either equal to E , F , or equimultiples of them.

First let GB , be equal to E . I say, that HD is also equal to F ; for to F let CK be put equal.



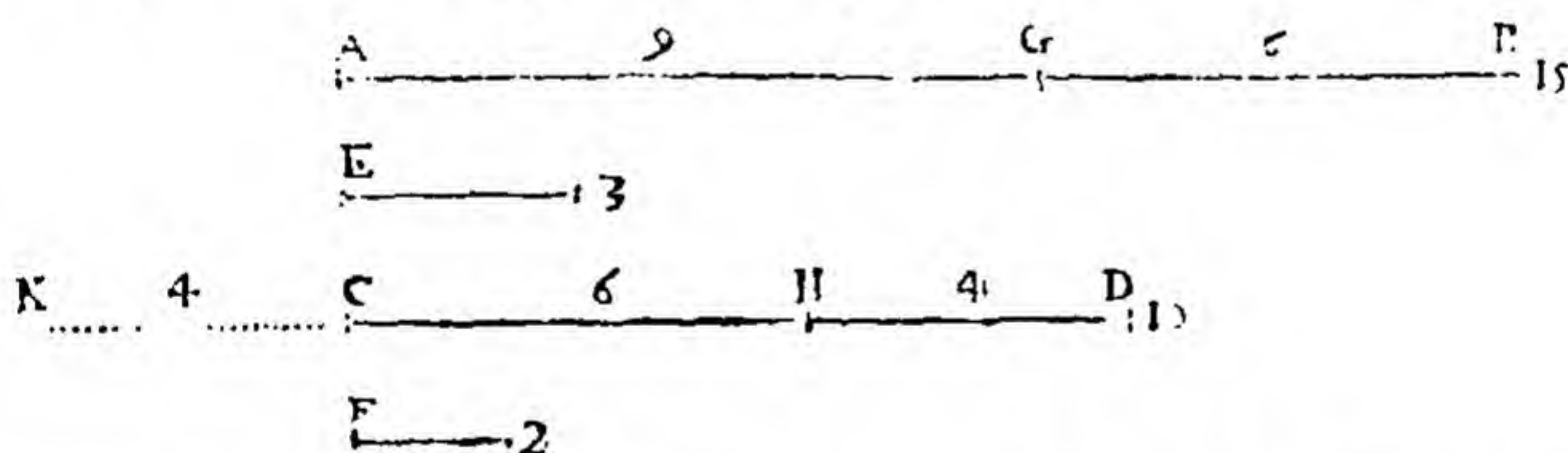
$AB.E::CD.F.$ and $AG.E::CH.F.$ and if $GB=E$. Then $HD=F$.
 $15.3::10.2.$ and $12.3::8.2.$ and if $3=3$. Then $2=2$.

And because AG the first is equimultiple of E the second, as CH the third is of F the fourth; and GB a fifth is equal to E the second; and CK a sixth is equal to F the fourth; Therefore AB the first and fifth together, are equimultiple of E , as KH the third and sixth are of F the fourth [by Prop. 2.].

But AB is by Supposition equimultiple of E , as CD of F ; therefore KH is equimultiple of F , as CD of F . Because now KH, CD , each is equimultiple of F ; therefore KH is equal to CD [by Ax. 6.].

Let CH common to both be detracted, therefore the Remainder KC is equal to the Remainder HD . But KC was put equal to F ; therefore HD is also equal to F . If therefore GB be equal to E , then shall HD be equal to F .

In like manner shall be demonstrated, that if GB be multiple of E , also HD shall be as equally multiple of F .



$AB.E::CD.F.$ and $AG.E::CH.F.$ and if $GB=2E$. Then $HD=2F$.
 $15.3::10.2.$ and $9.3::6.2.$ and if 6.6 . Then 4.4 .

If therefore two Magnitudes be equimultiple of two Magnitudes, and some parts of them detracted be equimultiple, &c. Which was to be demonstrated.

This here in multiple proportion is at Prop. 24th. universally demonstrated in all kinds of proportion.

These few Propositions, which hitherto only concern multiple proportion, are all again demonstrated in general. But they were to be premised as subservient to many of the following demonstrations,

monstrations, before the affections of Proportionals could be universally demonstrated according to all proportions Rational and Irrational whatsoever.

PROPOSITION VII.

Equal Magnitudes have the same proportion to the same Magnitude: And the same Magnitude hath to equal Magnitudes the same proportion.

Let the equal Magnitudes be A, B , and any other Magnitude be c . I say, that each of these A, B , have the same proportion to c . And again, that c hath to each, A and B , the same proportion. For of A, B , let be taken any equimultiples D, E : and of c any multiple whatsoever F . Whereas then D is equimultiple of A , as E of B ; and that A is equal to B , therefore D is also equal to E [by Ax. 6.]; wherefore if D exceeds F , also E shall exceed F ; if equal, equal; if less, less. But D, E are any equimultiples of A, B [the first and third], also F is any multiple whatsoever of c [the second and fourth, for c is instead of two Magnitudes]; therefore as A to c , so B to c [Def. 5. El. V.].

Upon the same construction we may alike demonstrate that D is equal to E : and therefore if F exceeds D , it also exceeds E ; if equal, equal; if less, less. But F is any multiple of c [the first and third]; and D, E are any equimultiples of A, B [the second and fourth]; therefore as c to A , so c to B .

Wherefore equals have to the same Magnitude the same proportion: and the same hath to equals, &c. Which was to be demonstrated.

ANNOTATIONS.

This 7th. with the 8th, 9th, 10th, 11th, and 12th. Propositions following, are taken by *Tacquet* as meer Axioms, to be received without demonstration. They are indeed evident truths, especially in Numbers: but because these are to be applied in general to Magnitudes of all sorts, as Lines, Planes, and Solids, commensurable, or incommensurable indifferently; therefore being capable of a just demonstration, they ought to be demonstrated; as *Euclide* hath upon good reason done with an admirable subtilty of Ratiocination used in these Propositions, and throughout this whole Element; which *Tacquet* might have well perceived, if his vanity had not mislead him. And yet to help out his own method he was constrained to use these Propositions in the nature of Axioms, or common Notions: whereas *Borellus* also hath thought fit to demonstrate them, as well as *Euclide* had before.

But *Borellus* likewise hath his failures, who is forced in his method to make his demonstrations apart, some for commensurable Magnitudes, some for incommensurable: whereas Magnitudes, when compared together in this Elementary Doctrine of Proportions and Proportionals, are not proposed in particular, some-

times commensurable, sometimes incommensurable; but as Magnitudes in general. Wherefore *Euclid* frames his demonstrations with such artifice, as at once to comprehend alike all Magnitudes commensurable or incommensurable, in any kind of proportion without restriction, or any mention of commensurability, or incommensurability, to be distinctly considered. And therefore in this point *Borellus* his method is defective, and comes much short of *Euclid's* general way of demonstration in this excellent Element: where the demonstrations extend themselves to all things in nature capable of proportion.

PROPOSITION VIII.

O*F unequal magnitudes the greater hath to the same a greater proportion, than the less; And the same hath to the less a greater proportion, than to the greater.*

Let the unequal Magnitudes be AB, c , and AB the greater. [$AB, 5$. $c, 3$]. Let also D be any other Magnitude [as 2].

I say, that AB [5] hath to D [2] a greater proportion, than c [3] to D [2]. And D [2] hath to c [3] a greater proportion than to AB [5].

Whereas AB is greater than c , let in AB be put a Magnitude equal to c , namely BE [3].

Now the less of these two AE, EB , being multiplied, shall at length be greater than D .

First, let AE (the excess of AB above c) be less than EB (which is equal to c), and multiply AE [2] till the multiple thereof be made greater than D [2], and let this multiple of AE be FG [4] greater than D [2].

Now Quotuple FG [4] is of AE [2], Totuple let GH [6] be of EB [3], and K [6] of c [3]. So that EB being put equal to c , they shall likewise have their equimultiples GH, K , equal to one another, by Ax. 6.

And now of D [2] let be taken the duple L [4], and the triple M [6], and so onward more by one, till the multiple of D [2] be the first greater than K [6] the multiple of c [3]. Let now this multiple of D be taken, and let it be N [8] the quadruple of D [2], and the first greater than K [6] the multiple of c [3]. So that K [6] is the first less than N [8].



Forasmuch then as K , is the first less than N the quadruple of D ; therefore K is not less than M the triple of D [but either equal, or greater].

And whereas FG is equimultiple of AE , as GH of EB ; therefore the whole FH is equimultiple of the whole AB , as FG of AE [by Prop. 1. El. V.]. But FG is equimultiple of AE , as K of c [by Construction]; wherefore FH is equimultiple of AB , as K of c ; and therefore FH and K , are equimultiples of AB , and c .

Again, because GH is equimultiple of EB , as K of c , and EB is put equal to c [by Construction]; therefore GH shall be equal to K : [by Ax. 6.]

* But K is not less than M , therefore GH is not less than M , and FG is greater than D [by Construction]; wherefore the whole FH is greater than D, M together.

But D, M together are equal to N ; therefore FH exceeds N ; but K exceeds not N [by Construction] and FH, K are equimultiples of AB, c , and N is any whatever multiple of D ; therefore AB hath to D a greater proportion, than c to D [by Def. 7. El. V.].

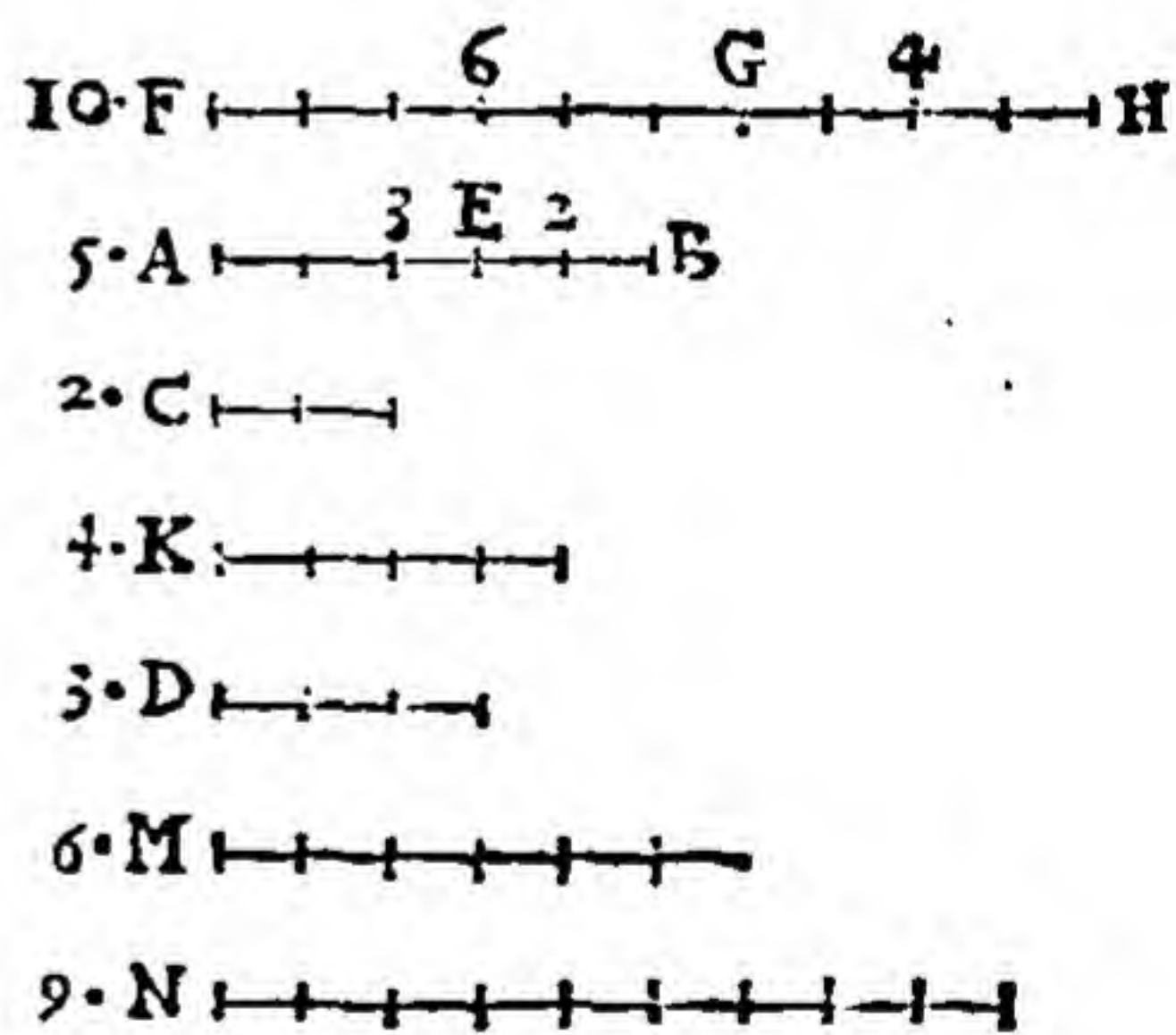
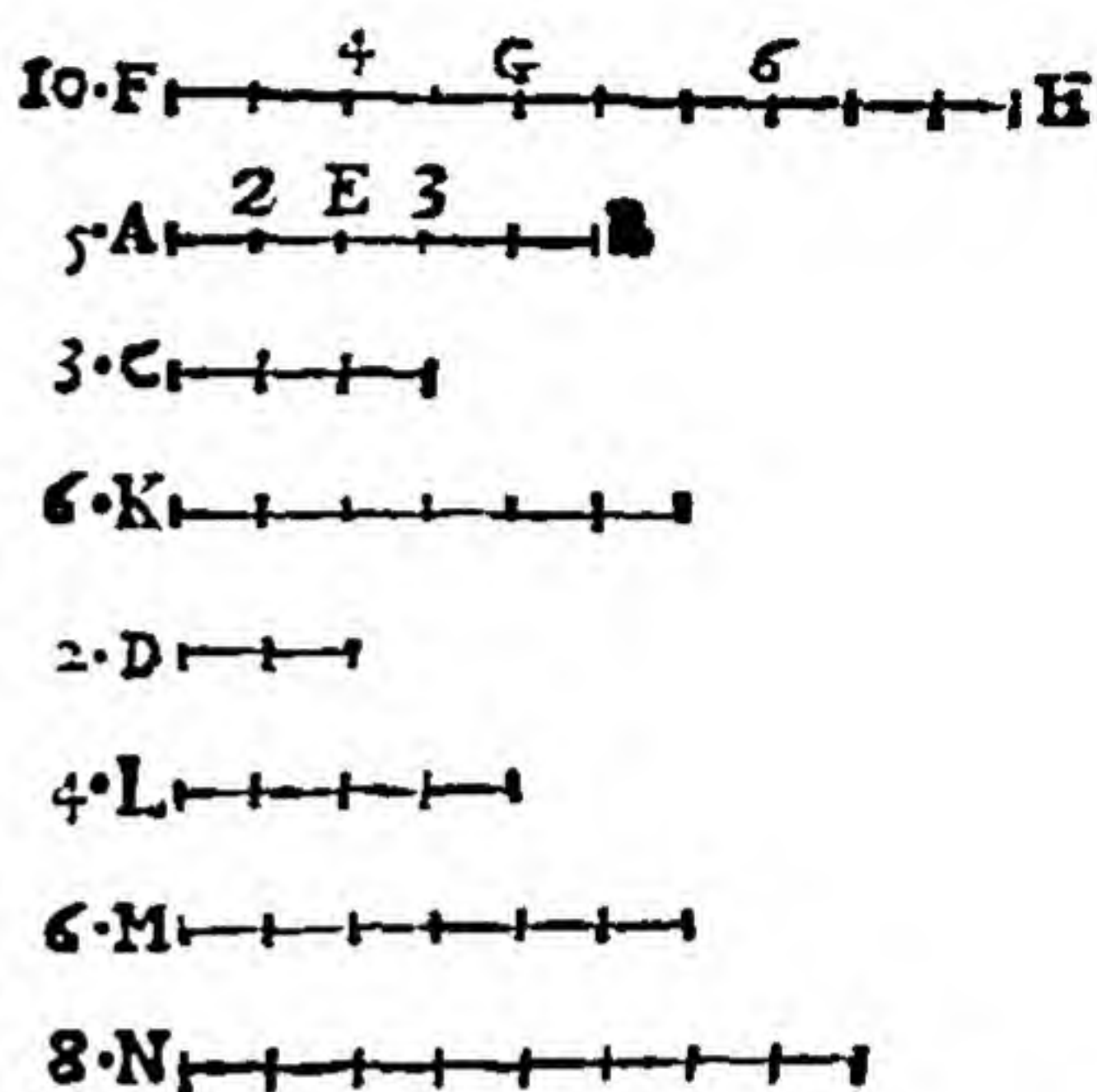
I say moreover, that D hath a greater proportion to c the less, than D to AB the greater. For on the same Construction we shall in like manner demonstrate, that N exceeds K , but exceeds not FH ; and that N is any multiple of D , and FH, K , are any whatever equimultiples of AB, c ; therefore D hath to c the less, a greater proportion, than hath D to AB the greater.

But again, of the unequal Magnitudes AB, c , when in the greater AB there is taken a Magnitude equal to c the less: Let AE the excess of AB , above c be greater than EB , which is put equal to c [$AE, 3$; $EB, 2$]. Now EB the less being multiplied, shall at length be greater than D . Let EB [2] be multiplied, and let GH [4] the multiple of EB (equal to c) be greater than D [3].

And Quotuple GH [4] is of EB [2], Totuple let FG [6] be of AE [3] and K [4] of c [2]. So that EB being put equal to c , they shall likewise have their equimultiples GH, K equal to one another, by Ax. 6. And also from the same Axiom it follows, that of these equimultiples, * FG the equimultiple of AE is greater than GH , that is, K the equimultiple of EB, c ; because AE is now supposed greater than EB , which is equal to c . For as of equals the equimultiples are equal, so of unequals the equimultiples are unequal, the greater of the greater, the less of the less.

Now after the former manner we shall demonstrate that FH [10], and K [4], are equimultiples of AB [5], and c [2].

And as before, let of D [3] be taken the multiples, till N be the first multiple of D , that is greater than FG [6] the multiple of AE [3]. Where-



Wherefore again, FG [6] is not less than M [6], and GH [4] is greater than D [3] by Construction; therefore the whole FH [10], exceeds D, M together; that is N [9]. But K exceeds not N , because FG being greater than GH which is equal to K , exceeds not N , by Construction. And FH, K , are equimultiples of AB, c ; and N is any whatever multiple of D ; therefore AB [5], hath to D [3] a greater proportion, than c [2] to D [3]. by Def. 7. El. V.

Wherefore of unequal Magnitudes the greater hath to the same a greater proportion, than the less; And the same hath to the less a greater proportion, than to the greater. Which was to be demonstrated.

ANNOTATIONS.

In this Proposition there are in effect four Magnitudes, whether AB , and C be compared to D , or D to AB , and C . For in each of these Cases D is twice taken: and in the first as a common Consequent to AB , and C : in the latter as a common Antecedent to the same Magnitudes.

The stress of this subtil demonstration lyes in three remarkable points. *First*, of the two unequal Magnitudes AB, C , there is in the greater AB taken a part EB equal to C , the less Magnitude: so that the Remainder AE is the excess of AB above C .

Secondly, of these two parts AE, EB , That, which happens to be the less, is to be multiplied, till the multiple thereof is allowed to exceed D the third Magnitude.

Thirdly, D is to be multiplied, till its multiple first exceeds the multiple of the greater part in AB , whether it be either AE , or EB . Upon these Constructions the demonstration chiefly depends.

Lastly, note that of the two unequal Magnitudes, each may be either greater, or less than the third; or one greater, and the other less. As for instance, let a Pound, and a Crown be compared to a Shilling; or a Crown, and a Shilling to a Pound; or a Pound, and a Shilling to a Crown: And the demonstration serves alike to all.

Advertisement.

In this Proposition are exposed two unequal Proportions that have the same Consequents: and again contrarily, two unequal proportions that have the same Antecedents. As $\frac{6}{2}$, and $\frac{4}{2}$, that is, 6 to 2, and 4 to 2. And contrarily $\frac{6}{4}$, and $\frac{4}{2}$. Now in this 8th. Proposition is only demonstrated which in each comparison is the greater proportion. But a farther enquiry may be made, in what proportion one is greater than the other. This question is, *De proportione Proportionum*, of the proportion of Proportions, for the discovery whereof take these two Rules.

1. Two unequal proportions having the same Consequents, are to one another as their Antecedents. Thus $\frac{6}{2}$ is to $\frac{4}{2}$ as 6 to 4, or $1\frac{1}{2}$, in proportion Sesquialteral, one greater than the other: and thus in general signified, $\frac{A}{B} : \frac{C}{D} :: A : C$. $\frac{6}{2} : \frac{4}{2} :: 6 : 4$.

2. Two unequal proportions having the same Antecedents, are to one another Reciprocally as their Consequents. Thus $\frac{6}{2}$ is to $\frac{4}{2}$, as 6 to 4; That is Subduple proportion is to Subtriple in proportion Sesquialteral, so that $\frac{6}{2}$ is in such a proportion greater than $\frac{4}{2}$, and in general, $\frac{A}{B} : \frac{C}{D} :: D : B$. $\frac{6}{2} : \frac{4}{2} :: 2 : 4$. But now if the two unequal proportions have different Antecedents, and Consequents, as 9 to 3, and 4 to 2. In this Case to discover which of them is the greater proportion, and in what proportion one is greater than the other, the Terms of these proportions are to be so changed, that the same unequal proportions may have the same Consequents, which is called *Reduction of proportions to a common Consequent*: and performed by this Rule.

Reduction of Proportions to a common Consequent.

Multiply the Consequents into one another; and each Antecedent into the others Consequent.

As $\frac{3}{4}$ and $\frac{2}{3}$ are brought to $\frac{6}{12}$ and $\frac{4}{6}$ by multiplying the Consequents 3 and 2 into one another: And the Antecedent 9 into 2 the others Consequent; likewise the Antecedent 4 into 3 the formers Consequent.

Now this 8th. Proposition demonstrates, that 18 hath to 6 a greater proportion than 12 to 6: and 6 to 12 a greater than 6 to 18.

Moreover in both Cases 'tis shewn by the two foregoing Rules, in what proportion one proportion is greater than the other. Thus $\frac{2}{3}$ is greater than $\frac{3}{4}$, and $\frac{3}{4}$ greater than $\frac{4}{5}$, in the same proportion of 18 to 12, or 1 $\frac{1}{2}$. But the proportions of $\frac{3}{4}$ and $\frac{2}{3}$ are the same with $\frac{6}{8}$ and $\frac{4}{6}$, which is thus demonstrated: $\frac{3}{4}$ multiplied by 2 make $\frac{6}{8}$, and $\frac{2}{3}$ multiplied by 3 make $\frac{4}{6}$: But if a number multiply two numbers, the Products shall be in the same proportion with the multiplied numbers, by Prop. 17. El. VII. Again, of the proportions $\frac{3}{4}$ and $\frac{2}{3}$ the Consequents 3 and 2 multiplied into one another, shall make one and the same common Consequent. For 3 into 2, or 2 into 3 make the same number 6, by Prop. 16. El. VII. If two numbers multiply each the other, the Products shall be equal to one another; therefore $\frac{3}{4}$ and $\frac{2}{3}$ are the same proportions with $\frac{6}{8}$ and $\frac{4}{6}$. Thus the Rule for Reduction of Proportions to a common Consequent is demonstrated.

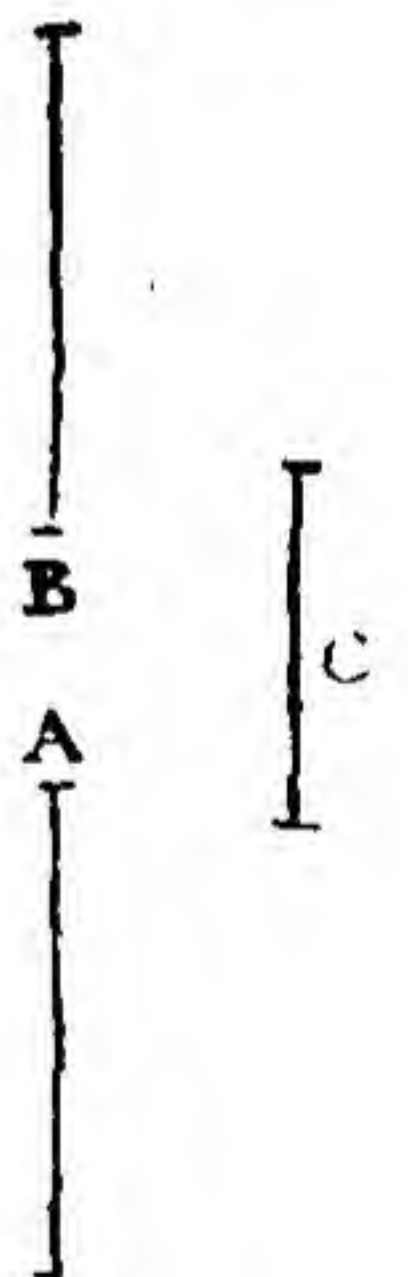
PROPOSITION IX.

Magnitudes which have the same proportion to the same magnitude, are equal to one another: And to what magnitudes the same magnitude hath the same proportion, they also are equal to one another.

Let each of the Magnitudes A, B, have to c the same proportion. I say, that A is equal to B. For if not, then each of those Magnitudes A, B, should not have to c the same proportion [by Prop. 8. El. V.]. But each have; therefore A is equal to B.

Again, let c have to each of the Magnitudes A, B, the same proportion. I say, that A is equal to B. For if not, then c should not have to A and B the same proportion [by Prop. 8. El. V.]. But it hath; therefore A is equal to B.

Wherefore Magnitudes which have the same proportion to the same Magnitude, are equal to one another. &c. Which was to be demonstrated.



PROPOSITION X.

Of Magnitudes having a proportion to the same magnitude, that which hath the greater proportion, is the greater. And to what magnitude the same magnitude hath a greater proportion, that is the less magnitude.

Let A have to c a greater proportion than B to c. I say, that A is

is greater than B: For if not, then A is either equal to B, or less. But A is not equal to B. For then each of the magnitudes A, B should have to c the same proportion [by Prop. 7. El. V.]. But each of them have not; therefore A is not equal to B: neither also is A less than B. For then A should have to c a less proportion than B to c [by Prop. 8. El. V.]. But it hath not; therefore A is not less than B: And it hath been demonstrated that it is not equal; therefore A is greater than B.

Again, let c have to B a greater proportion than c to A. I say, that B is less than A. For if not: it is either equal, or greater; but B is not equal to A; for then c should have to A and B the same proportion [by Prop. 7. El. V.]. But it hath not; therefore B is not equal to A: neither also is B greater than A. For then c should have to B a less proportion than to A [by Prop. 8. El. V.].

But it hath not; therefore B is not greater than A: And it hath been demonstrated that it is not equal: therefore B is less than A.

Wherefore of magnitudes having a proportion to the same magnitude, that which hath the greater proportion is the greater. &c. Which was to be demonstrated.

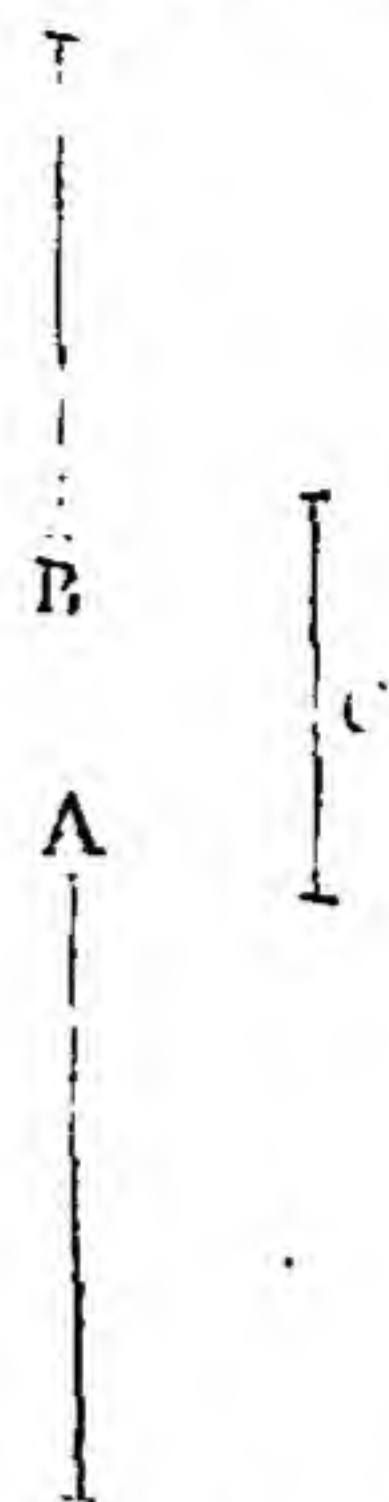
PROPOSITION XI.

Proportions which are the same to the same proportion, are the same to one another.

Let A be to B as c to D: also E to F as c to D. I say, that A is to B, as E to F. For let be taken of A, C, E (the Antecedents) any equimultiples G, H, K: and of B, D, F (the Consequents) any whatever equimultiples L, M, N. Now because it is as A to B, so c to D; and of A, c are taken equimultiples G, H: also of B, D any whatever equimultiples L, M. If therefore G exceeds L, then H exceeds M; and if equal, equal; if less, less. [Def. 5. El. V.]

Again, because E is to F as c to D: and of E, c are taken equimultiples K, H: as also of F, D any whatever equimultiples N, M. If therefore K exceeds N, then H exceeds M; and if equal, equal; if less, less. But if H exceeds M, then G exceeds L; and if equal, equal; if less, less [for by Supposition A is to B as c to D]; wherefore if G exceeds L, then K exceeds N; and if equal, equal; if less, less. But G, K are equimultiples of A, E: and L, N any whatever equimultiples of B, F; wherefore as A to B, so E to F [Def. 5. El. V.].

Therefore proportions which are the same to the same proportion, are the same to one another. Which was to be demonstrated.



$$\begin{array}{l} G \cdot H \cdot K \\ \frac{A}{B} :: \frac{C}{D} :: \frac{E}{F} \\ L \cdot M \cdot N \end{array}$$

PROPOSITION XII.

I*F magnitudes how many soever be Proportionals: it shall be, as one of the Antecedents to one of the Consequents, so all the Antecedents to all the Consequents.*

Let there be Proportional magnitudes how many soever, A, B, C, D, E, F; And as A to B, so C to D, and E to F. I say, that as A to B, so A, C, E, to B, D, F.

For of A, C, E (the Antecedents) let be taken any equimultiples G, H, K; and of B, D, F, (the Consequents) any whatever equimultiples L, M, N.

Because therefore it is as A to B, so C to D, and E to F: and there hath been taken of A, C, E any equimultiples G, H, K: and of B, D, F, any whatever equimultiples L, M, N.

G . H . K

$\frac{A}{B} :: \frac{C}{D} :: \frac{E}{F}$

L . M . N

If therefore G exceeds L, then H exceeds M, and K exceeds N; and if equal, equal; if less, less.

So that if G exceeds L, then G, H, K, exceeds L, M, N; and if equal, equal; if less, less.

But G, and G, H, K, are equimultiples of A, and of A, C, E. [For if there be magnitudes how many soever equimultiples of as many other magnitudes, each, of each, Quotuple one is of one, Totuple shall all be of all.]

By the same reason also L, and L, M, N, are equimultiples of B, and of B, D, F.

Therefore it is, as A to B, so A, C, E, to B, D, F.

Wherefore if magnitudes how many soever be proportionals: it shall be, as one of the Antecedents to one of the Consequents, so all the Antecedents to all the Consequents. Which was to be demonstrated.

To this Proposition answers in Numbers Prop. 12th. El. VII.

ANNOTATIONS.

What is here now universally proved of any kind of proportion, the same hath been before in the first Proposition of this Element demonstrated only of multiple proportions, that as one of the Antecedents, is to one of the Consequents, so are all to all. And note, that when the Antecedents are thus annexed to Antecedents, and Consequents to Consequents, This is called Addition of proportions; whose chief use is declared in this 12th. Proposition, that as one of the Antecedents is to one of the Consequents, so is the sum of the Antecedents to the sum of the Consequents. But when Antecedents are multiplied into Antecedents, and Consequents into Consequents, then it is said to be a Composition of Proportions: Which is of frequent use in the Mathematics.

PROPO-

PROPOSITION XIII.

IF the first hath to the second the same proportion, that the third hath to the fourth, and the third to the fourth hath a greater proportion, than the fifth to the sixth; also the first hath to the second a greater proportion, than the fifth to the sixth.

Let *A* the first have to *B* the second the same proportion, that *c* the third hath to *D* the fourth: and *c* the third a greater proportion to *D* the fourth, than *E* the fifth to *F* the sixth.

I say, *A* the first shall have to *B* the second a greater proportion, than *E* the fifth to *F* the sixth.

Forasmuch as *c* hath to *D* a greater proportion than *E* to *F*; therefore there are some equimultiples of *c*, *E*, and again some other whatever equimultiples of *D*, *F*, where the multiple of *c* exceeds the multiple of *D*, and the multiple of *E* exceeds not the multiple of *F* [Def. 7. El. V.]. Let them be taken, and of *c*, *E*, let *G*, *H*, be equimultiples; and of *D*, *F* any whatever equimultiples *K*, *L*; so that *G* exceeds *K*, but *H* exceeds not *L*.

Now Quotuple *G* is of *c*, Totuple let *M* be of *A*: $M : G : H$
And Quotuple *K* is of *D*, Totuple also let *N* be of *B*. $A : B :: C : D > \frac{E}{F}$

Because therefore it is as *A* to *B*, so *c* to *D*; and of *A*, *c* are taken equimultiples *M*, *G*; and of *B*, *D*, any whatever equimultiples *N*, *K*; then if *M* exceeds *N*, also *G* exceeds *K*; and if equal, equal; if less, less.

Now *G* exceeds *K*, therefore also *M* exceeds *N*; but *H* exceeds not *L* [by Construction], and *M*, *H* are equimultiples of *A*, *E* [the Antecedents], and *N*, *L* any whatever equimultiples of *B*, *F* [the Consequents]; therefore *A* hath to *B* a greater proportion than *E* to *F* [Def. 7. El. V.].

If therefore the first hath to the second the same proportion, that the third hath to the fourth, and the third hath to the fourth a greater proportion, than the fifth to the sixth, &c. Which was to be demonstrated.

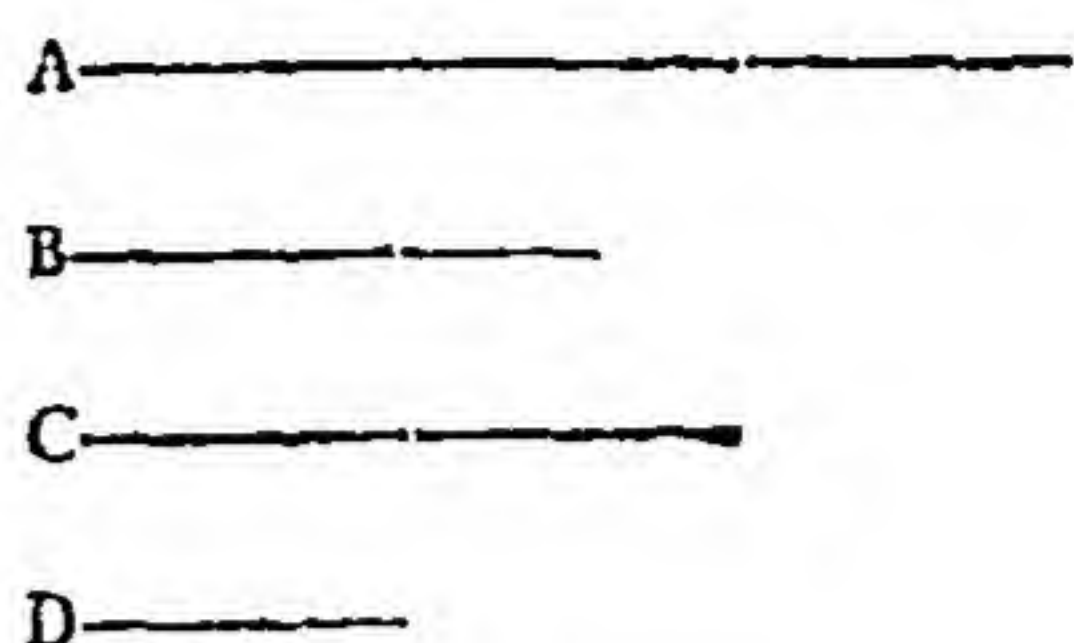
PROPOSITION XIV.

IF the first hath to the second the same proportion that the third hath to the fourth; and the first be greater than the third; also the second shall be greater than the fourth: And if equal, equal; if less, less.

Let *A* the first have to *B* the second the same proportion that *c* the third hath to *D* the fourth; and let *A* be greater than *c*. I say, that *B* is greater than *D*. For whereas *A* is greater than *c*, and there is an other magnitude *B*; therefore *A* hath to *B* a greater proportion

than

than c to B [Prop. 8. El. V.] But as A to B , so c to D ; therefore also c hath to D a greater proportion than c to B . But to what magnitude the same hath a greater proportion, that is the less [Prop. 10. El. V.]; therefore D is less than B : so that B is greater than D . In like manner we shall demonstrate that if A be equal to c , B shall also be equal to D ; and if A be less than c , B shall be also less than D .



If therefore the first hath to the second the same proportion, that the third hath to the fourth; and the first be greater than the third, &c. Which was to be demonstrated.

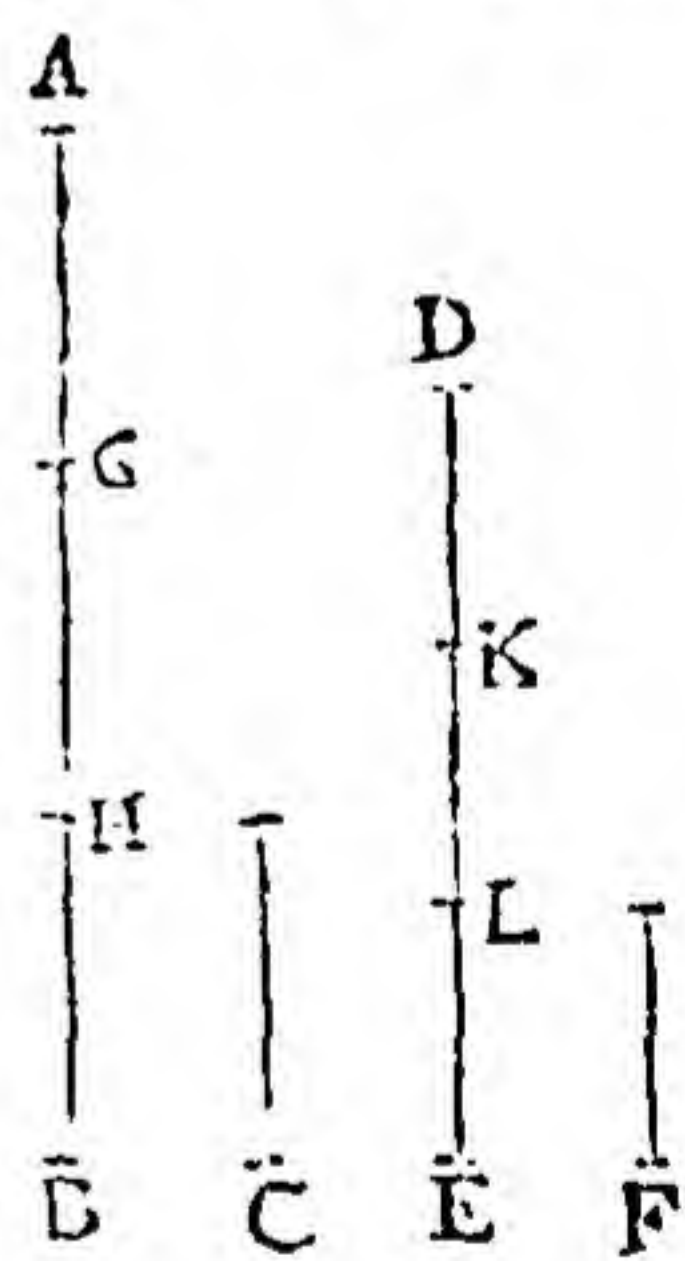
ANNOTATIONS.

Euclid thought fit to demonstrate, that in proportionals if one Antecedent were greater, equal, or less than the other Antecedent; then one Consequent shall be greater, equal, or less than the other Consequent. But needless it was to prove, that if one Antecedent were greater, equal, or less than the Consequent, the other likewise is the same. For that this is of it self implied in the very notion of proportionals; where the proportions are always either of equality, or of the greater or less inequality: So that the Antecedents are either equal to their Consequents, or greater, or less.

PROPOSITION XV.

Parts compared to one another have the same proportion with their equimultiples.

Let AB be equimultiple of c , as DE of F . I say, that it is as c to F , so AB to DE . For because AB is equimultiple of c , as DE of F , therefore how many magnitudes are in AB equal to c , so many are in DE equal to F . Let AB be divided into the magnitudes equal to c , namely AG, GH, HB ; And DE into magnitudes equal to F , as DK, KL, LE : therefore the multitude of these AG, GH, HB , shall be equal to the multitude of DK, KL, LE . Forasmuch then as AG, GH, HB , are equal to one another: and DK, KL, LE , are also equal to one another; wherefore as AG to DK , so GH to KL , and HB to LE [Prop. 7. El. V.]. Equal magnitudes have to the same (or to equal magnitudes) the same proportion.



And because as one of the Antecedents to one of the Consequents, so all the Antecedents to all the Consequents [Prop. 12. El. V.]; therefore as AG to DK , so AB to DE . But AG is equal to c , and DK to F [by Construction]; therefore as c to F (part to part), so AB to DE (equimultiple to equimultiple).

Parts therefore compared to one another, have the same proportion with their equimultiples. Which was to be demonstrated.

ANNO-

ANNOTATIONS.

To this Proposition answers in numbers Prop. 17th. El. VII. "If a number multiplying two numbers make some numbers, their Products shall have the same proportion with the multiplied numbers.

For let 4 multiplying two numbers 3 and 2, make 12 and 8, then the Products 12 to 8 shall have the same proportion with 3 to 2: Both in Sefquialteral proportion, 1 $\frac{1}{2}$.

These Propositions are very near to common Notions, and therefore likewise ought to be as carefully remarked, in regard of their general use in proportions between magnitudes to magnitudes, and numbers to numbers. Also this Prop. 17th. El. VII. is the ground of bringing all proportions to a common Consequent, by which is discovered what proportions are greater, or less one than an other. As hath been before noted upon Proposition 8th. Which Annotations review, and fully consider with relation to these Propositions.

PROPOSITION XVI.

IF four magnitudes be proportional, they shall also be alternly proportional.

Let four magnitudes A, B, C, D, be proportional, as A to B, so C to D. I say, that they shall also be alternly proportional, as A to C, so B to D. For of A, B, let be taken equimultiples E, F: and of C, D any whatever equimultiples G, H. Forasmuch then that E is equimultiple of A, as F of B; but parts have to one another the same proportion with their equimultiples [Prop. 15. El. V.]; therefore as A to B, so E to F: but as A to B, so C to D; therefore as C to D, so E to F [Prop. 11. El. V.]

Again, because G, H are equimultiples of C, D; therefore as C to D, so G to H: but as C to D, so E to F; therefore as E to F, so G to H. But if four magnitudes be proportional, and the first be greater than the third, the second shall be greater than the fourth, and if equal, equal; if less, less [Prop. 14. El. V.]. If therefore E exceeds G, also F exceeds H; and if equal, equal; if less, less. But E, F are equimultiples of A, B (the first, and third) also G, H are any whatever equimultiples of C, D (the second, and fourth): Therefore as A to C, so B to D [Def. 5. El. V.].

Wherefore if four magnitudes be proportional, they shall also be alternly proportional. Which was to be demonstrated.

To this Proposition answers in numbers Prop. 13th. El. VII.

ANNOTATIONS.

It is in the first place to be noted, that the alternation of proportions can only be used between Homogeneous magnitudes, where the four proportional Terms are either all Lines, or all Planes, or all Solids, as Def. 3. El. V. declares. For magnitudes of Heterogeneous quantities cannot in nature admit of mutual comparison according to quantity.

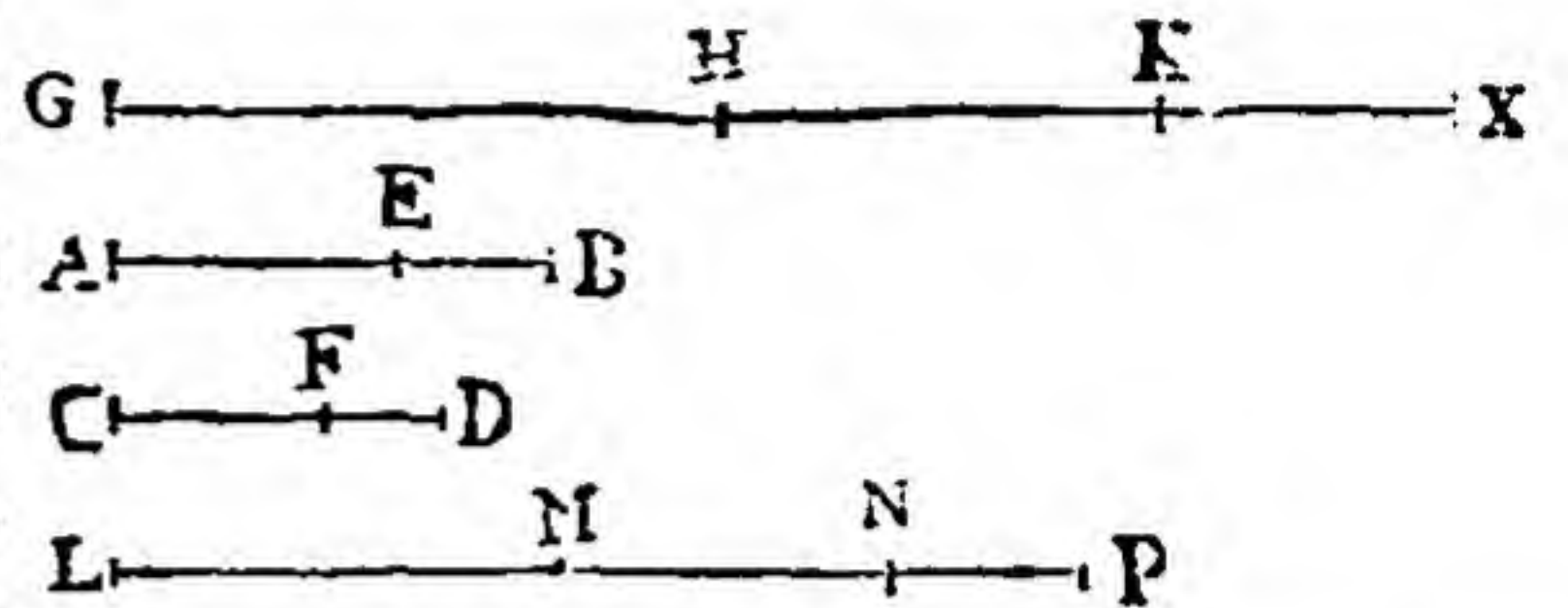
PROPO.

PROPOSITION XVII.

If compounded magnitudes be proportional, they shall also divided, be proportional.

Let the Compounded proportional magnitudes be AB, BE, CD, DF ; as AB to BE , so CD to DF . I say, that also divided they shall be proportional, as AE to EB , so CF to FD . For of AE, EB ; CF, FD , let be taken equimultiples GH, HK ; LM, MN . And of EB, FD , any whatever equimultiples KX, NP .

Forasmuch then as GH is equimultiple of AE , as HK of EB ; there-
 fore GH is equimultiple of AE , as AE to EB , as AB to BE (the whole) AB of the whole AB [Prop. I. El. V.]. But GH is equimultiple of AE , as LM of CF [by Construction]; therefore GK is equimultiple of AB , as LM of CF [Prop. II. El. V.].



Again, because LM is equimultiple of CF , as MN of FD , therefore LM is equimultiple of CF , as (the whole) LN of (the whole) CD [Prop. I. El. V.]. But LM was equimultiple of CF as GK of AB ; therefore GK is equimultiple of AB , as LN of CD [Prop. II. El. V.].

Again, forasmuch as [by Construction] HK (the first) is equimultiple of EB (the second) as MN (the third) of FD (the fourth): And also KX (a fifth) is equimultiple of EB (the second), as NP (a sixth) of FD (the fourth); therefore HX (the first and fifth) together, is equimultiple of EB (the second), as MP (the third and sixth) together, is of FD (the fourth) [Prop. 2. El. V.].

Now because it is as AB to BE , so CD to DF ; and of AB, CD are taken equimultiples GK, LN ; also of EB, FD any whatever equimultiples HX, MP . If therefore GK exceeds HX , also LN exceeds MP ; and if equal, equal; if less, less [Def. 5. El. V.]. Let therefore GK exceed HX , and HK common to both being detracted, then shall GH exceed KX . But if GK exceeds HX , also LN exceeds MP , and MN common to both being detracted, then shall LM exceed NP : wherefore if GH exceed KX , also LM exceeds NP .

In like manner shall we prove, that if GH be equal to KX , also LM shall be equal to NP , and if less, less. But GH, LM are equimultiples of AE, CF ; and KX, NP any whatever equimultiples of EB, FD ; therefore as AE to EB , so CF to FD .

If therefore compounded magnitudes be proportional, they shall also divided be proportional. Which was to be demonstrated.

ANNOTATIONS.

In this Proposition there is with great subtilty demonstrated from compounded proportions, the Analogy of divided proportions, as Division of proportion is by *Euclid* defined in the 16th. Definition.

PROPO.

PROPOSITION XVIII.

If divided magnitudes be proportional, they shall also compounded be proportional.

Let the divided magnitudes be $AE, EB; CF, FD$; and as AE to EB , so CF to FD . I say, that also compounded, they shall be proportional, as AB to BE , so CD to DF .

For if it be not as AB to BE , so CD to DF , then shall it be as AB to BE , so CD either to a less than DF , or to a greater.

First, let it be to a less, as DG . Forasmuch then that it is as AB to BE , so CD to DG ; therefore these compound magnitudes are proportional. So that divided, they shall also be proportional [Prop. 17. El. V.]. It is therefore as AE to EB , so CG to GD . But by Supposition as AE to EB , so CF to FD : wherefore as CG to GD , so CF to FD . But CG the first is greater than CF the third; therefore also GD the second is greater than FD the fourth [Prop. 14. El. V.]; but also 'tis less, which is impossible; therefore it is not as AB to BE , so CD to DG . In like manner shall we prove, that CD cannot be to any magnitude greater than DF : therefore it is as AB to BE , so CD to DF .

Wherefore if divided magnitudes be proportional, they shall also compounded be proportional. Which was to be demonstrated.

ANNOTATIONS.

In this Proposition is demonstrated from Divided proportions the Analogy of Compounded proportions, as Composition of proportion is explained in the 15th. Definition.

PROPOSITION XIX.

If it be as the whole to the whole, so a part detracted to a part detracted, then the Remainder shall be to the Remainder as the whole to the whole.

For let it be as the whole AB to the whole CD , so a part detracted AE , to a part detracted CF . I say, that the Remainder EB shall be to the Remainder FD , as the whole AB to the whole CD .

Forasmuch that it is as the whole AB to the whole CD , so AE to CF : therefore alternly as BA to AE , so DC to CF . And because compounded magnitudes are proportional, therefore divided they shall be proportional: wherefore as BE to EA , so DF to FC ; and therefore alternly it is as BE to DF , so EA to FC . But as AE to CF , so by Supposition, is the whole AB to the whole CD : therefore also the Remainder EB shall be to the Remainder FD , as the whole AB to the whole CD .

I i

Where.

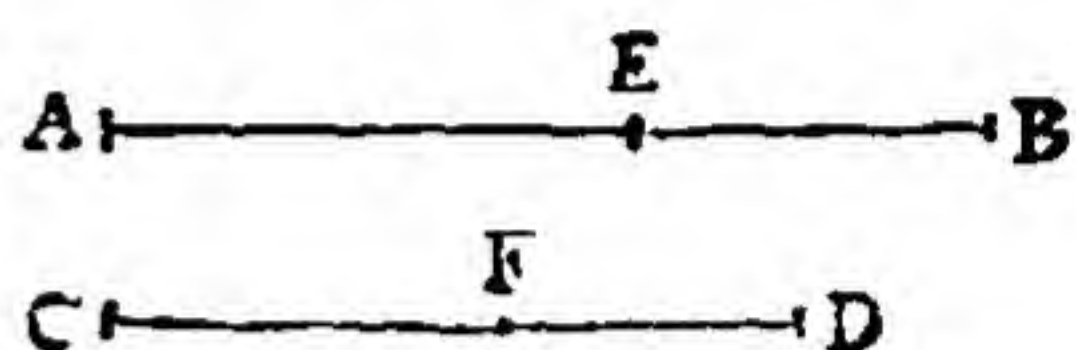
Wherefore if it be as the whole to the whole, so a part detracted to a part detracted, &c. Which was to be demonstrated.

Corollary

Of Converse Proportion.

From hence 'tis manifest, that if compounded magnitudes be proportional, they shall also by Conversion be proportional.

For let the compounded magnitudes be proportional as AB to BE , so CD to DF ; therefore divided, it shall be as AE to EB , so CF to FD ; And by inversion as BE to EA , so DF to FC ; therefore compounded as BA to AE , so DC to CF : which is Conversion of proportion, according to Def. 17. EL V. For AE is the excess of the Antecedent AB above the Consequent BE : and CF the excess of the Antecedent CD above the Consequent DF .



But this Corollary (however it hath hapned to be in this place) does more properly follow the 18th. Proposition. As we have now shewn how Converse proportion is immediately deduced from the Composition and Division of Proportions, without any relation, or dependence on this 19th. Proposition: where the four proportional Terms are restrained, and necessarily supposed to be all Homogeneous magnitudes. Whereas in Conversion of proportion the third and fourth Terms may be magnitudes of a different kind, from the first and second Terms, and one proportion be in Lines, and the other correspondent proportion in Planes, or Solids, as the following Annotations shall farther declare.

ANNOTATIONS.

This 19th. Proposition and the two foregoing have a close correspondence with one another, as appears by their demonstrations; yet there is a notable difference to be observed between them. For here is compared the whole to the whole, and each part of one whole to each part of the other respectively, so that these magnitudes are put Homogeneous, according to the Definition of proportion. But in Composition, Division, Conversion of proportion, each whole is separately compared to a part of it self, and each part to the other part of the same whole. So that the four proportional Terms may be all of the same kind, either Lines, Planes, or Solids: or else of different: that is, the first and second Terms, may be Lines, and the third and fourth be Planes, or Solids. As the whole AB may be a Lineal magnitude, and by consequence the parts AE , EB are Lines. But again, the whole CD may be either a Plane, or a Solid, and by consequence the parts CF , FD are accordingly Planes, or Solids; yet it is demonstrated, as AB to AE , a Line to a Line, so is CD to CF , whether it be a Plane to a Plane, or a Solid to a Solid: Which general Analogy may be used in Composition, Division, and Conversion of proportion.

But in this 19th. Proposition, the whole AB is compared to the whole CD , and the parts to the parts: so that these are here supposed Homogeneous magnitudes, interchangeably compared to one another.

PROPOSITION XX.

If there be three magnitudes in one Rank, and as many in an other, taken two and two in the same proportion: and *ex æquo* the first be greater than the third, also the fourth shall be greater than the sixth; and if equal, equal; if less, less.

Let the three magnitudes be A, B, C, in one rank, and D, E, F, as many in an other, taken two and two in the same proportion, as A to B, so D to E; and as B to C, so E to F: and *ex æquo* let A be greater than C. I say, that also D shall be greater than F: and if equal, equal; if less, less.

Forasmuch as A is greater than C, and there is a third magnitude B; And that the greater hath to the same a greater proportion than the less [Prop. 8. El. V.]; therefore A hath to B a greater proportion than C to B. But as A to B, so D to E: wherefore D hath to E a greater proportion than C to B [Prop. 13. El. V.]. But as C to B, so F to E by inversion; therefore D hath to E a greater proportion than F to E. But of magnitudes having a proportion to one and the same, that which hath the greater proportion, is the greater [Prop. 10. El. V.]: Therefore D is greater than F.

In like manner we shall demonstrate, that if A be equal to C, also D shall be equal to F; and if less, less.

Wherefore if there be three magnitudes in one rank, and as many in an other, &c. Which was to be demonstrated.

PROPOSITION XXI.

If there be three magnitudes in one rank, and as many in an other, taken two and two in the same proportion; and the *Analogy* be *Perturbate*: if then *ex æquo* the first be greater than the third, also the fourth shall be greater than the sixth; and if equal, equal; if less, less.

Let the three magnitudes be A, B, C, in one rank, and D, E, F, as many in an other, taken two and two in the same proportion. And let the *Analogy* be *Perturbate*, as A to B, so E to F, and as B to C, so D to E: then *ex æquo* let A be greater than C. I say, that D shall be greater than F: and if equal, equal; if less, less.

Forasmuch as A is greater than C; and there is a third magnitude B; therefore A hath to B a greater proportion than C to B [Prop. 8. El. V.]. But as A to B, so E to F; wherefore E hath to F greater proportion

$$12 \quad . \quad 9 \quad . \quad 6$$

$$A \quad . \quad B \quad . \quad C$$

$$:: \quad ::$$

$$D \quad . \quad E \quad . \quad F$$

$$8 \quad . \quad 6 \quad . \quad 4$$

$$18 \quad . \quad 12 \quad . \quad 4$$

$$A \quad . \quad B \quad . \quad C :: *$$

$$::$$

$$* :: D \quad . \quad E \quad . \quad F$$

$$27 \quad . \quad 9 \quad . \quad 6$$

than c to B . But because it is as B to c , so D to E ; therefore by inversion as c to B , so E to D . But E hath F a greater proportion than c to B ; therefore E hath to F a greater proportion than E to D . But to what magnitude the same hath a greater proportion, that is the less [Prop. 10. El. V.]; therefore F is less than D , and D greater than F : wherefore if A be greater than c , also D shall be greater than F .

$$18 \cdot 12 \cdot 4$$

$$A \cdot B \cdot c :: *$$

$$::$$

$$* :: D \cdot E \cdot F$$

$$27 \cdot 9 \cdot 6$$

In like manner we shall demonstrate, if A be equal to c , also D shall be equal to F ; and if less, less.

Wherefore if there be three magnitudes in one rank, and as many in an other, taken two and two in the same proportion, &c. Which was to be demonstrated.

PROPOSITION XXII.

If there be magnitudes how many soever in one rank, and as many in an other, taken two and two in the same proportion; also *ex æquo* they shall be in the same proportion.

Let the magnitudes how many soever be A, B, c , in one rank, and D, E, F in another, taken two and two in the same proportion, as A to B , so D to E ; and as B to c , so E to F . I say, that also *ex æquo* they shall be in the same proportion, as A to c , so D to F .

For of A, D let be taken equimultiples G, H .

And of B, E any whatever equimultiples K, L : also of c, F any whatever equimultiples M, N .

$$G \cdot K \cdot M$$

$$A \cdot B \cdot c$$

$$:: ::$$

$$D \cdot E \cdot F$$

$$H \cdot L \cdot N$$

Forasmuch then that it is as A to B , so D to E , and of A, D are taken equimultiples G, H : Also of B, E any whatever equimultiples K, L ; therefore it is as G to K , so H to L . [Prop. 4. El. V.]. By the same reason it is also as K to M , so L to N .

Forasmuch now as there are three magnitudes in one rank, G, K, M ; and as many in another, H, L, N , taken two and two in the same proportion; therefore *ex æquo* if G exceeds M , also H exceeds N ; and if equal, equal; if less, less [Prop. 20. El. V.]: And G, H are equimultiples of A, D ; also M, N any whatever equimultiples of c, F ; therefore it is as A to c , so D to F [Def. 5. El. V.].

Wherefore if there be magnitudes how many soever in one rank, and as many in an other, taken two and two in the same proportion, &c. Which was to be demonstrated.

ANNOTATIONS.

When there is a Concatenation of two, three, or more proportions, as of A to B, of B to C, of C to D, either in the same proportion, or in proportions different from one another, then in both these Cases the proportion of the extremes, as of A to D, is in Def. 10th. and 11th. El. V. said to be compounded of A to B, of B to C, and of C to D.

Now in this Proposition there is put a double Series of Concatenate proportions, in each whereof the first Term is to the second, as the first to the second, and the second to the third, as the second to the third, and so forth, in an Ordinate Analogy: Then the extremes in each Series taken *ex æquo*, at equal distance, as the first to the third, or the first to the fourth, &c. are here demonstrated to be proportional to one another, which is in effect to prove, that *two proportions compounded of equal proportions are equal to one another.*

PROPOSITION XXIII.

IF there be three magnitudes in one rank, and as many in another, taken two and two in the same proportion, and the Analogy be Perturbate: also *ex æquo* they shall be in the same proportion.

Let the three magnitudes be A, B, C, in one rank, and D, E, F, as many in another, taken two and two in the same proportion; and the Analogy be Perturbate, as A to B, so E to F; and as B to C, so D to E. I say, that *ex æquo* as A to C, so D to F.

Of A, B, D, let be taken equimultiples G, H, K; Also of C, E, F, any whatever equimultiples L, M, N.

G . H . L

A . B . C :: *

::

* :: D . E . F

K . M . N

Forasmuch as G, H are equimultiples of A, B; and parts have the same proportion with their equimultiples [Prop. 15. El. V.]; therefore it is as A

to B, so G to H. And by the same reason, as E to F, so M to N. But as A to B, so E to F; therefore as G to H, so M to N [Prop. 11. El. V.].

Again, because it is as B to C, so D to E; and of B, D are taken equimultiples H, K: also of C, E any whatever equimultiples L, M; therefore it is as H to L, so K to M [Prop. 4. El. V.].

Whereas now there are three magnitudes G, H, L, in one rank, also K, M, N, in another, taken two and two in the same proportion, and the Analogy is Perturbate, as G to H, so M to N; and as H to L, so K to M; therefore *ex æquo* if G be greater than L, also K shall be greater than N; and if equal, equal; if less, less [Prop. 21. El. V.].

But G, K are equimultiples of A, D, and L, N, are any whatever equimultiples of C, F; therefore it is as A to C, so D to F [Def. 5. El. V.].

Wherefore if there be three magnitudes in one rank, and as many in another, taken two and two in the same proportion, and the Analogy be perturbate: also *ex æquo* they shall be in the same proportion. Which was to be demonstrated.

ANNOTATIONS.

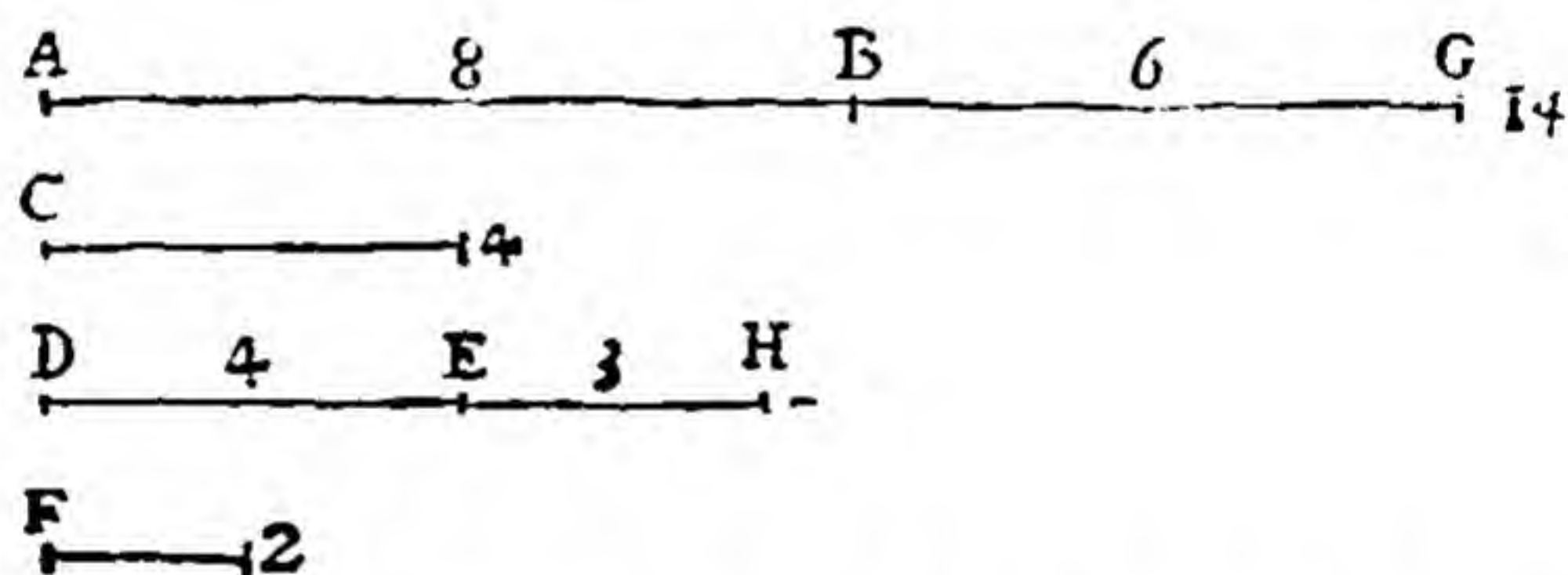
Euclide here proposes only three magnitudes; altho' they might be *quocunque*, like as in Prop. 22^d. But because in proportion *ex æquo* when the Analogy is Perturbate, the Geometrician never has occasion to use more than three Concatenate Terms, or two proportions; therefore our Elementator puts only three magnitudes in this Proposition: as taking special care to be no where in words or matter, defective or superfluous, and ever has a prospect to what is only and generally useful. Yet otherwise there is not any difference between Ordinate, and Perturbate Analogy in the Conclusions of these two Propositions; for as numbers, viz. 2, 3, 4, 5, &c. taken in any interchangeable order, and added together shall always give the same sum, or multiplied into one another, make the same Product; so in proportion *ex æquo*, the Terms placed either Ordinately, or Perturbately, shall notwithstanding ever have their extremes compounded of the same intermedial proportions; and therefore the extremes taken *ex æquo*, shall always be proportional to one another: as here in magnitudes, and in Prop. 14th. and 22^d. El. VII. is in numbers demonstrated.

PROPOSITION XXIV.

If the first hath to the second the same proportion, that the third hath to the fourth: and a fifth hath to the second the same proportion, that a sixth hath to the fourth: Also the first and fifth together, shall have to the second the same proportion, that the third and sixth hath to the fourth.

Let the first AB have to the second c , the same proportion that the third DE hath to the fourth F . Also let a fifth BG have to the second c , the same proportion, that a sixth EH hath to the fourth F .

I say, that AG the first and fifth together, hath to c the second, the same proportion, that DH the third and sixth together, hath to F the fourth.



$AB . C :: DE . F$. and $BG . C :: EH . F$. Then $AB + BG . C :: DE + EH . F$.

$8 . 4 :: 4 . 2$. and $6 . 4 :: 3 . 2$. Then $8 + 6 . 4 :: 4 + 3 . 2$.

Forasmuch as it is as BG to c , so EH to F ; therefore by inversion as c to BG , so F to EH .

And because it is as AB to c , so DE to F , and as c to BG , so F to EH ; therefore *ex æquo* as AB to BG , so DE to EH [Prop. 22. El. V.].

Now divided magnitudes being proportional, shall also compounded be proportional [Prop. 18. El. V.]; therefore as AG to GB , so DH to HE .

But also it is as GB to c , so E to F ; therefore *ex æquo* as AG to c , so DH to F .

If therefore the first hath to the second the same proportion, that the third hath to the fourth: and a fifth hath to the second, &c. Which was to be demonstrated.

ANNOTATIONS.

The 2^d. Proposition of this El. is the same with this 24th, excepting that it was there confined only to multiple proportions, whereas here is comprehended all kind of proportions whatsoever. The present Example is exposed in Duple, and Sesquialteral proportions added to one another: As 8 the first in Duple proportion to 4 the second; and 6 the fifth in Sesquialteral proportion to 4 the second: where 8 the first, and 6 the fifth added together, make 14 to 4, or $\frac{7}{2}$; that is, Triple Sesquialteral proportion, or $3\frac{1}{2}$. So again, 4 the third to 2 the fourth; and 3 the sixth to 2 the fourth, are in the same proportions as before 8 was to 4, and 6 to 4: And here likewise 4 the third, and 3 the sixth added together, make 7 to 2, the same Triple Sesquialteral proportion, or $3\frac{1}{2}$. For 7 contains 2 thrice, and one half of 2. This Proposition we have here both in *Species* and Numbers, represented after this manner.

$$\begin{array}{l} 8 . 4 :: 4 . 2 . \\ \text{If } A . B :: C . D . \text{ and} \\ E . B :: F . D . \text{ Then} \\ 6 . 4 :: 3 . 2 . \\ A + E . B :: C + F . D \\ 8 + 6 . 4 :: 4 + 3 . 2 \end{array}$$

Note farther hereupon, that the 18th. Proposition is in some part, of the same nature with this 24th. For there the Consequents added to the Antecedents, are as the fifth, and sixth Terms added to the first, and third, making also four other proportionals: As thus Appears.

$$\begin{array}{l} 9 . 3 :: 6 . 2 . \\ \text{If } A . B :: C . D . \text{ Then} \\ A + B . B :: C + D . D \\ 9 + 3 . 3 :: 6 + 2 . 2 . \end{array}$$

Or to set forth more fully the agreement between the 18th. Proposition, and this 24th. As in this form.

$$\begin{array}{l} \text{If } A . B :: C . D . \text{ and} \\ B . B :: D . D . \text{ Then} \\ A + B . B :: C + D . D . \end{array}$$

The difference here between them is, that in the 18th. only the Consequents are added to the Antecedents, and together compared to the same Consequents: but in the 24th. any magnitudes, or numbers, which are in the same proportion to the given Consequents, may be taken; and added to the given Antecedents, are together compared to the same Consequents. And therefore in Prop. 18. the Consequents being supposed as the fifth and sixth Terms compared to themselves as the second and fourth Terms, they are always in proportion of equality: but in this 24th. the fifth and sixth Terms, may be taken in any proportion whatsoever.

Addition of Proportions.

Moreover, from this 24th. Proposition, the Addition of Proportions is plainly discovered: and how several proportions, which have the same common Consequent, are united into one single proportion, by adding the Antecedents to one another, and making the sum of them an Antecedent to the common Consequent: As 3 to 1, and 3 to 1, added together make 6 to 1, or $\frac{6}{1}$: That is, two Triple proportions, which have the same common Consequent added together in this manner, do make Sextuple proportion. So again, 4 to 3, 5 to 3, 9 to 3 added together, make 18 to 3, or $\frac{18}{3}$: that is, Sextuple proportion is by addition of the Antecedents made of Sesquitertrial, Sesquibitertial, and Triple proportions, when they have such a common Consequent.

But now when several proportions have several Consequents, then for to add these together, they are first to be brought to have the same common Consequent: As the proportions of 4 to 3, and of 3 to 2 cannot be added together, unless they be reduced unto a common Consequent. We have before in the Annotations upon the 8th. Proposition, given a Rule how to bring proportions to a common Consequent, like as Fractions are brought to others of the same denomination. For the Addition, Subtraction, Multiplication, and Division of proportions, are performed after the same manner as in Fractions: And proportions are as properly noted by placing the Antecedent above, and the Consequent beneath the interposed strait line, as the Numerator and Denominator are used to be in Fractions. And therefore this particular Doctrine is to be sought after among the Arithmetical Authors.

Now in the foregoing Instance to bring the proportions of 4 to 3, and of 3 to 2, or $\frac{4}{3}$ and $\frac{3}{2}$ unto a common Consequent, First multiply the Consequents 3 and 2 into one another; that is, 3 into 2, or 2 into 3, which by Prop. 16. El. VII. make the same number 6, and this now is to be put for a common Consequent. Then multiply each Antecedent into the others Consequent, that is, 4 into 2, making 8 for one Antecedent, and 3 into 3, making 9 for the other Antecedent. Thus the proportions of $\frac{4}{3}$ and $\frac{3}{2}$ are brought unto $\frac{8}{6}$ and $\frac{9}{6}$, the very same proportions as before, and having a common Consequent: For $4 : 3 :: 8 : 6$ in proportion Sesquitertrial, $1 : \frac{1}{3} :: 3 : 2$ and $3 : 2 :: 9 : 6$ in proportion Sesquialteral, $1 : \frac{1}{2}$.

The reason of this practice is manifest, for $\frac{4}{3}$ multiplied by the same number 2, make the Products $\frac{8}{6}$ in the same proportion, by Prop. 17th. El. VII. or by Prop. 15. El. V. So $\frac{3}{2}$ multiplied by 3, make the same proportion $\frac{9}{6}$.

Then of these two proportions $\frac{8}{6}$ and $\frac{9}{6}$, add the Antecedents 8 and 9 together, which make 17 the Antecedent to the common Consequent 6, or $\frac{17}{6}$: that is, $2\frac{5}{6}$, and named *Proportio dupla superquintu-partiens sextas*.

Therefore by reduction of the proportions $\frac{4}{3}$ and $\frac{3}{2}$ into a common Consequent, it now appears that these proportions, $1 : \frac{1}{3}$ and $1 : \frac{1}{2}$ added together make $2\frac{5}{6}$. The like practice is to be used in the Addition of all other proportions: but very little use there is made of it in the Mathematics, farther than what is shewn in the 18th. and 24th. Propositions of this Fifth Element. From whence we have taken occasion to touch briefly upon the Addition of proportions, and in what manner it can be only made. But the Multiplication of proportions is more remarkable: which therefore we shall here by the way explain in a few words, for that there will be a necessity to consider farther of this matter at the 5th. Definition of the Sixth Element: And in this place also to shew the difference between the Multiplication, and Addition of proportions.

Multipli-

Multiplication of Proportions.

Multiplication of proportions is made by multiplying the Antecedents into the Antecedents, and the Consequents into the Consequents; whose Products shall make some certain proportion. As of the proportions $\frac{4}{3}$ and $\frac{3}{2}$, the Antecedents 4 and 3 multiplied into one another, and the Consequents 3 and 2 multiplied into one another, shall make $\frac{12}{4}$, or Duple proportion.

Whereas the same proportion $\frac{4}{3}$ and $\frac{3}{2}$, added to one another make $2\frac{1}{2}$; as we have before shewn. So $\frac{4}{3}$ and $\frac{3}{2}$, multiplied together make $\frac{12}{4}$ or $\frac{3}{1}$, Triple proportion. But added make $\frac{7}{2}$, or $3\frac{1}{2}$, proportion Triple Sesquialteral. Again, $\frac{4}{3}$ added to $\frac{3}{2}$, make $\frac{17}{6}$; but multiplied together make $\frac{12}{4}$. Likewise $\frac{3}{2}$ added to $\frac{4}{3}$ make $\frac{17}{6}$; but multiplied make $\frac{12}{4}$. Thus it appears, how proportions may by Addition arise to be sometimes greater, sometimes less than by Multiplication: which is a Property peculiar to proportions, and somewhat remarkable.

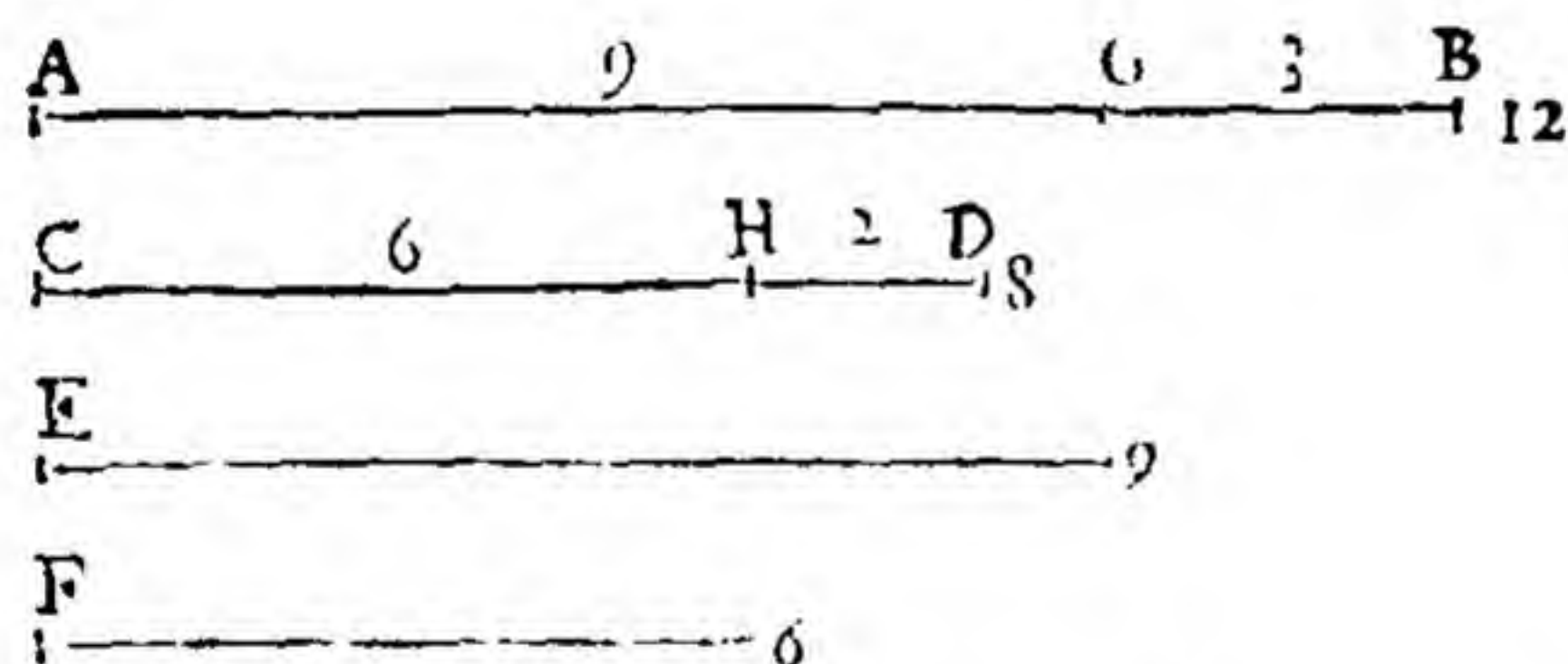
PROPOSITION XXV.

If four magnitudes be proportional, the greatest and least are greater than the other two.

Let the four proportional magnitudes be AB, CD, E, F ; and as AB to CD , so E to F . Now let the greatest of them be AB , and the least F . I say, that AB, F , are greater than CD, E .

For to E let AG be put equal; and to F be put equal CH : forasmuch therefore that it is as AB to CD , so E to F : and that AG is equal to E , and CH to F ; therefore as AB to CD , so AG to CH [Prop. 11. El. V.].

Also because it is as the whole AB to the whole CD , so the part detracted AG , to the part detracted CH ; therefore the Remainder GB , shall be to the Remainder HD , as the whole AB to the whole CD [Prop. 19. El. V.]. But the whole AB is [by Supposition] greater than the whole CD ; therefore also the Remainder GB is greater than the Remainder HD .



And forasmuch as AG is equal to E , and CH to F ; therefore AG and F are equal to CH , and E . For if equals be added to equals, the wholes are equal [Ax. 2.].

But again, if to unequals be added equals, the wholes shall be unequal [Ax. 4.]: Therefore if to the unequals GB, HD be added these equals, F, AG , and E, CH , namely, F , and AG added to GB the greater, and E , and CH to HD the less; then shall AB , and F be greater than CD , and E .

If therefore four magnitudes be proportional, the greatest, and least are greater than the other two. Which was to be demonstrated.

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Here *Euclide* ends this Element: but the Moderns have (out of *Pappus*) annexed several other Propositions; which because they are only manifest Confectaries arising from the foregoing Propositions, and not of so frequent use, we have thought fit not to overcharge the younger Students with. For *Peletarius* here very well says, *Quæ (propositiones) per se claræ sunt, locum tantum occupant: Ingenium etiam onerant, & multitudo tedium parit.* What *Euclide* hath delivered is abundantly sufficient in this matter.

ANNOTATIONS.

This property of proportional magnitudes hath been formerly intimated in Prop. 35th. and 36th. El. I. where Parallelograms on the same, or equal bases in the same Parallels, are demonstrated to be equal; that is, to be to one another in the same proportion of Equality. And there also it hath been observed, that of those equal Parallelograms their *Perimeters* are unequal, and of two equal Parallelograms the longest, and shortest sides, are greater than the other sides. Now this Proposition demonstrates the like Affection of four proportionals universally in all kinds of proportion, and in all kinds of magnitude, either Lines, Planes, or Solids: and that in each of them the greatest, and least Terms added together, shall always be greater than the other two.

A
SYNOPSIS
OF THE
PRINCIPAL PROPOSITIONS
OF THE
FIFTH ELEMENT.

Prop. 1, and 12. If $A . B :: C . D :: E . F$. Then

$$A . B :: A + C + E . B + D + F.$$

Prop. 2, and 24. If $A . B :: C . D$. and

$E . B :: F . D$. Then

$$A + E . B :: C + F . D.$$

Prop. 3, 22, and 23. Proportionals *ex æquo* Ordinate and Perturbate.

Prop. 4. If $A . B :: C . D$. Then

$$3A . 5B :: 3C . 5D \&c.$$

Likewise $3A . B :: 3C . D \&c.$

and $A . 3B :: C . 3D . \&c.$

Let $\mathcal{A} = A + E$. and $\mathcal{a} = a + e$

Prop. 5, and 19. If $\mathcal{A} . \mathcal{a} :: A . a$. Then

$$E . e :: \mathcal{A} . \mathcal{a}.$$

Prop. 15. $A . B :: 3A . 3B \&c.$

Primary Proportionals, $A . E :: B . C$.

Corol. Prop. 4. Inversion, $E . A :: C . B$.

Prop. 16. Alternation, $A . B :: E . C$.

Prop. 17. Composition, $A + E . E :: B + C . C$.

Prop. 18. Division, $A - E . E :: B - C . C$.

Corol. Prop. 19. Conversion, $A . A - E :: B . B - C$.

Note farther, that Composition, Division, Conversion of proportionals, may be again, and frequently are proportionally varied by Inversion, and Alternation after this manner.

Composition and Alternation, $A + E . B + C :: E . C$

Division and Alternation, $A - E . B - C :: E . C$. Therefore

Composition and Division, $A + E . B + C :: A - E . B - C$. by Prop. 11.

K k 2

Thus

Thus hath *Euclide* finished his general Doctrine of *Proportions*, and *Proportionals* in magnitudes: which we confess is more plainly set forth in his Elements of Numbers. For tho' the Properties of Proportionals be alike in Magnitudes and Numbers; yet are they demonstrated in those Elements from more natural Principles, and more obvious to the common notions of men in this matter. We have formerly expounded the different Definitions of proportional Magnitudes, and of proportional Numbers, and shewn the reasons thereof for want of a natural measure in Magnitudes. And therefore what evidence, or what demonstrations could in this Element be expected, or made from a certain measure between proportional Magnitudes (as it is in Numbers) where there is in nature no certain measure? Nay farther, where there cannot be put, or supposed any imaginable common measure. Number is only one way infinite by augmentation; but hath its indivisible Monade for a beginning, and common measure, the natural instrument of Demonstration in the Doctrine of proportions, and accordingly made use of in the Elements of Numbers. But Magnitude is both ways infinite, by Augmentation, and Diminution. It hath not a Monade, that *quid Minimum, unum & solum*, That one and sole measure of all things. What foundation it hath useful for Demonstration in this Subject, *Euclide* hath most ingeniously found out, and given us in the fifth Definition of proportional Magnitudes. And it is either plain Ignorance, or great Vanity in those, who charge this Element of Intricacy, and Imperfection, which is framed with so much Art, and in as clear a Manner, as the nature of Magnitude could admit.

The *Epichœremata*, or Attempts of those Learned Geometricians *Joannes Benedictus*, *Tacquet*, *Borellus*, &c. in this matter I leave to be at large examined by the Professors of Geometry in our Universities, and *Gresham College*.

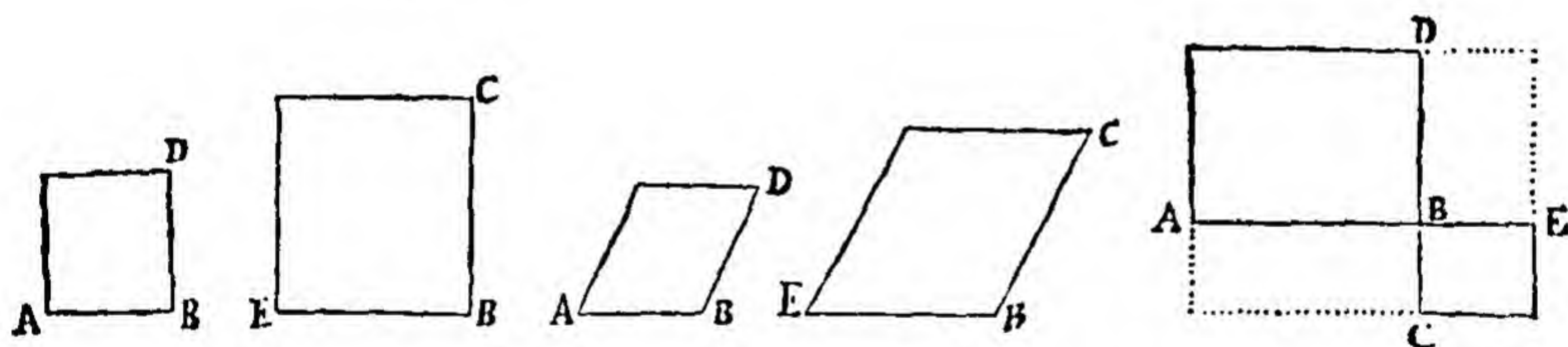
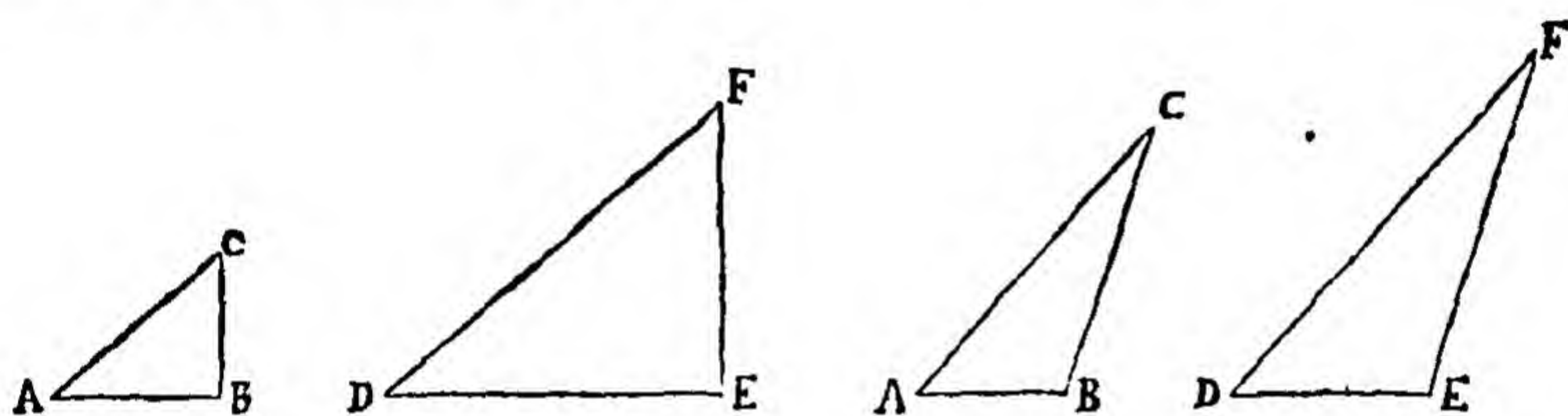
THE SIXTH ELEMENT.

DEFINITIONS.

DEFINITION I.

Like *strait-lined Figures* are those, which have their several angles equal, each to each, and the sides about the equal angles proportional.

In the Triangles ABC , DEF , if the several angles be equal, each to each, A to D , B to E , C to F , and the sides about the equal angles proportional, AB to BC , as DE to EF ; BC to CA , as EF to FD ; CA to AB , as FD to DE , then these are said to be like Triangles.

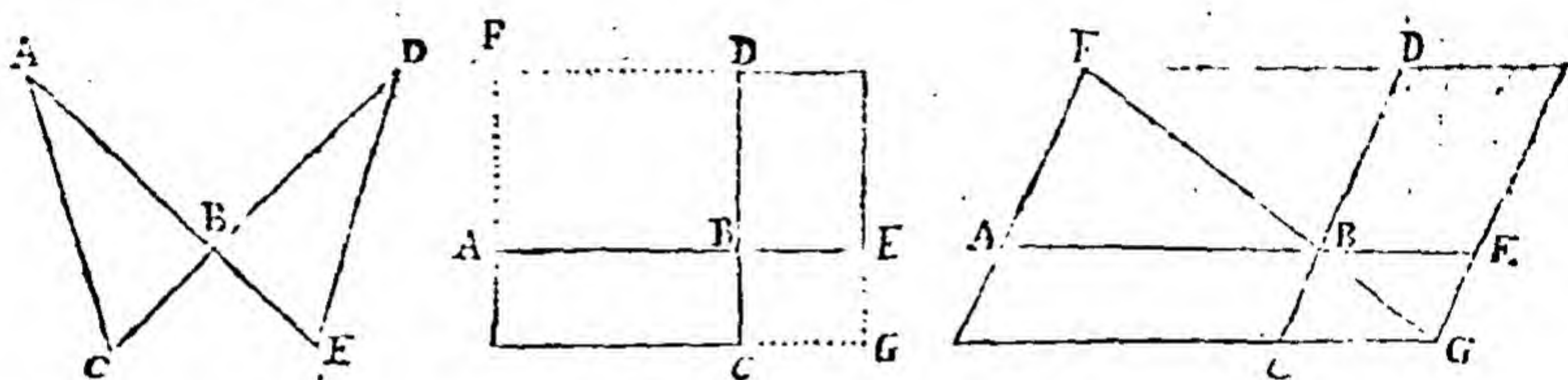


So again, in the Equiangular Parallelograms AD , EC , if AB be to BD as EB to BC , then these are called like Figures: And so forth in all Multilateral Figures, or Polygons. But Equiangular Parallelograms are not always like Figures; as it frequently happens in Right-angled Parallelograms, and Equiangular Rhomboids, which are not like Figures; unless they have also their sides about the equal Angles proportional. But every Square is to every Square, and every Equiangular Rhombus to every Equiangular Rhombus a like Figure: because they have their sides in the same proportion of Equality to one another.

DEFINITION II.

Reciprocal Figures are, when in each of the Figures there are Terms both Antecedent and Consequent.

As in the Triangles ABC , DBE , and the Parallelograms AC , DB , if AB is to BE , as back again DB is to BC , then these are called Reciprocal Figures. In which, note for distinction between Like and Reciprocal Figures, that the extreme Terms of the two proportions in Reciprocal Figures; namely the first Antecedent, and the last Consequent, shall ever be in the same figure: whereas in like figures the first Antecedent is in one figure, and the last Consequent always in the other. Moreover, in the foregoing Definition of like Figures, all the Angles are to be equal, each to each, and all the sides about the equal Angles directly proportional to one another. But it is not necessary in all Reciprocal Figures to have the several Angles equal, but only that Angle, which is contained by those sides which are Reciprocally proportional to one another. For in the other Angles the figures may be very often unlike, and not agree in equal Angles, and sides proportional, as hereafter in Triangular Figures the 15th. Proposition will manifestly shew.



Again, note that Reciprocal Figures are supposed always equal, tho' not always Equiangular; and like figures are commonly supposed unequal, and ever Equiangular. For if like figures be also equal in Area, they are as it were only the same figure seated in several places.

Moreover, the Parallelograms FG , BE are in this matter very remarkable Schemes, wherein the Complements AC , DE , are equal by Prop. 43. El. I; and also Reciprocal Figures (as at Prop. 14th. of this Element is demonstrated) having the side AB to the side BE , as Reciprocally DB is to BC . And again, the Parallelograms AD , CE , are like figures, having the side AB to the side BD , as EB is to BC in a direct proportion to one another. As is demonstrated in the following 24th. Proposition.

DEFINITION III.

A Strait line is said to be cut in extreme and mean proportion, when it is as the whole to the greater Segment, so the greater Segment to the less.

If the line AB be divided at the point C in such a proportion, that as the whole AB is to the greater Segment AC , so is the same AC to the less Segment CB , then the line AB is said to be cut in extreme and mean proportion, For that the whole line, and the less Segment are the two extremes, and the greater Segment is a mean or middle proportional Term between them. And so the whole line, the greater Segment, and the less are all three in the same continual proportion: Which by our incomparable Mr William Oughtred, is justly said to be *Sectio per Divina*, and how to effect this Section is demonstrated at Prop. 30th.

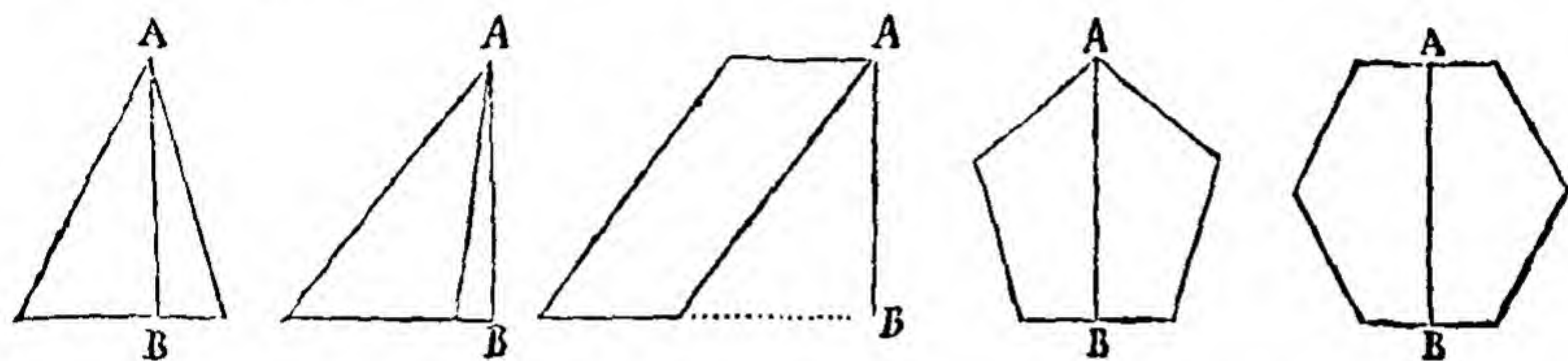
DEFI-

DEFINITION IV.

The *Altitude* of every Figure is a *strait line* drawn from the *Vertex* perpendicular to the *Base*.

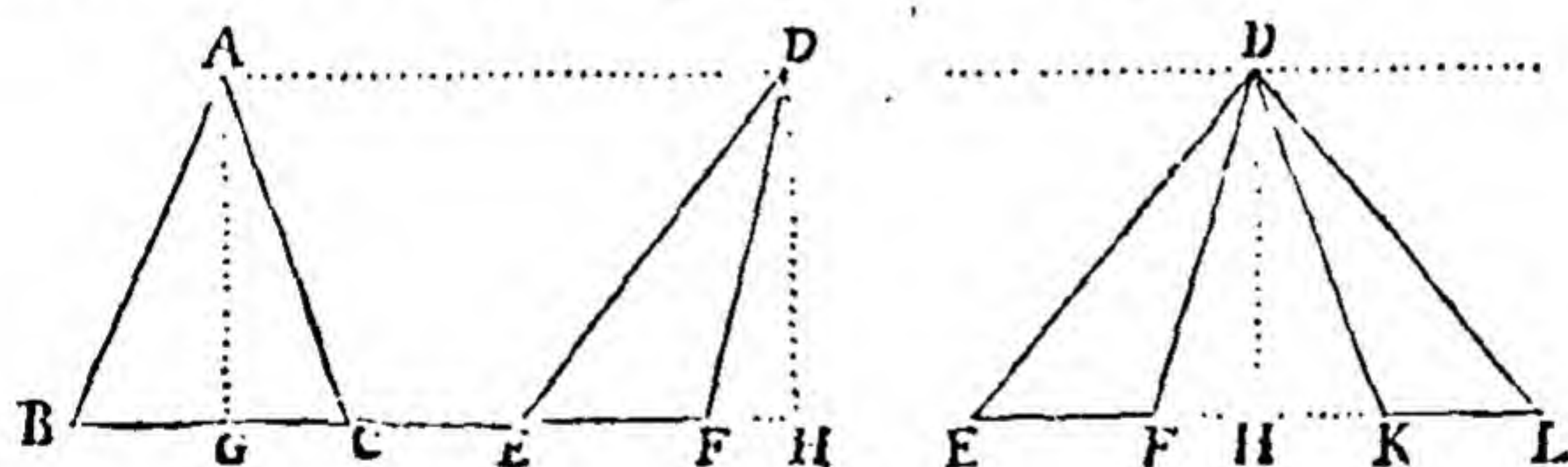
The Altitude, or Height of any thing, is vulgarly accounted to be the distance from the Top to the Bottom; and by distance is meant the shortest length between them, because the shortest length is the only Singular, Certain, and Determined Space between any two distant things.

So the Geometrician determines the Altitude of any Figure by a Perpendicular from the Vertex of the Figure to the opposite Base (produced if need be); as in these exposed Figures their Altitudes are the Perpendiculars AB , AB , &c; for that they are in common sense, and also demonstratively the shortest length between the Vertex and the Base.



Therefore when Figures are said to be of the same Altitude, we are to understand that the Perpendiculars from their Vertex to their Bases are equal to one another, and therefore may be placed in the same parallels.

For let the Triangles ABC , DEF , be of the same Altitude, having the Perpendiculars AG , DH equal to one another. I say, the Triangles ABC , DEF , are in the same parallels. For let be drawn the strait line AD : Then forasmuch as AG , DH are equal, and at right angles to GH ; therefore they are also parallel one to another [Prop. 28. El. I.]. But strait lines equal and parallel are bounded by equals, and parallels [Prop. 33. El. I.]; therefore AD is parallel to GH , and the Triangles ABC , DEF , are in the same parallels.



Again, if the Triangles DEF , DKL , have a common Vertex D , so by consequence a common Altitude DH , which is a Perpendicular to their continued Bases; it is then also obvious, that there may be drawn by the same Vertical point D , a parallel to the opposite Bases [by Prop. 31. El. I.].

So that in general, Triangles, and Parallelograms to be in the same Altitude, and in the same Parallels import the same thing.

DEFI-

DEFINITION V.

A Proportion is said to be compounded of proportions, when the Quantities of the proportions multiplied into themselves, do make some proportion (or some Quantities of a proportion).

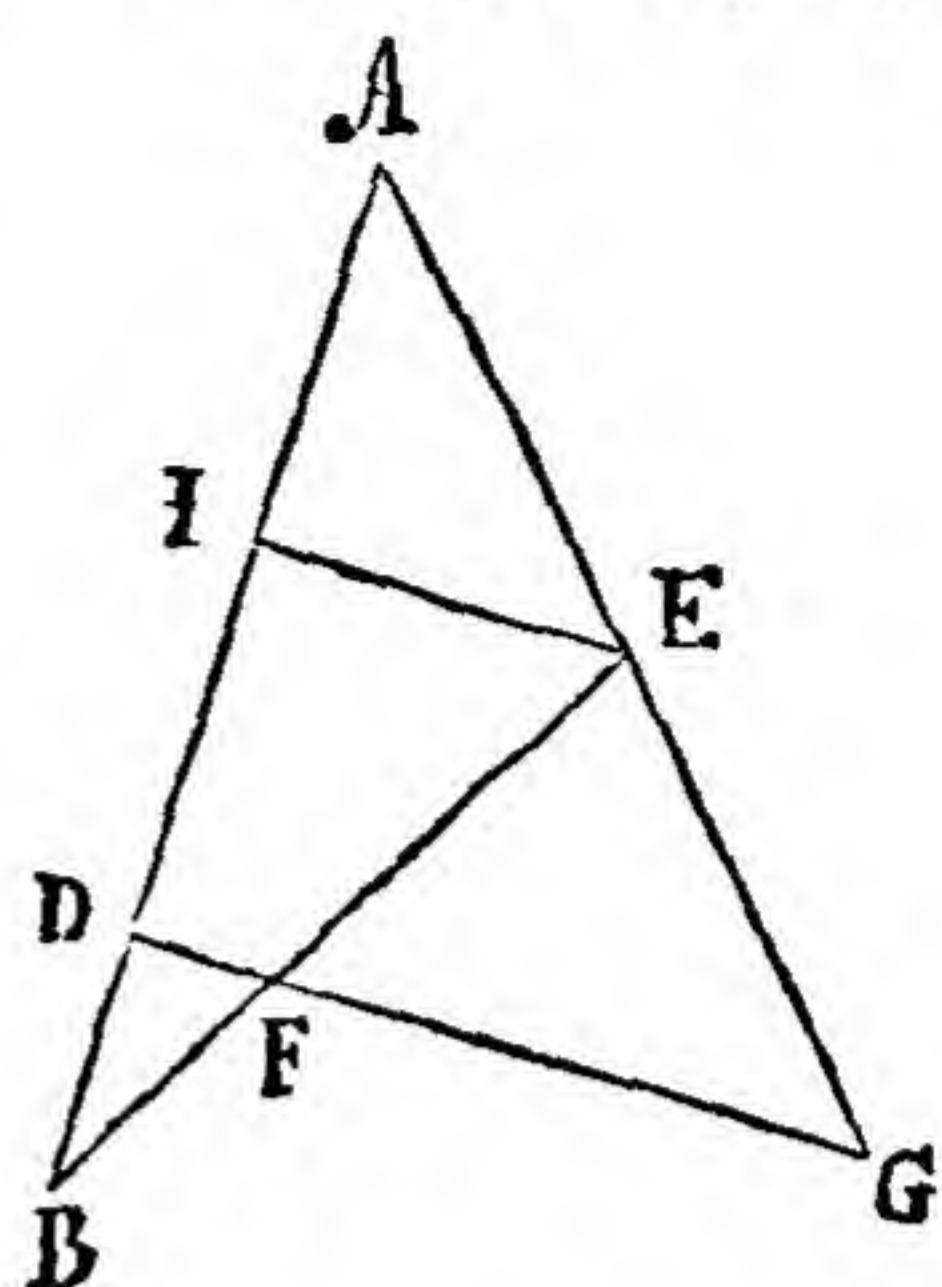
There are various Lectons of this place; Some Editions have it, *ποιῶσι πρὸς λόγον, do make some proportion*: but the more ancient Reading is, *ποιῶσι πρὸς, scil. πρὸς ἑαυτοῖς*, that is, the Quantities of the proportions multiplied into one another *do make some Quantities*, to wit, of a proportion: which proportion being thus produced from the multiplied proportions, is said to be compounded of them. As of the given proportions, let one be Sesquitertrial $1\frac{1}{3}$, and the other Sesquialteral $1\frac{1}{2}$: As 4 to 3, and 3 to 2, or $\frac{4}{3}$ and $\frac{3}{2}$, which multiplied into one another, that is, Antecedents into Antecedents, and Consequents into Consequents, do make $1\frac{2}{3}$, or 12 to 6, that is, some Terms, or Quantities of Duple proportion; which is here compounded of Sesquitertrial, and Sesquialteral proportions, *scil. $\frac{4}{3}$ compounded of $1\frac{1}{3}$ and $1\frac{1}{2}$* .

For by this Definition *whatsoever proportion arises from the multiplication of any other proportions, the same is said to be compounded of them*: And this is in general the meaning of this Definition.

Now farther to explain the words. By the *Quantities* of proportions our Commentators would have the Denominators to be understood: But it is not material whether by the *Quantities* we understand either the *Denominators*, or the *given, and exposed Numbers* of the proportions, or the *least Numbers* of the same proportions: For that each of these being multiplied together, do produce the same compound proportion, tho in different Numbers, and also in such, as are far enough from being the *Denominator*, or the *least Numbers* of that compound proportion; as the foregoing Instances do apparently shew: only the Arithmetical operations are performed with more ease in the least Numbers: otherwise the effects are still the same. And the most proper use of Denominators is to discern the Species of a proportion exposed in any Numbers whatsoever, and to denote the name. But *Euclid* in his Elements of Numbers never mentions Denominators, or the Quantities of proportions. He only shews in Prop. 35th. El. VII. how to bring any given Numbers into the least Numbers in the same proportion with them: upon which Reduction all his Demonstrations proceed: And thus much for the interpretation of the words of this Definition.

We are next to enquire, what use is made thereof in these Elements, in regard that *Euclid* had before otherwise defined compound proportion, as it is comprehended in Def. 10. El. V. And according to that Definition he demonstrates in Prop. 23. El. VI. *All equiangular Parallelograms to have to one another a proportion compounded of their sides*: And likewise in Prop. 5. El. VIII. *All Plain, or Superficial Numbers to have a proportion compounded of their sides*. The demonstrations both in Magnitudes and Numbers, are framed just after the same manner upon the 10th. Definition of the Fifth Element, without the least use, or any mention of the multiplication of proportions according to this 5th. Definition of this Sixth Element. And strange it seems to me, nay very absurd, to admit that for an Elementary Definition, which is never used either in these Elements of *Euclid*, or in the *Conics* of *Apollonius*, or elsewhere in *Archimedes*: but in all these, and other Geometrical Authors, Composition of proportion is ever taken in the sense, and notion of Def. 10th. El. V. It will not therefore be impertinent to examine how another useless Definition of Compound proportion, and which besides can only relate to numbers, came to be so *Ungeometrically* inserted in this place, where the proportions of Magnitudes are solely considered, and the use of numbers (as altogether improper, and also insufficient for demonstration in these 5th. and 6th. Elements) is by *Euclid* studiously avoided.

The most ancient account I have met with, is in *Theon's Commentaries on Ptolemy's Mathematical Syntax*, Lib. I. cap. 12. entitled *Prolambanómena*, where, says *Ptolemy*, Let from the point A be produced two lines AB, AG, and from the points B, G, be drawn BE, GD, cutting each other in F; and draw EI parallel to GD. I say, that the proportion of GA to AE is compounded of the proportions of GD to DF, and of FB to BE.



Now to prove this, he introduces an other extraneous line, viz. DF taken *ἐξωθεν* (says *Ptolemy*) *extrinsecus*, or *ab extra*, as it were *from without*; and interposing DF for a middle Term between GD, and EI, does assume that the proportion of GD to EI, is compounded of the proportions of GD to DF, and of DF to EI. As a compound proportion is understood by *Euclide* in Def. 10. El. V.

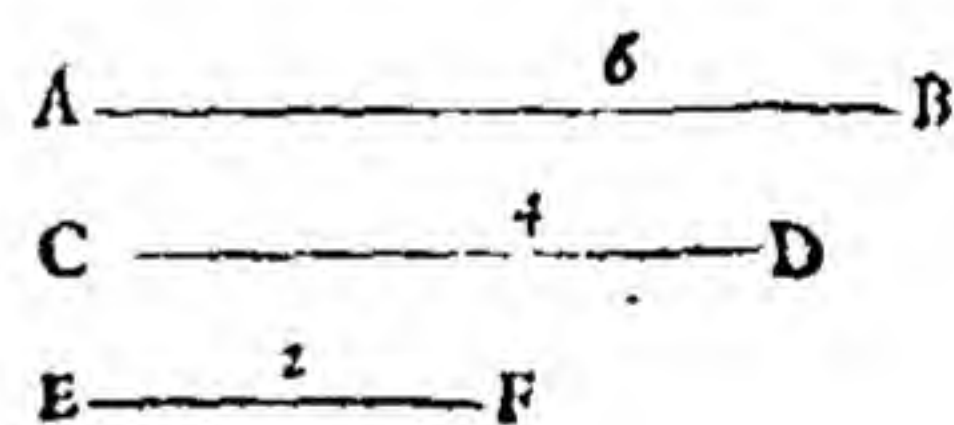
But now altho' in a Series of continued Terms, this be called by *Euclide* a compound proportion (as he might *pro arbitrio* give such a name to the proportion of the extreme Terms); Yet a question may be made, How doth it appear that the proportion of GD to EI, is really compounded of the Proportions of GD to DF, and of DF to EI?

In satisfaction to such demands, and for the explication of this place, and therewithal of *Euclid's* Definition, *Theon* in his Comment, Page 62, gives an other Definition of Compound proportion in these words.

"A proportion is said to be Compounded of two, or more proportions, when the Quantities of those proportions being multiplied do make some Quantity of a proportion.

The use now that *Theon* makes of this Definition, is to explain *Euclid's* 10th. Def. El. V. and to shew how both agree in producing one, and the same Compound proportion; as supposing his own Definition to be a more natural, and clearer Notion of a Compound proportion than that of *Euclid's*; and thereby to confirm the 10th. Def. of El. V. He therefore with *Euclide* puts two proportions in a Series of three continued Terms.

For (says he) Let AB have to CD any given proportion (as 6 to 4) and CD to EF any proportion (as 4 to 2). I say, that the proportion of AB to EF, is compounded of the proportions of AB to CD, and of CD to EF: that is, if the Quantity of the proportion of AB to CD be multiplied into the Quantity of the proportion of CD to EF, it shall make the Quantity of the proportion of AB to EF (that is of 6 to 2.).



For let the given Proportions be 6 to 4, and 4 to 2: then 6 multiplied into 4, make 24, the same Triple proportion with 8, which is the proportion of the first Term to the third. And this *Theon* universally demonstrates, viz. That in any three Terms (suppose A, B, C), if the proportion of A the first to B the second, be multiplied into the proportion of B the second to C the third, then the Product shall always make the same proportion with that of A the first to C the third. This proportion of the first Term to the third, *Euclide* in Def. 10th. El. V. calls a proportion Compounded of the two intermedial proportions exposed in three continued Terms, viz. Of the proportion of the first Term to the second, and of the second to the third: and owns no other notion of a Compound proportion in the demonstrations of Prop. 23^d. El. VI. and Prop. 5th. El. VIII. or elsewhere.

But because in *Euclid's* Series of three, four, or more continued Terms, the proportion of the first Term to the last, does not so immediately appear to be Compounded of the Intermedial, and Concatenate proportions; therefore *Theon* joyns the Philosopher with the Geometrician, and gives another very natural Definition

of a Compound proportion. For in *Theon's* multiplication of Antecedents into Antecedents, and Consequents into Consequents, there is manifestly made a natural Mixture, or Composition of proportions: which being thus involved into one another, do really produce a Compound proportion; and also such, as he demonstrates to be the very same with *Euclid's*, according to Def. 10th. El. V. which *Theon* hath from his Definition thus illustrated, and confirmed. So that this multiplication of proportions may well be allowed to a Commentator for a good Animadversion, and Explication of Def. 10th. El. V. as here an occasion was given to *Theon*: but by no means to stand so improperly, and uselessly for a Geometrical Definition of *Euclid's*, as it hath been by some *Scholias*t unadvisedly transferred into this Sixth Element.

The next we find is *Eutocius* in his Comment on *Archimedes*, who in Prop. 4. Lib. II. *de Sphæra, & Cylindro*, proposes to cut a Sphere so that the Segments may have to one another the same proportion with any given.

In the course of his demonstration he uses Compound proportion in *Euclid's* sense, according to Def. 10th. El. V. and assumes that the proportion of RL to LQ (*ααπλζ* as *Archimedes* words it) is connected by, or (as commonly said *συνεστη*) is compounded of the proportions, which RL hath to LD, and LD to LQ.

R ————— Q — D ————— L

Hereupon says *Eutocius* in his Comment, 'tis manifest that Composition of proportions is taken as in the Elements (meaning Def. 10th. El. V.) in that LD is interposed as a middle Term between RL, and LQ. For (says he) if between two Numbers, or Magnitudes, be taken any middle Term, the proportion of these first Numbers or Magnitudes, is compounded of the proportion which the first Term hath to the middle, and of the proportion which the middle hath to the third.

But farther adds, that this is spoken *ἀδιασπῆκτως*, somewhat inarticulately (or abruptly, as if matters were not well jointed together) *καὶ ἄλ' ὅτως, ὥστε τὴν ἐννοίαν ἀναπαύειν*, and not in so plain a manner, as fully to satisfy our understanding, (to wit) in the proper notion of a Compound proportion. Therefore to make *Euclid's* Composition of proportions more perspicuous, he cites this other Definition of a Compound proportion, as being also found by him in the Elements (so early it seems to have been transmitted into this place) and demonstrates that

"If between two Numbers or Magnitudes, be taken any middle Term, then the
"proportion of those first Numbers or Magnitudes, to one another, shall be the
"same with that Compound proportion, which is made out of the proportions
"of the first Term to the middle, and of the middle to the third, multiplied into
"one another.

Now by the demonstration of this agreement, *Eutocius* intended to make manifest, that in any three continued Terms, the proportion of the first to the third is rightly said by *Euclide* in Def. 10th. El. V. to be a Compound proportion of the first to the second, and of the second to the third.

Moreover to illustrate this matter, he gives several instances in Numbers. Let (says he) between 12 and 2, be interposed any number, as 4; then the proportion of 12 to 2, that is, Sextuple, is Compounded of the proportions of 12 to 4, and of 4 to 2, of Triple, and Duple proportions. For $\frac{12}{4}$ multiplied into $\frac{4}{2}$ makes $\frac{12}{2}$, that is 6, the same proportion with $\frac{12}{2}$ the first given Terms.

In this instance of *Eutocius*, note that the two proportions $\frac{12}{4}$, and $\frac{4}{2}$, are each of the greater inequality: and therefore the proportion $\frac{12}{2}$ resulting from their multiplication, is really a *totum Compositum*, a greater proportion consisting of two less proportions, as a whole of so many partial Components; that is, a Sextuple proportion is here rightly compounded, and made up of a Triple, and Duple proportion multiplied into one another. And every Compound proportion is always a *totum* of the like nature, when the intermedial proportions are all of the greater inequality.

But

But now on the contrary, in proportions of the less inequality it is to be noted, that if between 2 and 12, be again interposed 4; then $\frac{2}{4}$ and $\frac{4}{12}$, multiplied into one another, shall make $\frac{8}{48}$, or $\frac{1}{6}$, a Subsextuple proportion, which is emergent from Subduple, and Subtriple proportions; yet does not, as in the intermedial proportions of the greater inequality, consist of these intermedials, as a whole of its parts; but contrarywise is as it were Subcompounded, and becomes to be a less proportion derived from two greater. For $\frac{1}{6}$ is a proportion less than either $\frac{1}{4}$, or $\frac{1}{12}$, from both which it doth descend; and the like will be ever found, when all the intermedial proportions are of the less inequality.

Again says *Eutocius*, Let between 9, and 6 be interposed 4; then the proportion of 9 to 6, that is Sesquialteral $1\frac{1}{2}$, is compounded of 9 to 4, and of 4 to 6, that is, of Duple Sesquiquartal, $2\frac{1}{3}$, and of Subsesquialteral $\frac{2}{3}$. For $\frac{9}{4}$ multiplied into $\frac{4}{6}$ makes $\frac{36}{24}$, a proportion Sesquialteral, and the same with $\frac{3}{2}$, the first given Terms.

In this instance 'tis to be noted, that $\frac{36}{24}$, a proportion Sesquialteral resulting from $\frac{9}{4}$ multiplied into $\frac{4}{6}$, is less than $\frac{9}{4}$, a proportion Duple Sesquiquartal, and greater than $\frac{4}{6}$, a proportion Subsesquialteral. And the like will ever happen, that when of the intermedial proportions some are of the greater inequality, and some of the less, then the Compound proportion arising from them may be sometimes greater, sometimes less than some of the Components: yet it is truly made by a mixture, and as it were a temperature of all the intermedial proportions. For proportions of the less inequality are an allay to proportions of the greater inequality, and may so counterpoise one another, that the same proportion may from different mixtures be diversly compounded.

As again to instance with *Eutocius* in Sesquialteral proportion. Let between 6 and 4, or $\frac{6}{4}$, a proportion Sesquialteral, be interposed 2, then $\frac{6}{4}$ is compounded of $\frac{6}{2}$, and $\frac{2}{4}$, that is, of a Triple, and of a Subduple proportion. For $\frac{6}{2}$ multiplied in $\frac{2}{4}$ makes $\frac{12}{8}$, a proportion Sesquialteral, and the same with $\frac{3}{2}$, the first given Terms. Thus Sesquialteral proportion is here compounded of Triple, and Subduple proportions, which in the numbers of *Eutocius* was made out of Duple Sesquiquartal, and Subsesquialteral proportions.

There are therefore these three different Constitutions of a Compound proportion taken notice of by *Theon*, and *Eutocius*, as they arise from proportions, which are either all of the greater inequality, or all of the less, or of the greater, and less intermingled with one another: as the demonstrations of *Theon*, and *Eutocius* (which may readily be found in *Clavius*) have distinctly comprehended.

Lastly if all the intermedials be proportions of equality, then the proportion of the first Term to the last is said to be compounded of a Duplicate, or Triplicate, &c. proportion of the first to the second, according to the number of the intermedial proportions, as they happen to be two, three, four, five, or more indefinitely.

But these four several Constitutions of a Compound proportion (whether it arises from proportions of equality, or of the greater inequality, or of the less, or of the greater, and less intermixed) may in general be at once demonstratively made evident in *Species*, or Symbolical computation.

For let there be (according to Def. 13th. El. V.) a Series of continued Terms, as A, B, C, D, E, which represent any Concatenate proportions whatsoever. I say with *Euclide*, that the proportion of the extremes of A to E, is compounded of the proportions of A to B, of B to C, of C to D, of D to E, that is, the proportions of $\frac{A}{B}$, $\frac{B}{C}$, $\frac{C}{D}$, $\frac{D}{E}$, multiplied together, Antecedents into Antecedents, and Consequents into Consequents, shall make the same proportion with that of A to E, as *Theon* and *Eutocius*, have expounded *Euclid's* Definition.

For let the Antecedents A, B, C, D, be multiplied together, and also the Consequents B, C, D, E, after this form: Then the intermedial Antecedents B, C, D, being expunged by the intermedial Consequents B, C, D, there only stands the proportion of A to E, of the first Term to the last. In this Symbolical form is laid open at one view the whole Myltery (if any there be) of Compound proportions; which *Theon*, *Eutocius*, and *Pitellio Opticorum*, Lib. I. Prop. 13. have laboriously demonstrated.

To conclude therefore this matter, when there are put three magnitudes, as in Def. 10th. El. V. in a continued Series compared to one another; *Euclide* calls the proportion of the first to the third a proportion compounded of the proportion of the first to the second, and of the proportion of the second to the third. Now *Theon* explains this Definition by an other, in which the nature of a Compound proportion seemed to be made more evident: and shews that in *Euclid's* Series of three continued Terms,

“If the proportion of the first to the second be multiplied into the proportion of the second to the third, there shall ever be made the proportion of the first to the third.”

Here now *Theon* supposes his Definition to be an obvious, and natural notion of a Compound proportion, by which multiplication of proportions, and their agreement with *Euclid's* Composition of proportions laid down in a Series of continued Terms, he thought it a proper explanation and confirmation of *Euclid's* Definition, which indeed we readily acknowledge, as also that it was the first real, and secret ground of Def. 10th. El. V.

But yet *Euclide* found this kind of multiplication to be in no manner serviceable for demonstrating all Compound proportions, even no more than his Definition of Proportional Numbers could be applied to Proportional Magnitudes. Therefore he substitutes an other of a more general, and useful form, as firm and true, tho' not so perspicuous. For in Def. 10th. El. V. the Series of Concatenate proportions shews not so evidently the Genesis, and Production of a Compound proportion arising from them, as *Theon's* multiplication of proportions does most naturally, and immediately suggest to our common understanding. And we must confess that *Euclid's* two eminent Definitions of Proportional Magnitudes, and of Compound proportions, lye under the like difficult circumstances, and that both are taken up at the second hand upon meer necessity. Yet are they so admirably contriv'd for general demonstrations in those concerns, that *Euclide* hath no where else given a more manifest testimony of an exquisite judgement, and through insight into all the Mathematics, than in the invention of those two Definitions: which the ancient Geometricians who had searched this whole business to the very bottom saw just reason to receive, and use without exception, or any endeavour to amend them, or deviate from *Euclid's* method in this matter.

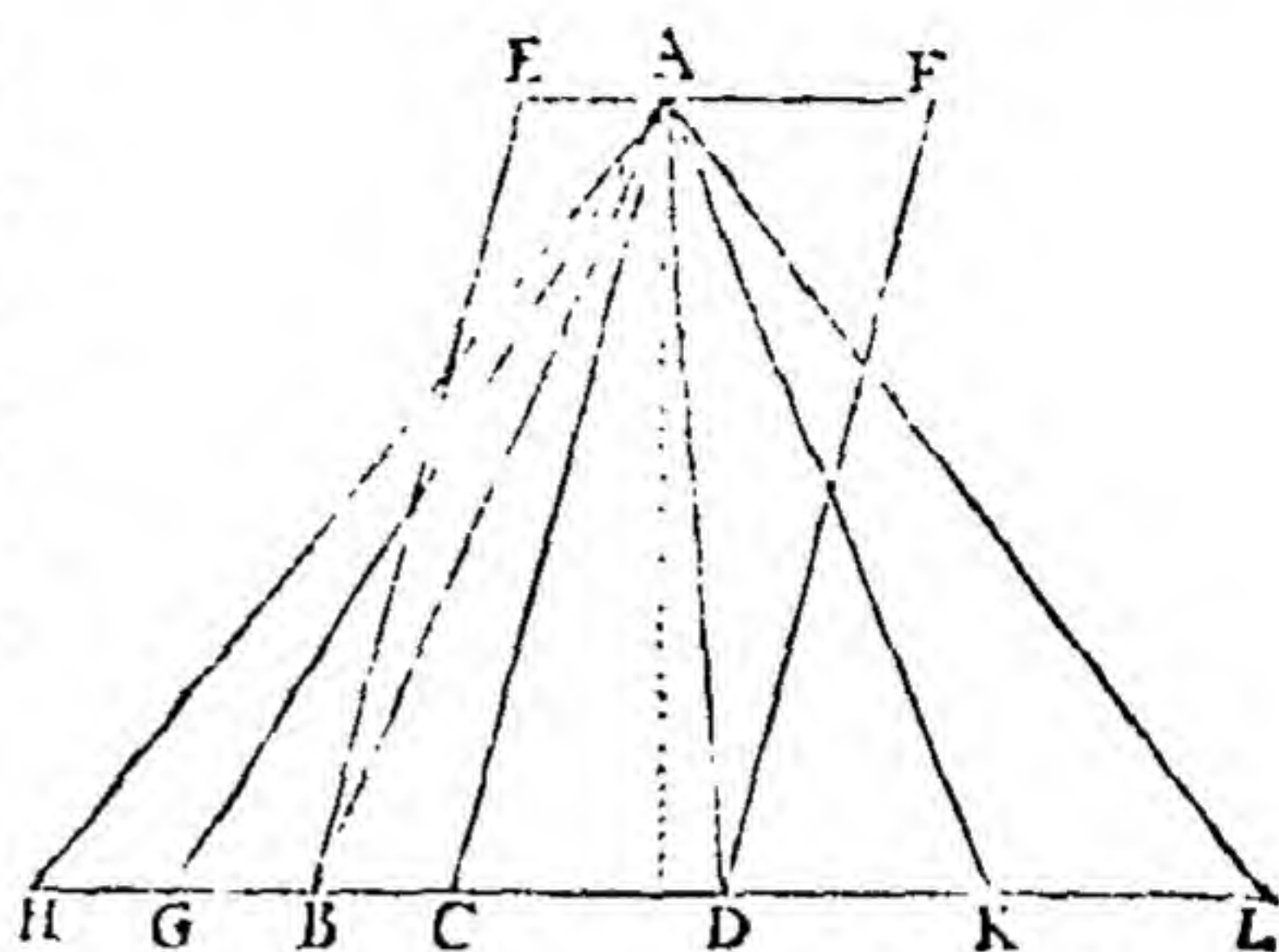
PR O P O -

PROPOSITION I.

Triangles and Parallelograms of the same Altitude, are to one another as their Bases.

Let the Triangles be ABC, ACD , and the Parallelograms EC, CF , having the same altitude, the perpendicular drawn from A to BD . I say, that as the base BC is to the base CD , so the Triangle ABC , is to the Triangle ACD ; and the Parallelogram EC to the Parallelogram CF .

Let BD be produced both ways to the points H, L ; and to the base BC let be put equals how many soever BG, GH . Again, to the base CD , let be put equals how many soever DK, KL ; and let be joyned AG, AH, AK, AL .



Now forasmuch as CB, BG, GH , are equal to one another; therefore the Triangles AHG, AGB, ABC , are equal to one another [Prop. 38. El. I.]; wherefore Quotuple the base HC is of the base BC , Totuple is the Triangle AHC of the Triangle ABC . By the same reason Quotuple the base LC is of the base CD , Totuple is the Triangle ALC of the Triangle ACD .

Now if the base HC be equal to the base CL , the Triangle AHC is also equal to the Triangle ALC [Prop. 38. El. I.]: and therefore if the base HC exceeds the base CL , the Triangle AHC does also exceed the Triangle ALC : And if the base be less, the Triangle is also less [by Corol. Prop. 38. El. I.].

There being then four magnitudes the two bases BC, CD , and the two Triangles ABC, ACD : and of the base BC , and of the Triangle ABC , are taken equimultiples (any whatsoever) the base HC , and the Triangle AHC : also of the base CD , and of the Triangle ACD , are taken other equimultiples (any whatsoever) the base CL , and the Triangle ALC .

And it hath been prov'd that if the base HC exceeds the base CL , the Triangle AHC does exceed the Triangle ALC : and if equal, 'tis equal; and if less, 'tis less. Therefore as the base BC is to the base CD , so the Triangle ABC is to the Triangle ACD [Def. 5. El. V.].

And now because the Parallelogram EC is double of the Triangle ABC [Prop. 41. El. I.], and the Parallelogram CF is double of the Triangle ACD ; and that parts have the same proportion with their equimultiples [Prop. 15. El. V.]; therefore as the Triangle ABC is to the Triangle ACD , so the Parallelogram EC is to the Parallelogram CF .

L1 3

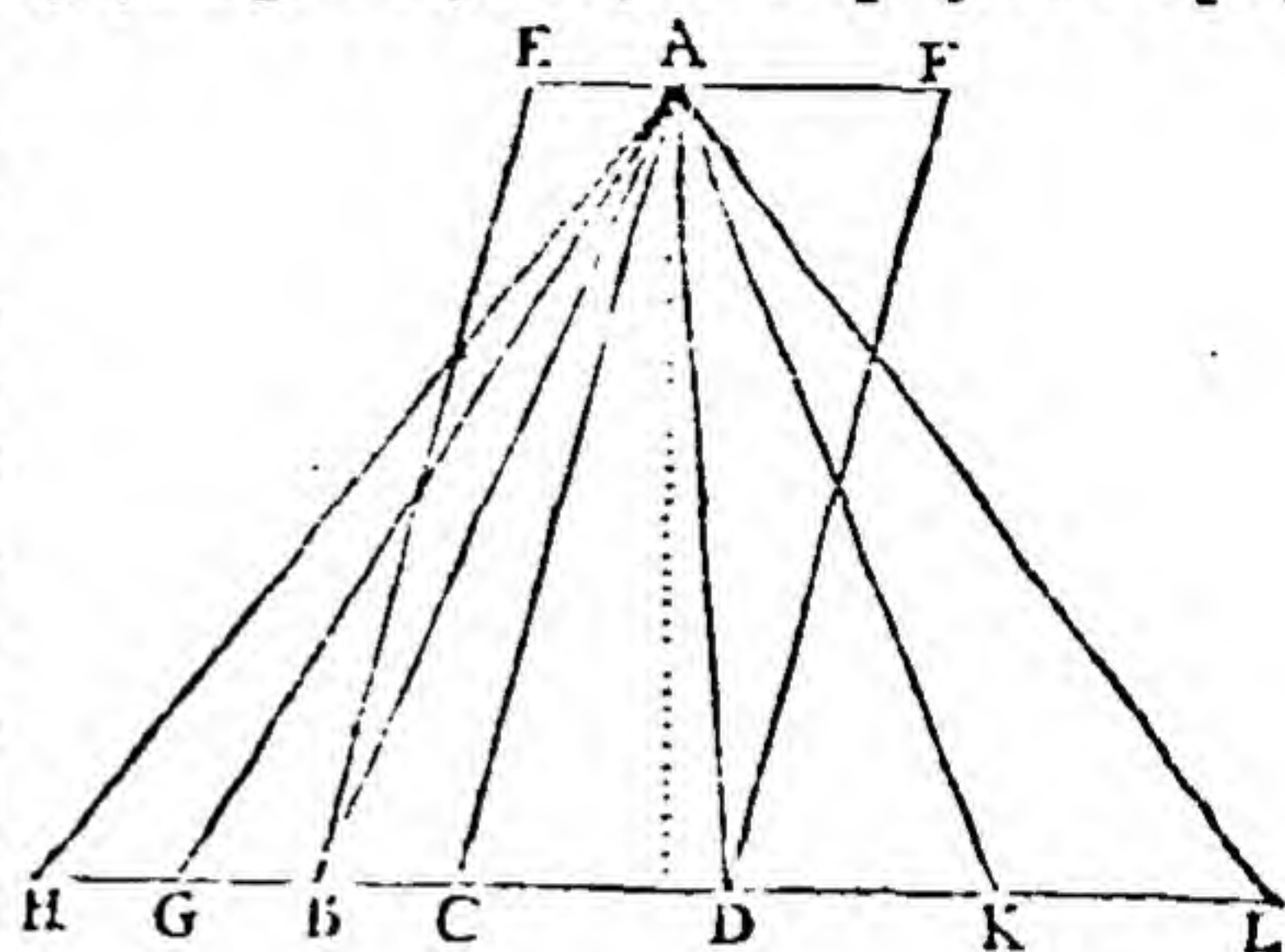
Because

Because therefore it has been prov'd, that as the base BC to the base CD , so the Triangle ABC to the Triangle ACD : And as the Triangle ABC to the Triangle ACD , so the Parallelogram EC to the Parallelogram CF : wherefore also as the base BC to the base CD , so the Parallelogram EC to the Parallelogram CF [Prop. 11. El. El. V.].

Therefore Triangles and Parallelograms of the same altitude are to one another as their bases. Which was to be demonstrated.

ANNOTATIONS.

On this Proposition depends the main part of this Element, as also the whole doctrine of proportions in Magnitudes throughout all Geometry: and by this demonstration it plainly appears, with what facility the 5th. Definition of El. V. is applied to Magnitudes. For what is of more easy Construction than to Duple, or Triple, &c. the base BC ; and then by joyning AG , AH , to Duple, or Triple, &c. the Triangle ABC , that is, the first Term, and the third, the Antecedents. Again, to Duple, or Triple, &c. the base CD , and accordingly the Triangle ACD , that is, the second Term, and the fourth, the Consequents, is the same obvious Construction. Neither is there elsewhere in *Euclide*, or *Archimedes* any greater trouble in the multiplication of magnitudes according to Def. 5th. El. V: And upon such like easy Constructions the demonstrations do as easily proceed, being always readily confirmed by some one single Proposition. As here 'tis evident by Prop. 38th. El. I. that Triangles in the same parallels, (that is to say, of the same altitude) and on equal bases are equal to one another: and therefore on unequal bases are unequal; on the greater base the greater Triangle, on the less, the less. So that if the bases be equal, greater, or less, one than the other; the Triangles likewise are the same in any multiplication whatsoever: therefore as base to base, so Triangle to Triangle, by Def. 5th. El. V. And we see here that there needs no trial of various or perplexed multiplications to prove this Proposition: but that the Analogy between the bases, and Triangles is at once apparent by one only Geometrical Proposition, well known aforehand, viz. Prop. 38th. El. I.



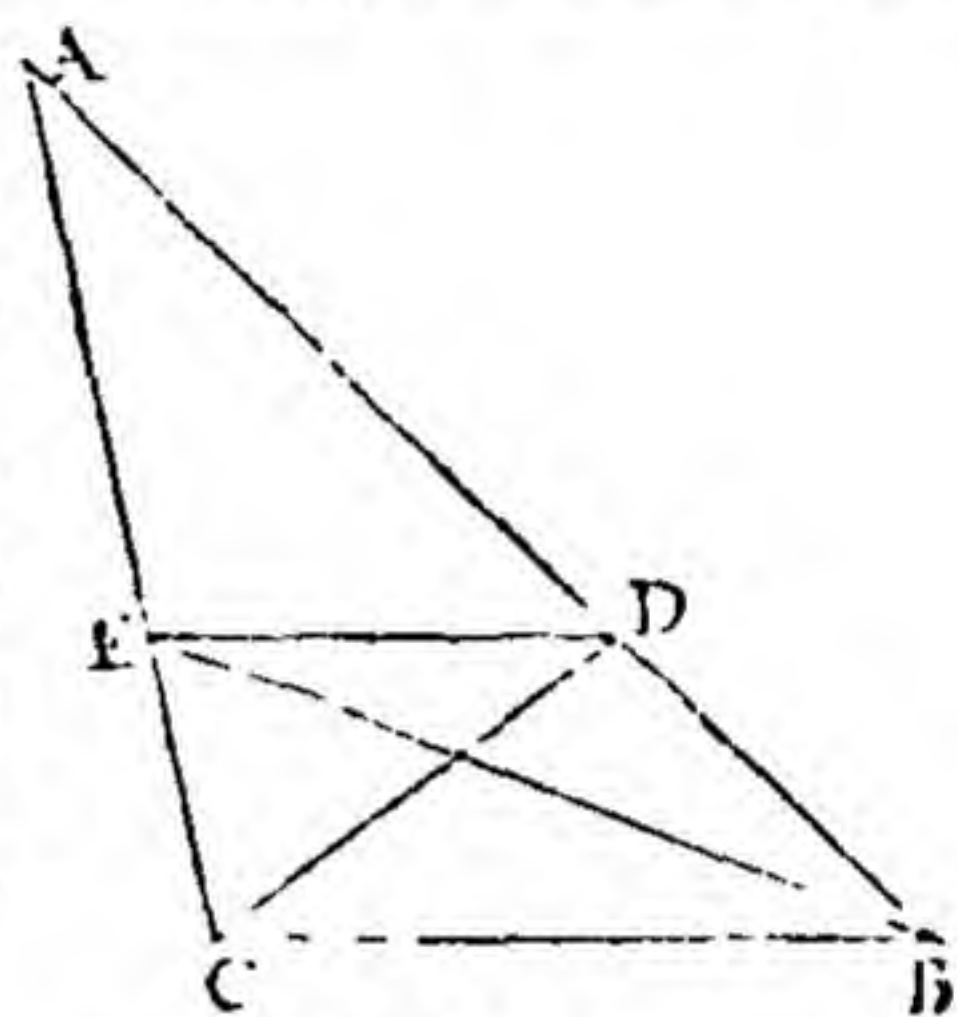
Now *Tacquet's* confidence in this matter is very remarkable, who so boldly prefers his own demonstration, which is encumbered with the divisions of the Consequents into *like aliquot parts*; and these again are to be subtracted from the Antecedents *quoties fieri potest*, &c. All which will undoubtedly seem to his *Mathematical Tyrones* (of whom he pretends to have a great care) a much more troublesome, and perplexed business, than this plain Construction, and Demonstration of *Euclide*, which he unjustly accuses of obscurity and intricacy; whereof he himself is most guilty: as will certainly appear to whoever shall compare both together. But I leave *Tacquet*, and *Borellus* with the rest, to the correction of the Geometrical Professors in our Universities, and *Gresham College*: this subject being too large for Elementary Annotations intended only for the instruction of Younger Students.

PROPOSITION II.

IF to one side of a Triangle a strait line be drawn parallel; it shall cut the sides of the Triangle proportionally.

And if the sides of a Triangle be cut proportionally, the strait line joyning the Sections, shall be parallel to the remaining side of the Triangle.

For in the Triangle ABC to one of the sides BC , let DE be drawn parallel. I say, that as BD is to DA , so CE is to EA . Let be joyned BE , CD : therefore the Triangle BDE , is equal to the Triangle CDE [Prop. 37. El. I.] For they are on the same base DE , and within the same parallels DE , BC . But moreover there is an other Triangle ADE : and because equals have to the same thing the same proportion [Prop. 7. El. V.]; therefore as the triangle BDE , is to the Triangle ADE , so the Triangle CDE is to the Triangle ADE : But as the Triangle BDE is to the Triangle ADE , so BD is to DA . For having the same Altitude, the perpendicular drawn from E to AB , they are to one another as their bases [Prop. 1. El. VI.]. And by the same reason, as the Triangle CDE is to the Triangle ADE , so CE is to EA ; therefore as BD is to DA , so CE is to EA [Prop. 11. El. V.].



But now of the Triangle ABC , let the sides AB , AC be cut proportionally in the points D , E , that as BD to DA , so CE to EA : and let be joyned DE . I say, that DE is parallel to BC . For the same Construction being made, because as BD is to DA , so CE is to EA : and as BD is to DA , so the Triangle BDE is to the Triangle ADE ; and as CE is to EA , so the Triangle CDE is to the Triangle ADE . Therefore as the Triangle BDE is to the Triangle ADE , so the Triangle CDE is to the Triangle ADE [Prop. 11. El. V.]: wherefore each of the Triangles BDE , CDE , have the same proportion to ADE ; therefore the Triangle BDE is equal to the Triangle CDE [Prop. 9. El. V.]. And they are on the same base DE ; but equal Triangles and on the same base, are within the same parallels [Prop. 40. El. I.]; therefore DE is parallel to BC .

If therefore to one side of a Triangle a strait line be drawn parallel; it shall cut the sides of the Triangle proportionally.

And if the sides of a Triangle be cut proportionally, the strait line joyning the Sections, shall be parallel to the remaining side of the Triangle. Which was to be demonstrated.

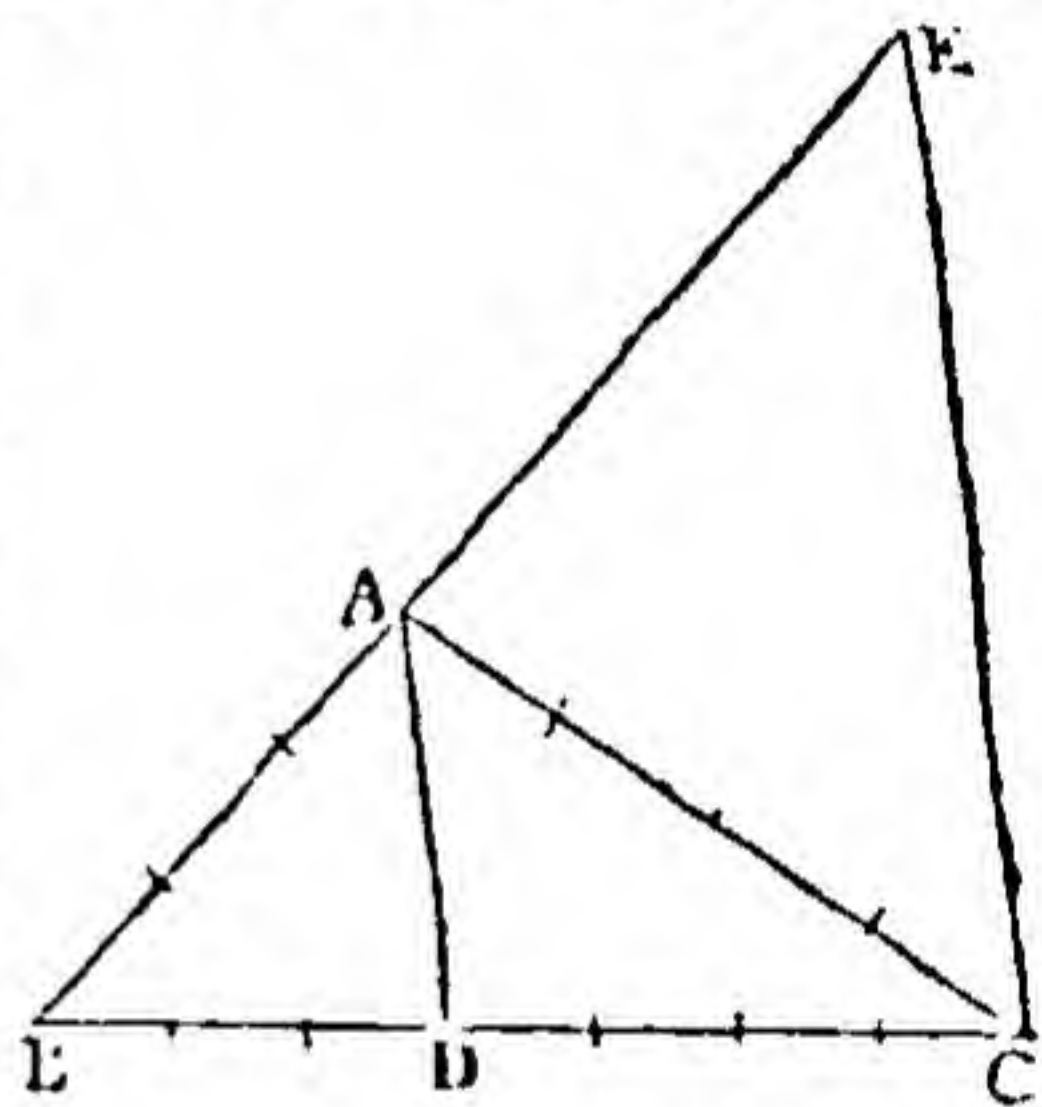
PROPO

PROPOSITION III.

IF an angle of Triangle be cut into halves, and the strait line cutting the angle does also cut the base: the Segments of the base shall have the same proportion with the remaining sides of the Triangle.

And if the Segments of the base have the same proportion with the remaining sides of the Triangle, the strait line drawn from the Vertex to the Section, does cut the angle of the triangle into halves.

Let the Triangle be ABC , and let the angle BAC be cut into halves by the line AD [by Prop. 9. El. I.] I say, that as BD is to DC , so BA is to AC . For by the point c let be drawn CE parallel to DA [Prop. 31. El. I.]; and let BA produced meet with the same in the point E . Now forasmuch as the strait line AC falls upon the parallels AD, EC ; therefore the angle ACE is equal to the alternate angle CAD [Prop. 29. El. I.]: but the angle CAD is put equal to the angle BAD ; therefore the angle BAD is also equal to the angle ACE . Again, because the strait line BAE falls upon the parallels AD, EC ; therefore the outward angle BAD is equal to the inward angle AEC . And it has been prov'd, that the angle ACE is also equal to the angle BAD ; therefore the angle ACE is also equal to the angle AEC , so that also the side AE is equal to the side AC [Prop. 6. El. I.]. And because to one side of the Triangle BCE , namely to EC , is drawn parallel AD ; therefore proportionally as BD to DC , so BA to AE [Prop. 2. El. VI.]. But AE is equal to AC ; therefore as BD to DC , so BA to AC [Prop. 7. El. V.].



But now let it be as BD to DC , so BA to AC : and let be joyned AD . I say, that the angle BAC is cut into halves by the strait line AD . For the same Construction being made, because it is as BD to DC , so BA to AC : and as BD is to DC , so BA is to AE : for to one side of the Triangle BCE , namely to CE , is drawn AD parallel; therefore as BA is to AC , so BA is to AE : therefore AC is equal to AE [Prop. 9. El. V.]; so that also the angle AEC , is equal to the angle ACE [Prop. 5. El. I.]. But the angle AEC is equal to the outward angle BAD [Prop. 29. El. I.]; and the angle ACE is equal to the alternate angle CAD ; wherefore the angle BAD is equal to the angle CAD ; therefore the angle BAC is cut into halves by the strait line AD .

If therefore an angle of a Triangle be cut into halves, and the strait line cutting the angle does also cut the base: the Segments of the base shall have the same proportion with the remaining sides of the Triangle.

And

And if the Segments of the base have the same proportion with the remaining sides, &c. Which was to be demonstrated.

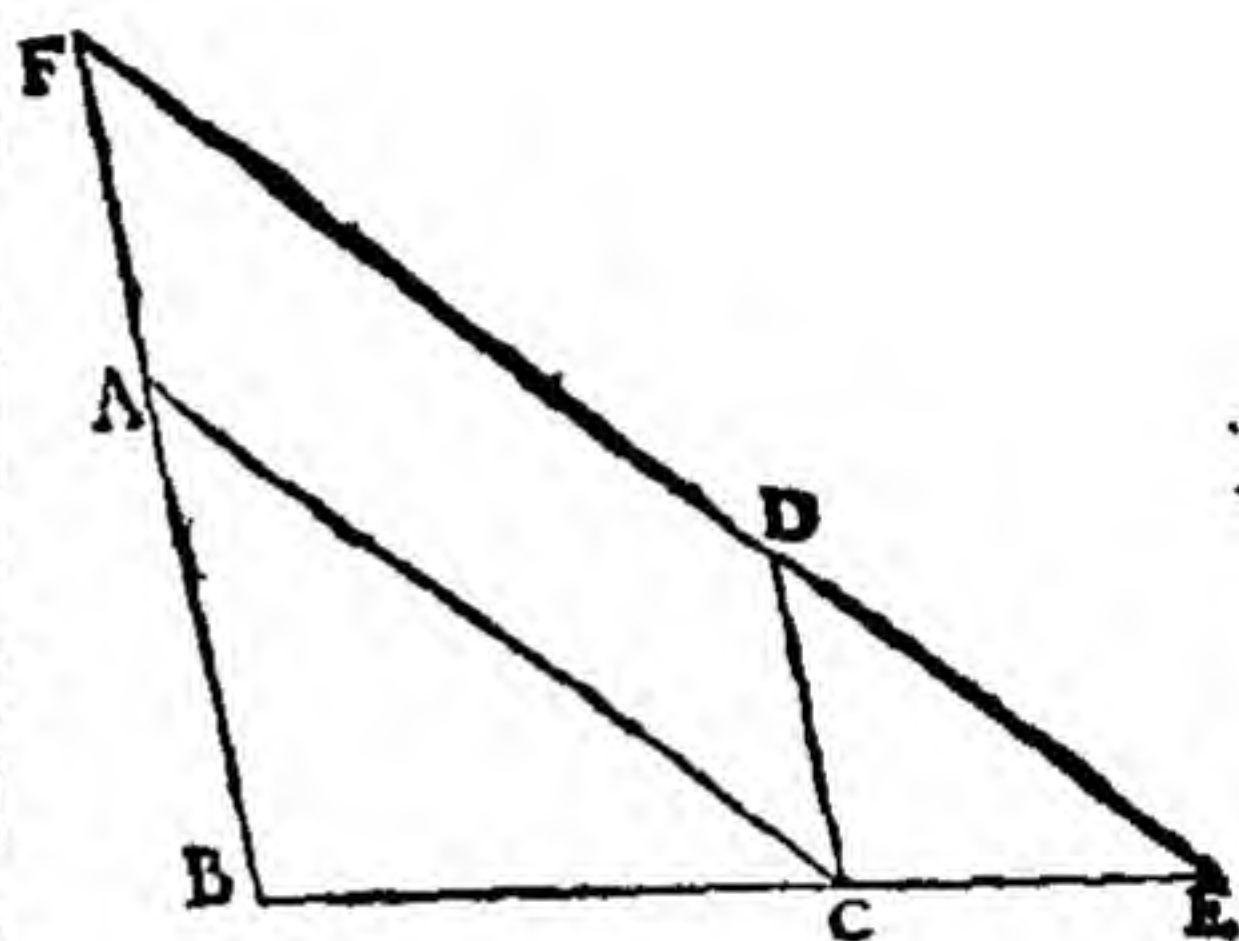
PROPOSITION IV.

Of equiangled Triangles the sides about the equal angles are proportional: And the sides subtended under the equal angles are Homologal.

Let the equiangled Triangles be ABC , DCE , having the angle ABC equal to the angle DCE , and the angle ACB to the angle DEC , and moreover the angle BAC to the angle CDE .

I say, that of the Triangles ABC , DCE , the sides about the equal angles are proportional, and the sides subtended under the equal angles are Homologal.

For let BC be put directly to CE , and because the angles ABC , ACB , are less than two right angles [Prop. 17. El. I.], and the angle ACB is [by Supposition] equal to the angle DEC ; therefore the angles ABC , DEC , are less than two right angles: wherefore BA , ED produced shall meet [by Postulate 3.]. Let them be produced, and meet in the point F .



Now forasmuch as the angle DCE is equal to the angle ABC [by Supposition]; therefore BF is parallel to CD [Prop. 28. El. I.]. (For that the outward angle DCE is equal to the inward and opposite ABC .) Again, because the angle ACB is equal to the angle DEC [by Supposition]; therefore AC is parallel to FE [Prop. 28. El. I.]. (For that the outward angle ACB is equal to the inward and opposite DEC .) Wherefore $FACD$ is a Parallelogram; therefore AF is equal to CD , and AC to FD [Prop. 34. El. I.]. And now because of the Triangle FBE , to one of the sides FE , the line AC is parallel; therefore it is as BA to AF , so BC to CE [Prop. 2. El. VI.]. But AF is equal to CD ; as therefore BA to CD , so BC to CE [Prop. 7. El. V.]; and alternately, as AB to BC , so DC to CE .

Again, because of the same Triangle FBE , to the side BF , the line CD is parallel; therefore it is, as BC to CE , so FD to DE [Prop. 2. El. VI.]. But FD is equal to CA ; therefore as BC to CE , so CA to ED : Alternately therefore as BC to CA , so CE to ED .

And forasmuch as it has been proved, that as AB to BC , so DC to CE , and as BC to CA , so CE to ED : therefore by equality, as BA to AC , so CD to DE .

Therefore of equiangled Triangles the sides about the equal angles are proportional: And the sides subtended under the equal angles are Homologal. Which was to be demonstrated.

Mm

“ Euclide

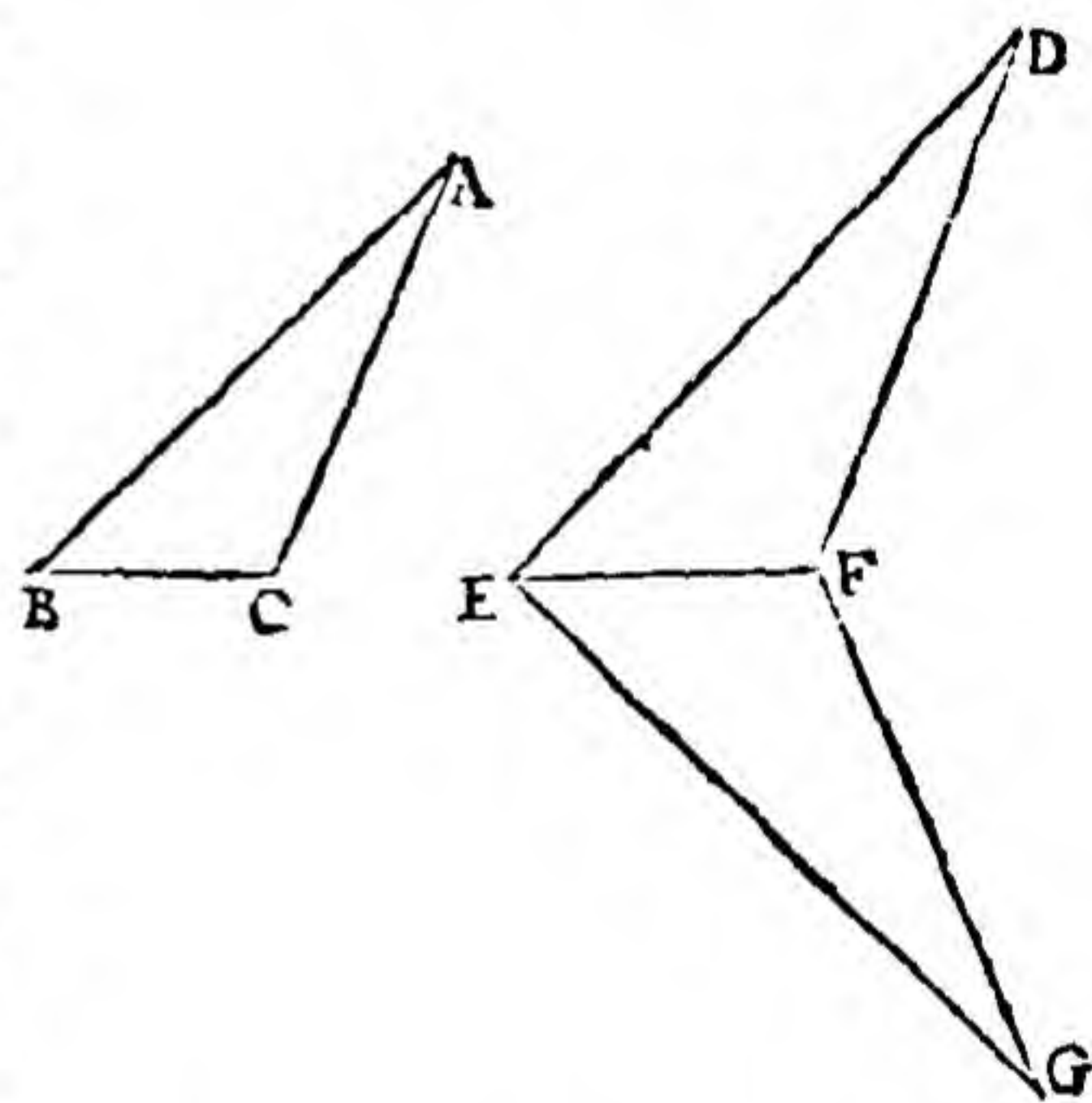
“*Euclide* here takes no notice of the Homologal sides, because
 “it manifestly appears in the course of the demonstration, that those
 “sides which are subtended under the equal angles are Homologal,
 “and correspondent Antecedents and Consequents.

PROPOSITION V.

If two Triangles have their sides proportional, the Triangles
 shall be equiangled: And shall have those angles equal under
 which are subtended Homologal sides.

Let the two Triangles be ABC , DEF , having their sides proportional, as AB to BC , so DE to EF : and as BC to CA , so EF to FD : and moreover, as BA to AC , so ED to DF . I say, that the Triangle ABC is equiangled to the Triangle DEF : And they shall have those angles equal under which are subtended Homologal sides; namely the angle ABC to the angle DEF , and BCA to EFD , and moreover BAC to EDF .

For to the strait line EF , and to the points in the same $E.F$, let be constituted the angle FEG , equal to the angle CBA , and the angle GEF , equal to the angle BCA [by Prop. 23. El. I.]; therefore the remaining angle BAC is equal to the remaining angle EGF : wherefore the Triangle ABC is equiangled to the Triangle GEF ; therefore of the Triangles ABC , GEF , the sides about the equal angles



are proportional, and the sides subtending the equal angles are Homologal: wherefore as AB is to BC , so GE is to EF . But as AB to BC , so by Supposition, DE is to EF ; therefore as DE to EF , so GE to EF : wherefore each of the lines DE , GE , have the same proportion to EF ; and therefore DE is equal to GE [Prop. 9. El. V.].

By the same reason, DF is equal to GF . Now forasmuch as DE is equal to EG , and EF common: therefore there are the two lines DE , EF , equal to the two lines GE , EF ; and the base DF is equal to the base GF ; therefore the angle DEF , is equal to the angle GEF , and the Triangle DEF , is equal to the Triangle GEF , and the remaining angles equal to the remaining angles under which are subtended equal sides [Prop. 8. El. I.]; therefore also the angle DFE is equal to the angle GFE , and EDF to EGF . And because the angle DEF is equal to the angle GEF , and the angle GEF to the angle ABC ; therefore the angle ABC is equal to the angle DEF . By the same reason, the angle ACB is equal to the angle DFE , and also the angle

angle at A , is equal to the angle at D : wherefore the Triangle ABC is equiangular to the Triangle DEF .

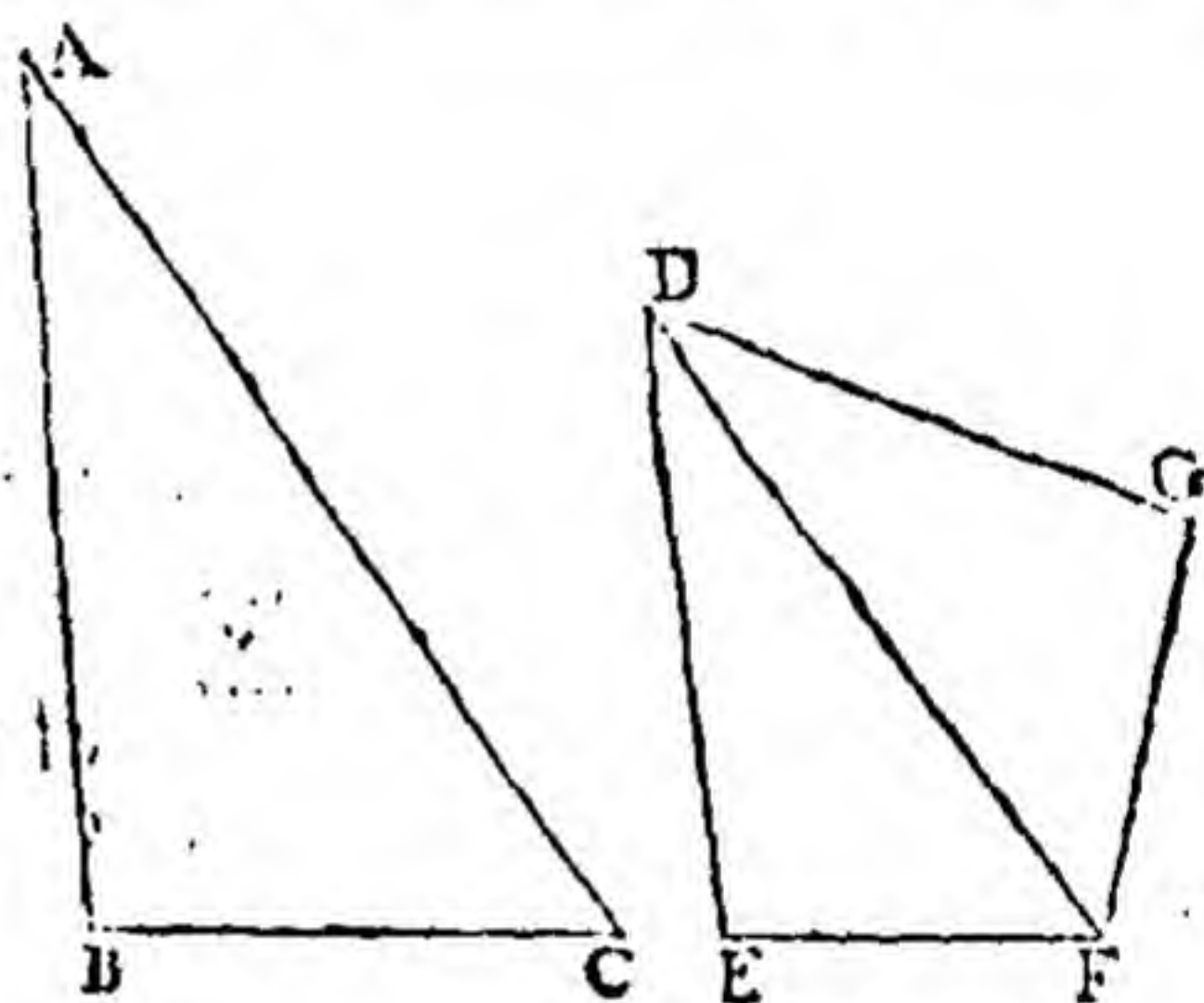
If therefore two Triangles have their sides proportional, the Triangles shall be equiangular: And shall have those angles equal under which are subtended Homologal sides. Which was to be demonstrated.

PROPOSITION VI.

If two Triangles have one angle equal to one angle, and about the equal angles the sides proportional; the Triangles shall be equiangular, and shall have those angles equal, under which are subtended Homologal sides.

Let the two Triangles be ABC , DEF , having one angle BAC equal to one angle EDF , and about the equal angles the sides proportional, as BA to AC , so ED to DF . I say, that the Triangle ABC is equiangular to the triangle DEF : And they shall have the angle ABC equal to the angle DEF , and the angle ACB to the angle DFE .

For to the strait line DF , and to the points in the same D , F , let be constituted the angle FDG , equal to either of the angles BAC , or EDF [Prop. 23. El. I.]; and the angle DFG equal to the angle ACB ; therefore the remaining angle at B , is equal to the remaining angle at G : wherefore the triangle ABC is equiangular to the triangle DGF : it is therefore proportional as BA to AC , so GD to DF . But by supposition as BA to AC , so ED to DF ; and therefore as ED to DF , so GD to DF : wherefore ED is equal to GD ; and DF common.



Therefore there are the two lines ED , DF , equal to the two lines GD , DF ; and the angle EDF is equal to the angle GDF ; wherefore the base EF is equal to the base GF : and the triangle DEF equal to the triangle DGF ; and the remaining angles shall be equal to the remaining angles, each to each, under which are subtended equal sides [Prop. 4. El. I.]; therefore the angle DFG is equal to the angle DFE , and the angle at G to the angle at E . But the angle DFG is equal to the angle ACB ; therefore the angle ACB is equal to the angle DFE . But the angle BAC is put equal to the angle EDF ; therefore also the remaining angle at B , is equal to the remaining angle at E : wherefore the triangle ABC is equiangular to the triangle DEF .

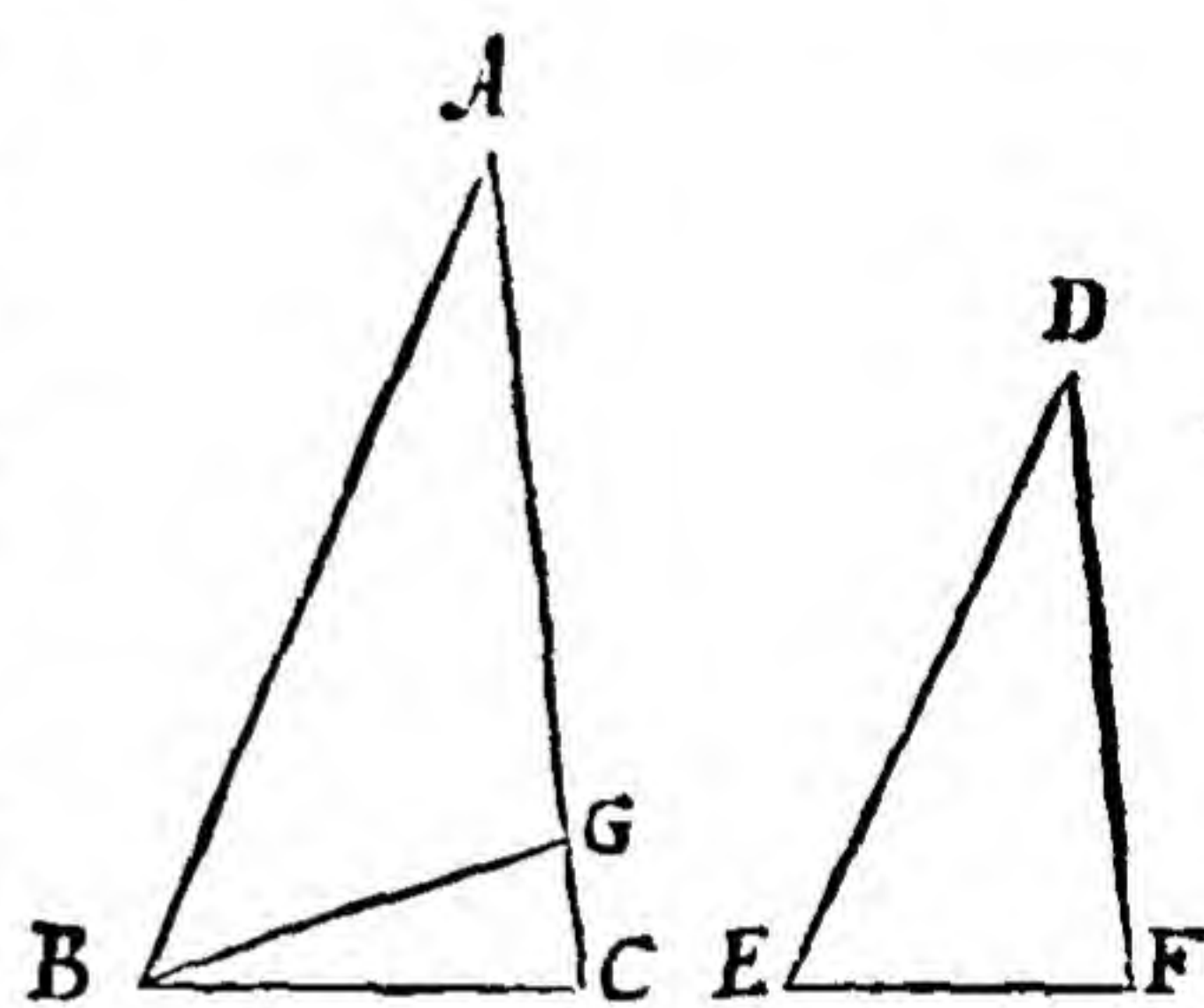
If therefore two triangles have one angle equal to one angle, and about the equal angles the sides proportional; the triangles shall be equiangular, and shall have those angles equal, under which are subtended Homologal sides. Which was to be demonstrated.

PROPOSITION VII

I*f two triangles have one angle equal to one angle, and about other angles the sides proportional; and have also each of the remaining angles either less, or not less than a right; the triangles shall be equiangled, and shall have those angles equal, about which are the proportional sides.*

Let the two triangles be ABC , DEF , having one angle equal to one angle, the angle BAC to the angle EDF ; and about other angles ABC , DEF , the sides proportional, as AB to BC , so DE to EF ; having also each of the remaining angles at C , F , first less than a right angle.

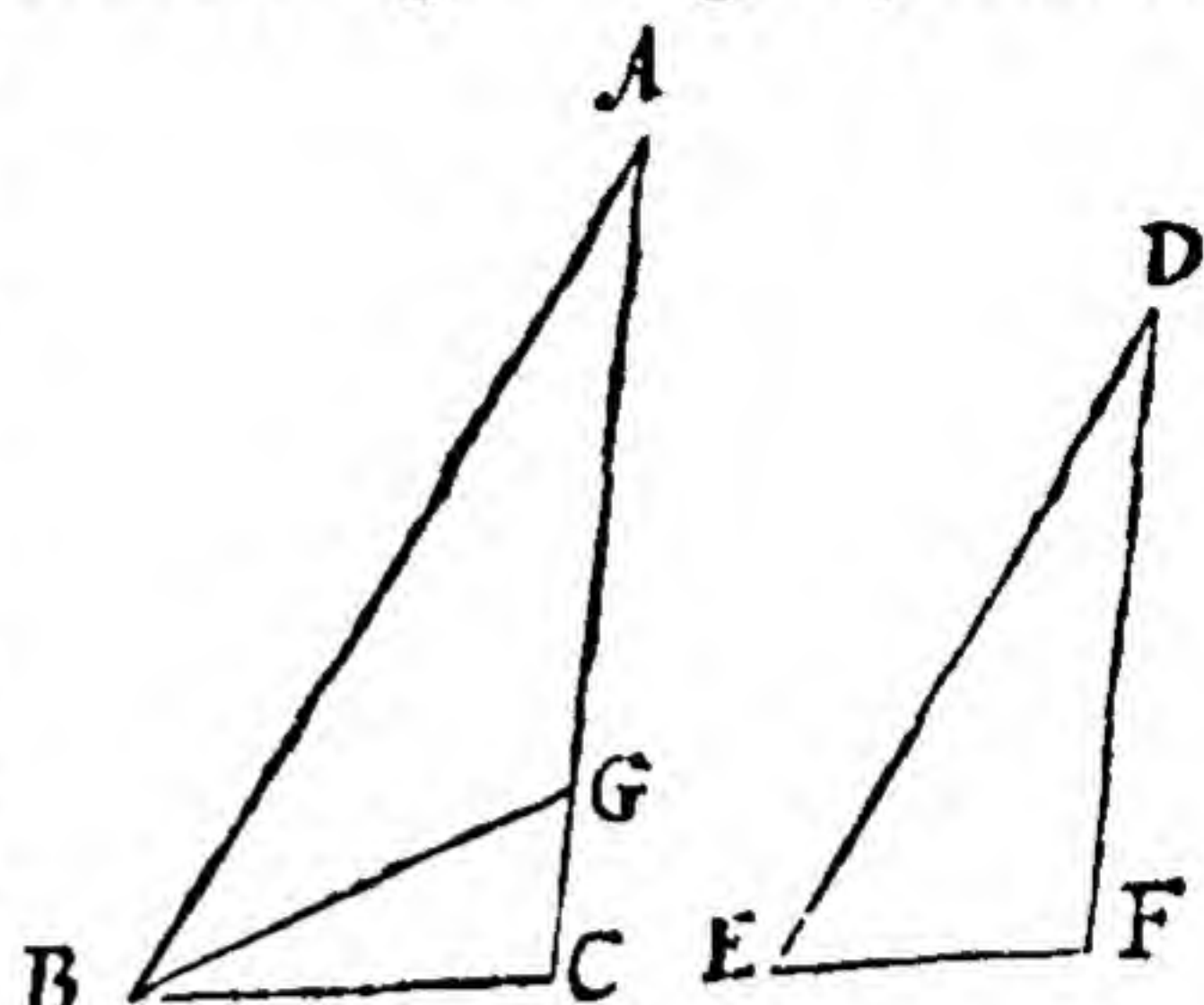
I say, that the triangle ABC is equiangled to the triangle DEF : and the angle ABC shall be equal to the angle DEF , and the remaining angle, to wit, at C , equal to the remaining angle at F . For if the angle ABC be unequal to the angle DEF , one of them is the greater. Let ABC be the greater; and to the strait line



AB , and to the point in the same B , let be constituted the angle ABG equal to the angle DEF [by Prop. 23. El. I.]. Now forasmuch as the angle A is equal to the angle D , and the angle ABG to the angle DEF ; therefore the remaining angle AGB is equal to the remaining angle DFE . Therefore the triangle ABG is equiangled to the triangle DEF : wherefore as AB to BG , so DE to EF [Prop. 4. El. VI.] But as DE to EF , so by Supposition is AB to BC ; and therefore as AB to BC , so AB to BG : wherefore AB has the same proportion both to BC and BG ; therefore BC is equal to BG [Prop. 9. El. V.]: so that also the angle BGC is equal to the angle BCG [Prop. 5. El. I.] But the angle at C is put less than a right angle; therefore also the angle BGC is less than a right angle: so that the consequent angle AGB is greater than a right angle: and it hath been proved equal to the angle at F ; therefore the angle at F is greater than a right angle. But it is put less than a right angle: which is absurd: wherefore the angle ABC is not unequal to the angle DEF : equal therefore it is. Now also the angle at A is equal to the angle at D [by Supposition], and therefore the remaining angle at C , is equal to the remaining angle at F : wherefore the triangle ABC is equiangled to the triangle DEF .

But

But again, let each of the angles at c, F be put not less than a right angle. I say, that in this case also the triangle ABC is equiangular to the triangle DEF . For the same Construction being made, we may in like manner demonstrate, that BC is equal to BG ; so that the angle at c is equal to the angle BGC . But the angle at c is put not less than a right angle; therefore BGC is not less than a right angle: wherefore of the triangle BGC there are two angles, and they not less than two right angles, which is impossible [Prop. 32. El. I.]; therefore again the angle ABC is not unequal to the angle DEF , equal therefore it is. Now also the angle at A is equal to the angle at D [by Supposition]: wherefore the remaining angle at c is equal to the remaining angle at F ; therefore the triangle ABC is equiangular to the triangle DEF .

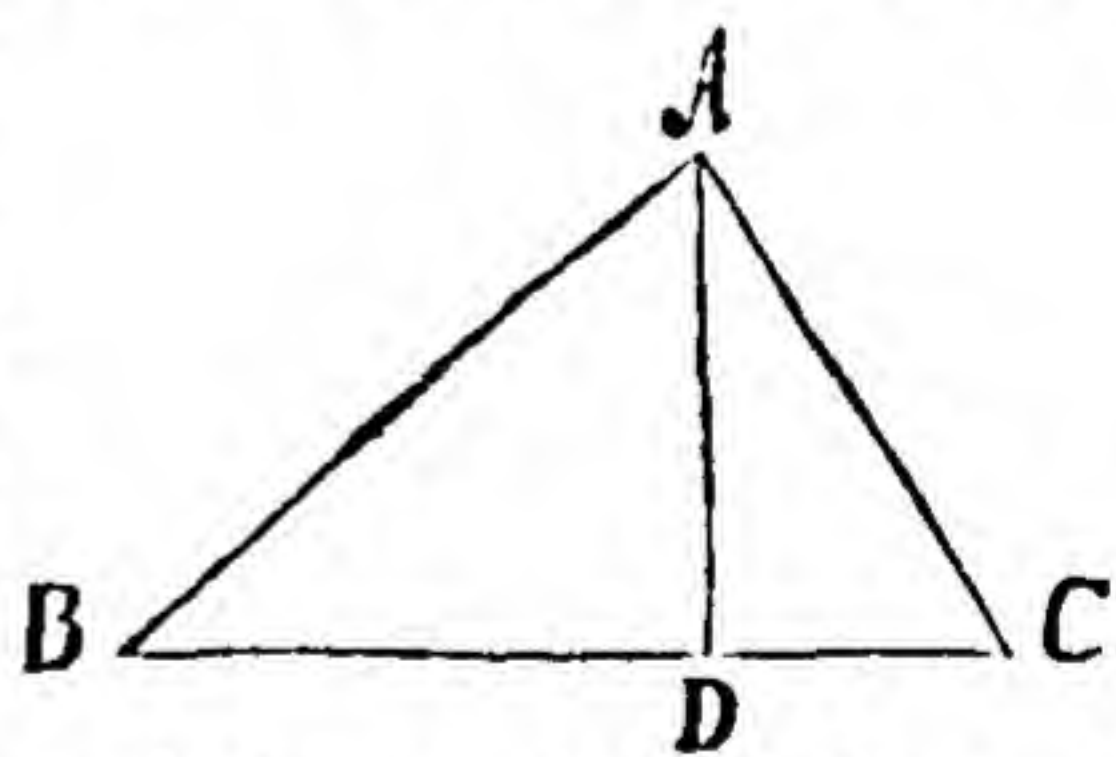


If therefore two triangles have one angle equal to one angle, and about other angles the sides proportional; and each of the remaining angles either less, or not less than a right angle; the triangles shall be equiangular, and shall have those angles equal, about which are the proportional sides. Which was to be demonstrated.

PROPOSITION VIII.

In a right angled triangle from the right angle be drawn a perpendicular to the base, the triangles at the perpendicular are like to the whole, and to one another.

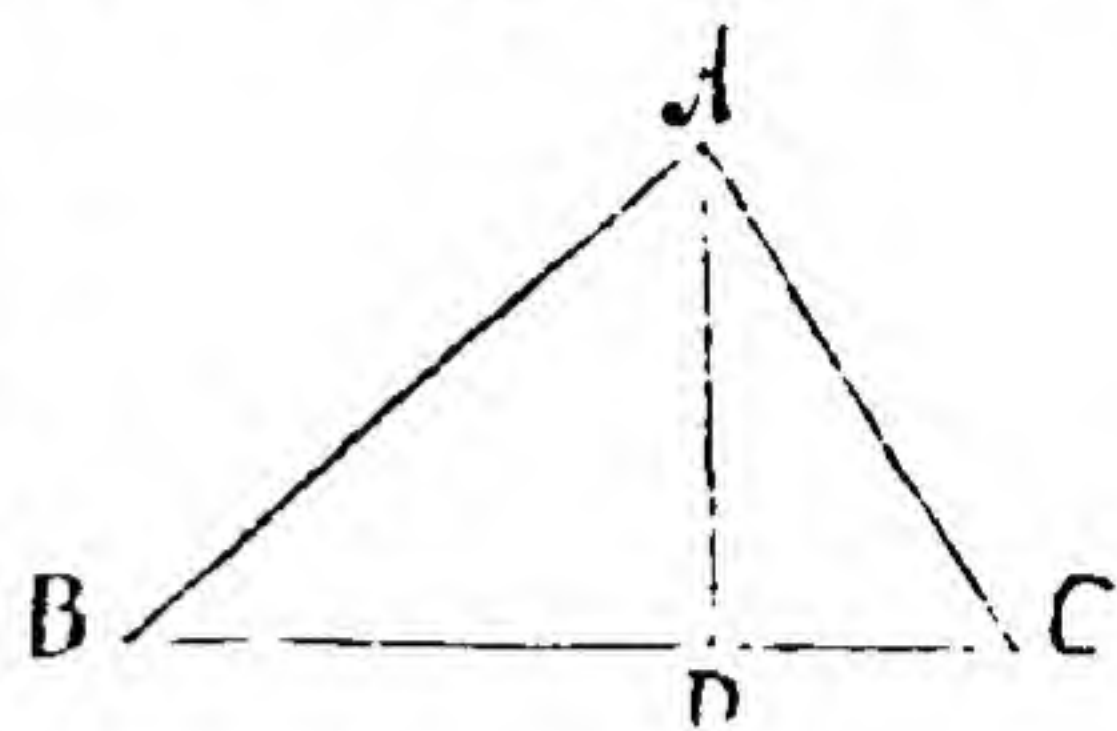
Let the right angled triangle be ABC , having the right angle BAC , and from the point A to the line BC , let AD be drawn perpendicular. I say, that each of the triangles ABD , ADC , is like to the whole triangle ABC , and also to one another. Forasmuch as the angle BAC is equal to the angle ADB , for each is a right angle; and the angle at B common to the two triangles ABC , ABD ; therefore the remaining angle ACB is equal to the remaining angle BAD ; therefore the triangle ABC is equiangular to the triangle ABD : wherefore as BC subtending the right angle of the triangle ABC , is to BA subtending the right angle of the triangle ABD , so the same AB subtending the angle at c of the triangle ABC , is to BD subtending the angle BAD of the triangle ABD , equal to the angle at c : and also as AC to AD subtending the angle at B , common to the two triangles: wherefore the triangle ABC is equiangular to the triangle ABD , and has the sides about the equal angles proportional.



tional; therefore the triangle ABC is like to the triangle ABD [Def. 1. El. VI.]. In like manner we may demonstrate, that also the triangle ADC is like to the triangle ABC : wherefore each of the triangles ABD , ADC , is like to the whole triangle ABC .

I say moreover, that the triangles ABD , ADC , are also like to one another. For because the right angle BDA is equal to the right angle ADC ; and also the angle BAD has been proved equal to the angle at c ; therefore the remaining angle at B is equal to the remaining angle DAC : wherefore the tri-

angle ABD is equiangled to the triangle ADC ; therefore as BD of the triangle ABD , subtending the angle BAD , is to DA of the triangle ADC , subtending the angle at c



equal to the angle BAD , so the same AD , of the triangle ABD , subtending the angle at B , is to DC subtending the angle DAC , of the triangle ADC , equal to the angle at B : and also BA subtending the right angle ADB to AC , subtending the right angle ADC : therefore the triangle ABD is like to the triangle ADC .

If therefore in a right angled triangle, from the right angle be drawn a perpendicular to the base; the triangles at the perpendicular are like to the whole, and to one another. Which was to be demonstrated.

Corollary.

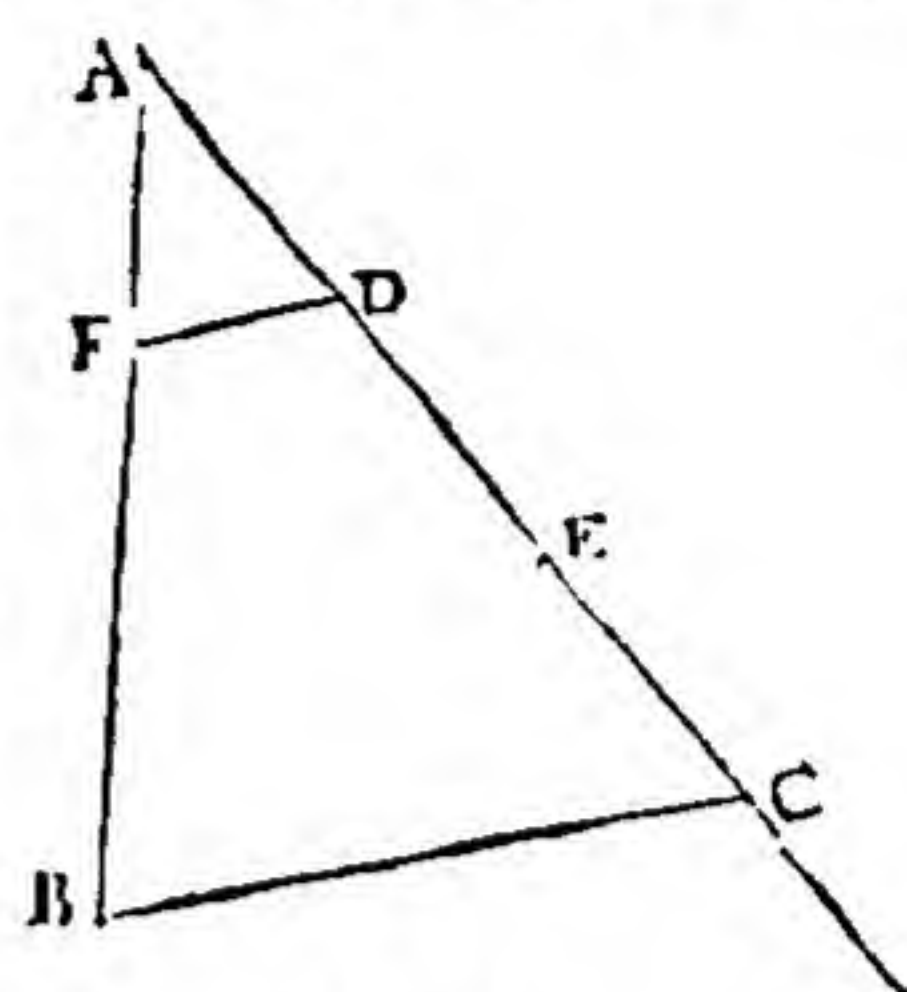
From hence 'tis manifest, that in a right angled triangle the perpendicular drawn from the right angle to the base, is a mean proportional between the Segments of the base. And moreover between the base and either one of the Segments, the side adjoined to that Segment is a mean proportional.

PROPO.

PROPOSITION IX.

From a given strait line to take off a demanded part.

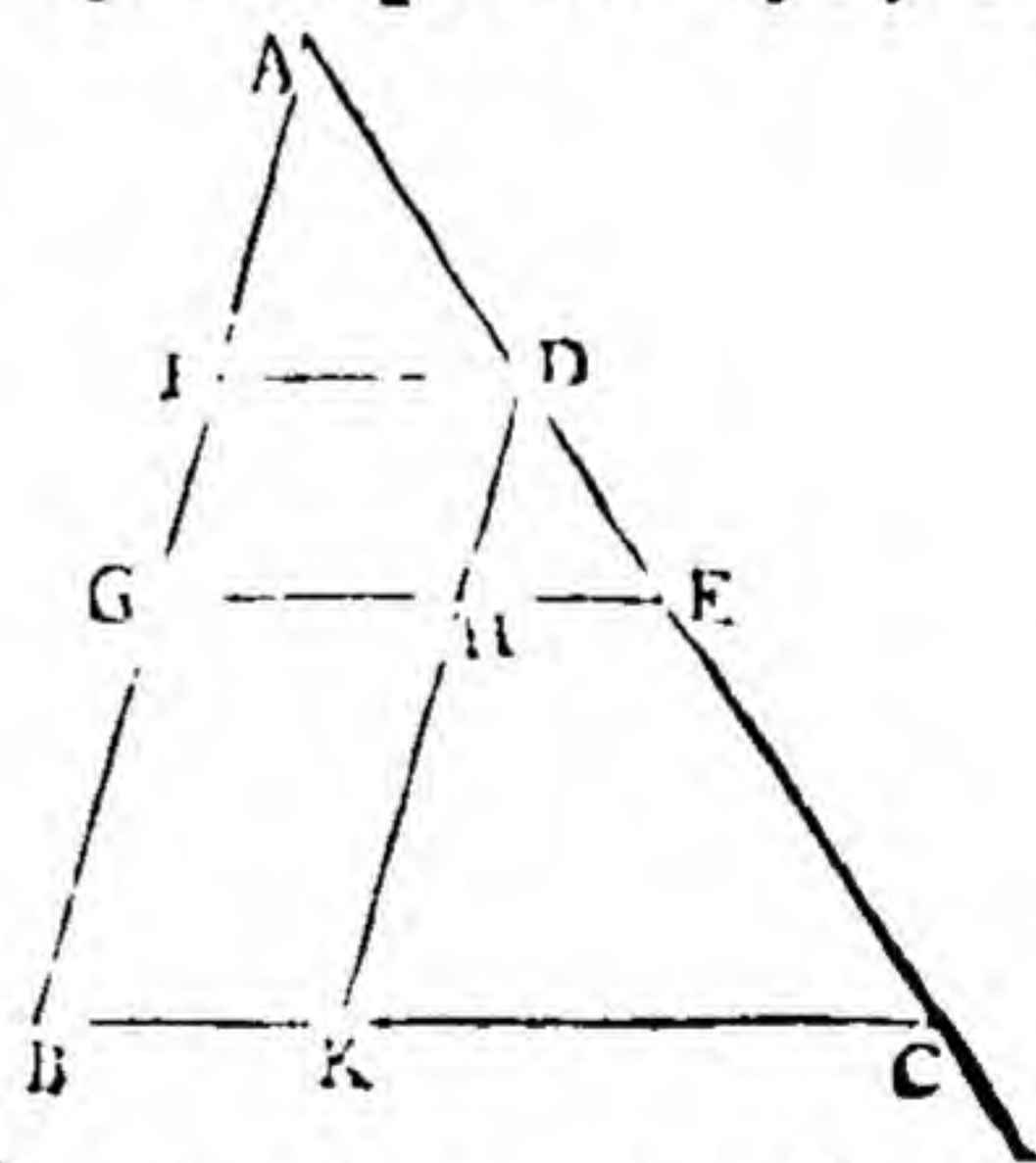
Let the given strait line be AB . It is required from AB to take off a demanded part. Let a third part be demanded; and from the point A let any strait line as AC be drawn, containing with AB any angle. And in AC let any point be taken as D : and to AD let DE, EC be put equal, and let be joyned BC : then by D let be drawn DF parallel to BC . Now forasmuch as to one side of the triangle ABC , namely to BC , there is drawn a parallel DF ; it is therefore proportionally, as CD to DA , so BF to FA [Prop. 2. El. VI.]. But CD is double of DA ; therefore also BF is double of FA : wherefore BA is Triple of AF . Therefore from the given strait line AB is taken off AF , a third part demanded. Which was to be done.



PROPOSITION X.

TO cut a given uncut strait line like to a given cut strait line.

Let the given uncut strait line be AB , and the cut line be AC . It is required to cut the uncut line AB like to the cut line AC . Let AC be cut in the points D, E ; and let AB, AC , be so put as to contain any angle, and let be joyned BC . Then by the points D, E , to BC let DF, EG , be drawn parallels, and by D to AB let DHK be drawn parallel; therefore each of the figures FH, HB , is a Parallelogram: wherefore DH is equal to FG , and HK to GB [Prop. 34. El. I.]. Now forasmuch as of the triangle DKC to one of the sides KC , the line HE is drawn parallel; it is therefore proportionally as CE to ED , so KH to HD [Prop. 2. El. VI.]. But KH is equal to BG , and HD to GF ; therefore as CE to ED , so BG to GF . Again, because of the triangle AGE , to one of the sides EG , the line FD is drawn parallel; it is therefore proportionally as ED to DA , so GF to FA . But it has been proved, as CE to ED , so BG to GF ; therefore as CE to ED , so BG is to GF , and as ED is to DA , so GF is to FA .



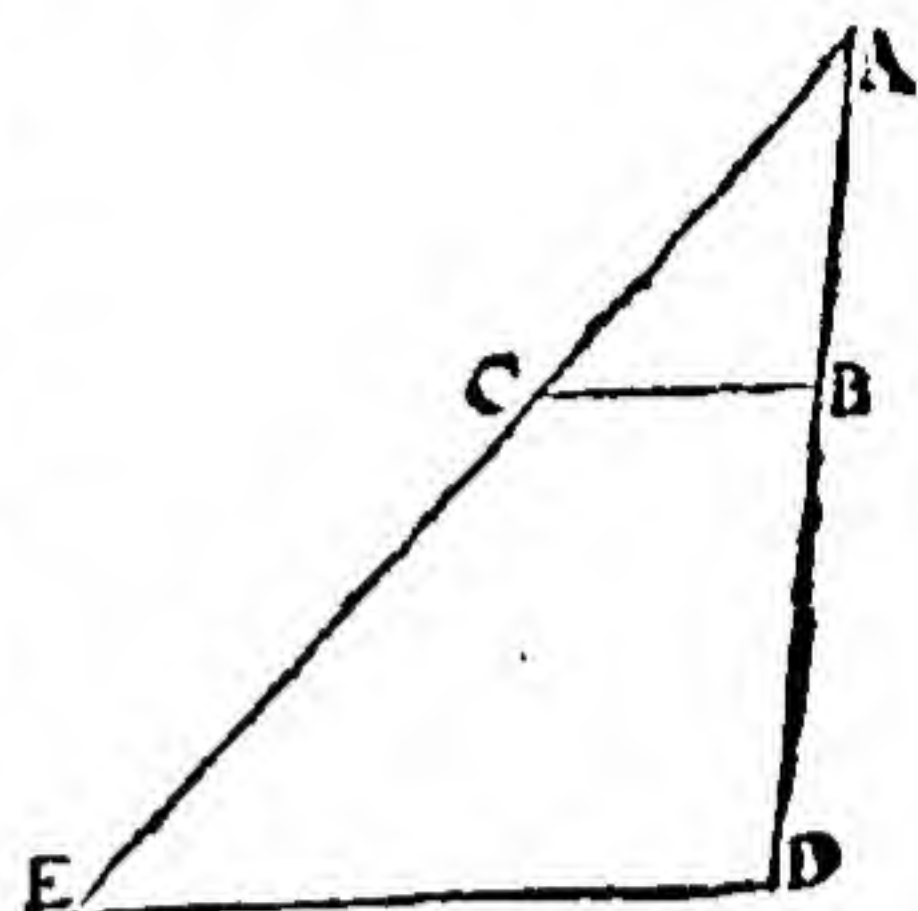
Wherefore the given uncut strait line AB , is cut like to the given cut strait line AC . Which was to be done.

PROPO-

PROPOSITION XI.

Two straight lines being given to find a third proportional.

Let the given straight lines be AB , AC ; and be they so put as to contain any angle. It is required unto AB , AC to find a third proportional. Let AB , AC be produced to the points D , E , and let BD be put equal to AC , and BC be joyned: then by D let DE be drawn parallel to BC . Now forasmuch as of the triangle ADE to one of the sides DE , the line BC is drawn parallel, it is proportionally as AB to BD , so AC to CE ; but BD is equal to AC ; therefore as BA is to AC , so AC is to CE .

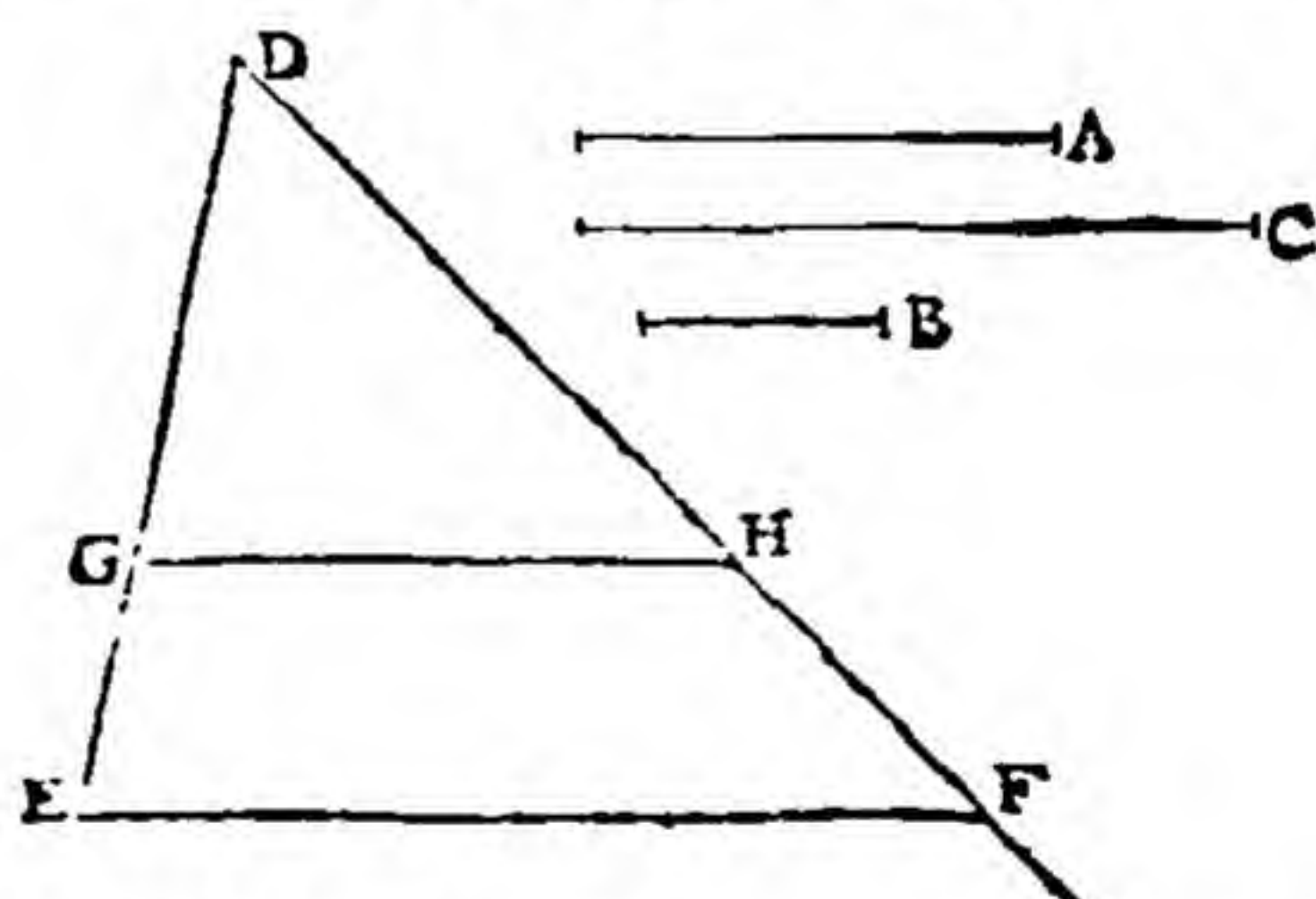


Wherefore to the two given straight lines AB , AC , is found CE a third proportional. Which was to be done.

PROPOSITION XII.

Three straight lines being given to find a fourth proportional.

Let the three given straight lines be A , B , C . It is required unto A , B , C , to find a fourth proportional. Let two straight lines DE , DF , be put, containing any angle as EDF ; and to A let DG be put equal; and to B , GE , and also to C let DH be put equal: then GH being joyned, let to the same be drawn by the point E , the line EF parallel. Now forasmuch as of the triangle DEF , to one of the sides EF , the



line GH is drawn parallel; therefore as DG is to GE , so DH is to HF [Prop. 2. El. VI.]. But DG is equal to A , and GE to B , and DH to C : therefore as A is to B , so C is to HF .

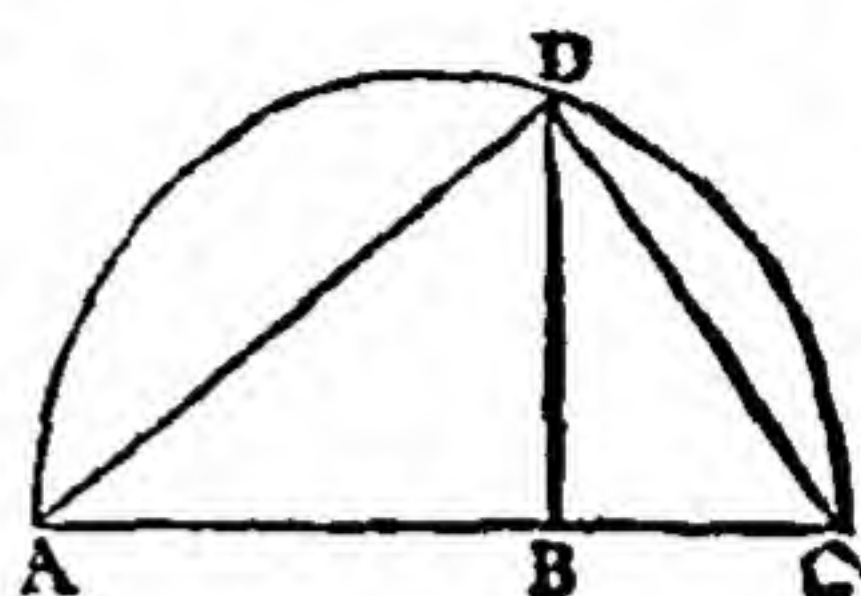
Wherefore to three given straight lines A , B , C , is found HF a fourth proportional. Which was to be done.

PRO-

PROPOSITION XIII.

Two strait lines being given to find a mean proportional.

Let the two given strait lines be AB, BC . It is required unto AB, BC , to find a mean proportional. Let AB, BC be put in a direct line, and on AC let be described the Semicircle ADC ; then from the point B to the strait line AC , let BD be drawn at right angles, and let be joyned AD, DC . Now forasmuch as in the Semicircle the angle ADC is a right angle [Prop. 31. El. III.], and because in the right angled triangle ADC , from the right angle to the base the perpendicular DB is drawn; therefore DB is a mean proportional to the Segments of the base AB, BC [Coroll. Prop. 8. El. VI.].

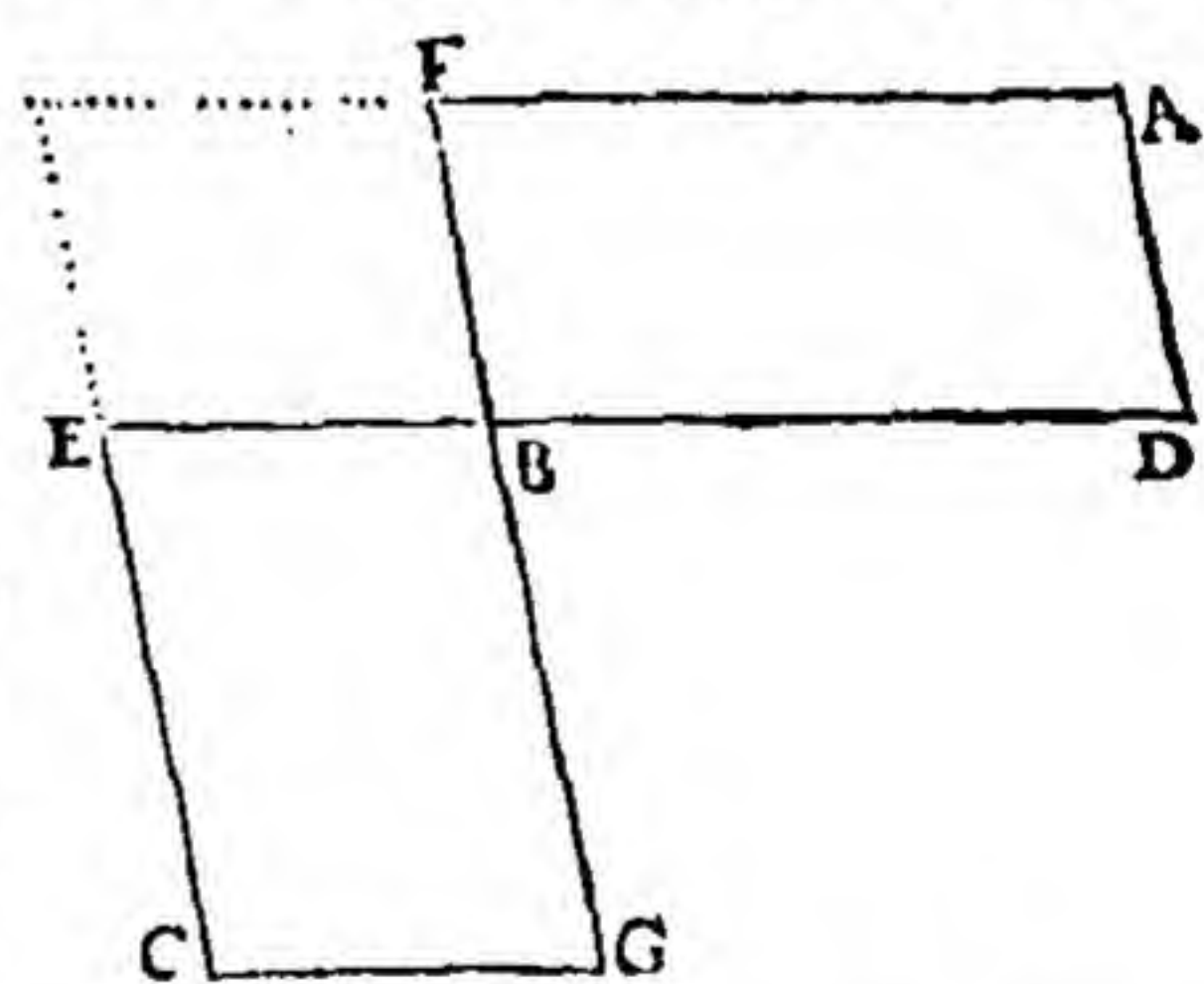


Wherefore to the two given lines AB, BC , is found DB a mean proportional. Which was to be done.

PROPOSITION XIV.

Of equal Parallelograms having one angle equal to one angle, the sides about the equal angles are reciprocally proportional. And Parallelograms having one angle equal to one angle, and the sides about the equal angles reciprocally proportional, are equal to one another.

Let the equal Parallelograms be AB, BC , having equal angles at B , and let DB, BE , be put directly to one another; therefore also FB, BG , are directly to one another. I say, that of the Parallelograms AB, BC , the sides about the equal angles are reciprocally proportional, that is, as DB is to BE , so GB is to BF . Let the Parallelogram FE be completed. Now forasmuch as the Parallelogram AB is equal to the Parallelogram BC , and FE is an other; therefore as AB is to FE , so BC is to FE [Prop. 7. El. V.]. But as AB to FE , so DB to BE , [Prop. 1. El. VI.] and as BC to FE , so GB to BF ; therefore as DB to BE , so GB to BF [Prop. 11. El. V.]: wherefore of the Parallelograms AB, BC , the sides about the equal angles are reciprocally proportional.



But now let the sides about the equal angles be reciprocally proportional, and let it be, as DB to BE , so GB to BF . I say, that the Parallelogram AB is equal to the Parallelogram BC . For because as DB is to BE , so GB is to BF ; and as DB to BE , so the Parallelogram AB to the Parallelogram FE [Prop. 1. El. VI.]; and as GB to BF , so

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the Parallelogram BC to the Parallelogram FE ; therefore as AB is to FE , so BC is to FE [Prop. 11. El. V.]: wherefore the Parallelogram AB is equal to the Parallelogram BC [Prop. 9. El. V.].

Therefore of equal Parallelograms having one angle equal to one angle, the sides about the equal angles are reciprocally proportional.

And Parallelograms having one angle equal to one angle, and the sides about the equal angles reciprocally proportional, are equal to one another. Which was to be demonstrated.

Animadversion.

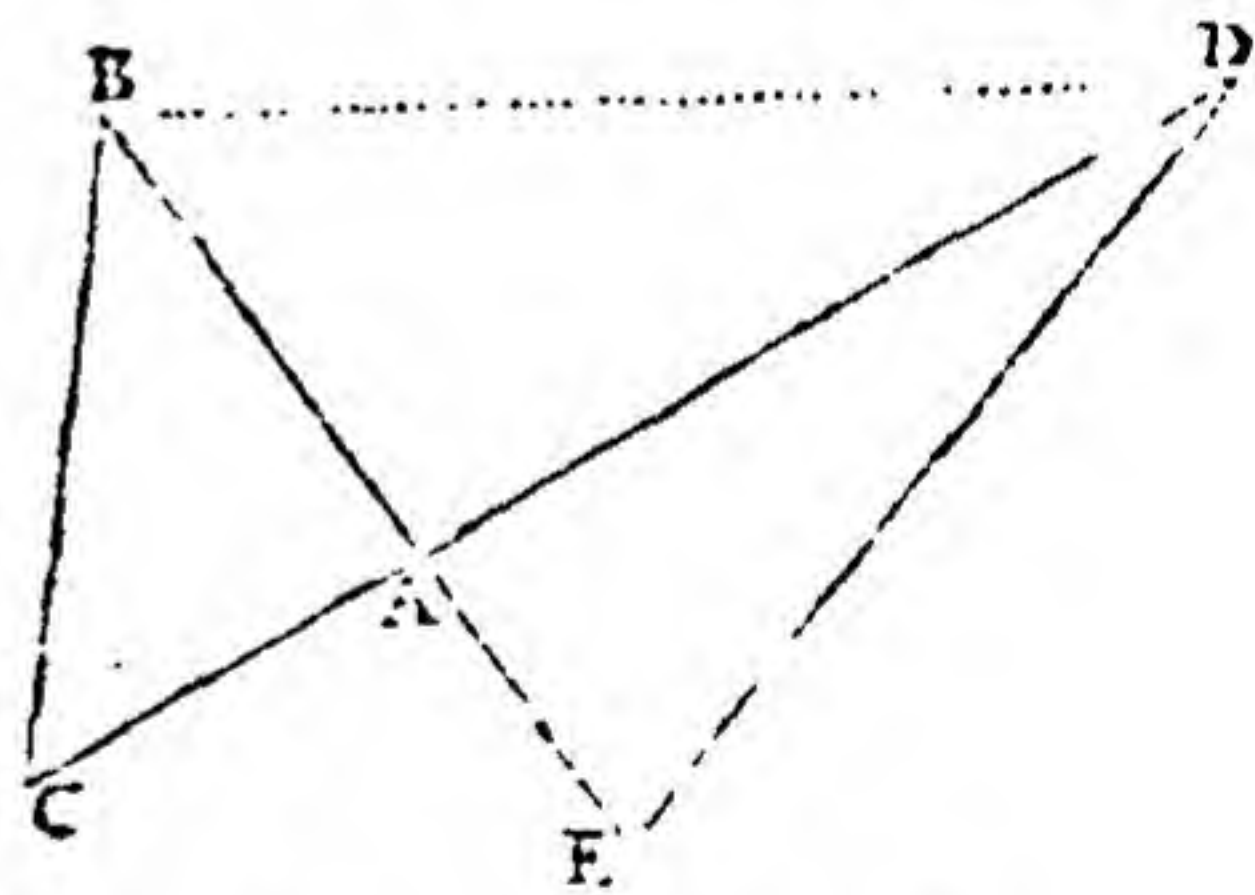
Forasmuch as the Complements of every Parallelogram are equal to one another [Prop. 43. El. I.]; therefore by this Proposition they have their sides reciprocally proportional.

PROPOSITION XV.

O*f equal triangles having one angle equal to one angle, the sides about the equal angles are reciprocally proportional.*

And triangles having one angle equal to one angle, and the sides about the equal angles reciprocally proportional, are equal to one another.

Let the equal triangles be ABC , ADE , having one angle equal to one angle, namely, the angle BAC to the angle DAE . I say, that of the triangles ABC , ADE , the sides about the equal angles are reciprocally proportional, that is, as CA is to AD , so EA is to AB . Let them be put so that CA be directly to AD ; therefore also EA is directly to AB , and let be joyned BD . Now forasmuch as the triangle ABC is equal to the triangle ADE ; and ABD is an other; therefore as the triangle CAB is to the triangle BAD , so the triangle ADE is to the triangle BAD [Prop. 7. El. V.]. But as CAB is to BAD , so CA is to AD [Prop. 1 El. VI.], and as the triangle EAD is to BAD , so EA is to AB ; therefore as CA is to AD , so EA is to AB [Prop. 11. El. V.]; wherefore of the triangles ABC , ADE , the sides about the equal angles are reciprocally proportional.



But now let the sides of the triangles ABC , ADE be reciprocally proportional, and let it be, as CA to AD , so EA to AB . I say, that the triangle ABC is equal to the triangle ADE . For again, BD being joyned; because as CA is to AD , so EA is to AB : and as CA is to AD , so the triangle ABC is to the triangle BAD : and as EA is to AB , so the triangle EAD is to the triangle BAD ; therefore as the triangle ABC is to the triangle BAD , so the triangle EAD is to the triangle BAD [Prop. 11. El. V.]; therefore each of the triangles ABC , EAD , have the

the same proportion to the triangle BAD : wherefore the triangle ABC is equal to the triangle EAD [Prop. 9. El. V.].

Therefore of equal triangles having one angle equal to one angle, the sides about the equal angles are reciprocally proportional.

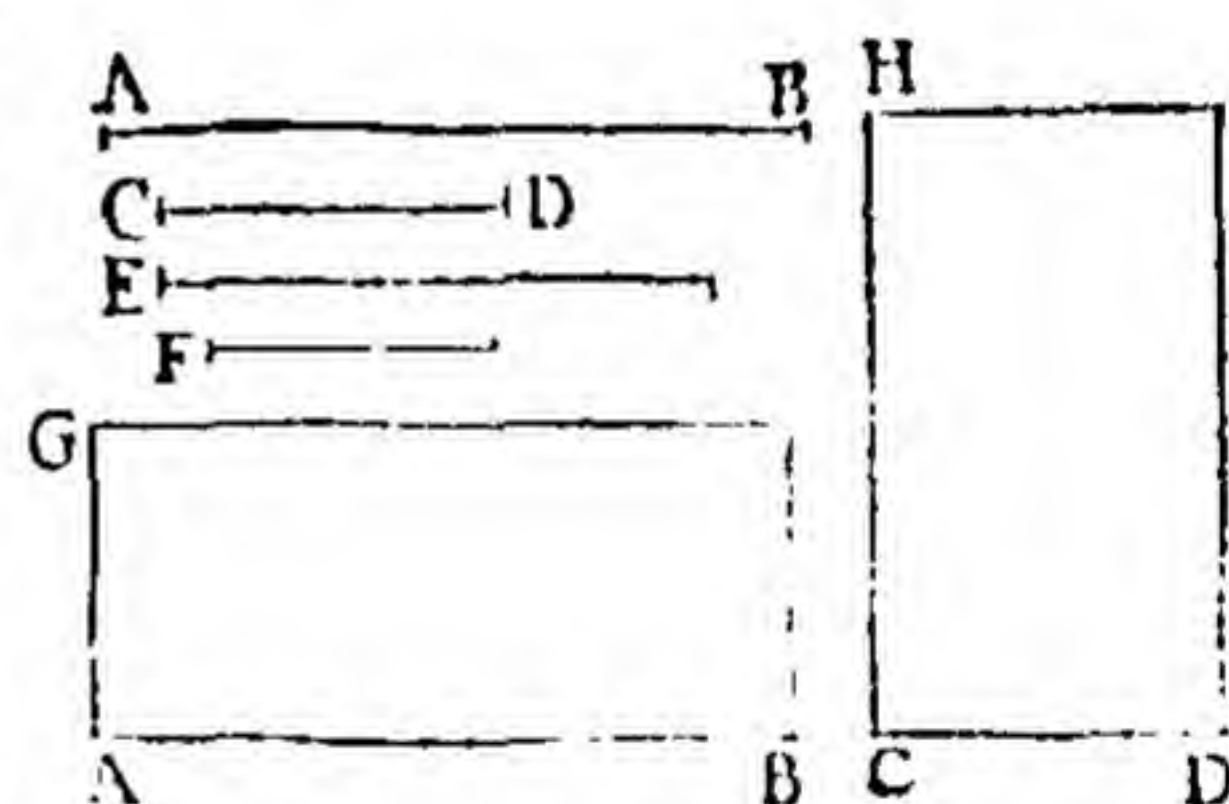
And triangles having one angle equal to one angle, and the sides about the equal angles reciprocally proportional, are equal to one another. Which was to be demonstrated.

PROPOSITION XVI.

If four straight lines be proportional, the Rectangle contained by the extremes is equal to the Rectangle contained by the means.

And if the Rectangle contained by the extremes be equal to the Rectangle contained by the means, the four straight lines shall be proportional.

Let four straight lines AB, CD, E, F , be proportional, as AB to CD , so E to F . I say that the Rectangle contained by AB, F , is equal to the Rectangle contained by CD, E . For from the points A, C to the straight lines AB, CD , let AG, CH be drawn at right angles, and let AG be put equal to F , and CH to E , and let the Parallelograms BG, DH , be completed. Now forasmuch as AB is to CD , so E is to F ; and E is equal to CH , and F to AG : therefore as AB is to CD , so CH is to AG : wherefore of the Parallelograms BG, DH , the sides about the equal angles are reciprocally proportional. But equiangular Parallelograms having the sides about the equal angles reciprocally proportional, are equal to one another [Prop. 14. El. VI.]; therefore the Parallelogram BG is equal to the Parallelogram DH . Now the Parallelogram BG is contained by AB, F ; for AG is equal to F : and the Parallelogram DH , is contained by CD, E ; for CH is equal to E ; therefore the Rectangle contained by AB, F ; is equal to the Rectangle contained by CD, E .



But now let the Rectangle contained by AB, F , be equal to the Rectangle contained by CD, E . I say, that the four straight lines shall be proportional, as AB to CD , so E to F . For the same Construction being made; because the Rectangle under AB, F , is equal to the Rectangle under CD, E ; and the Rectangle under AB, F , is BG ; for AG is equal to F : and the Rectangle under CD, E , is DH : for CH is equal to E : therefore the Rectangle BG is equal to the Rectangle DH , and they are equiangular. But of equal, and equiangular Parallelograms the sides about the equal angles are reciprocally proportional: wherefore as AB is to CD , so CH is to AG : but CH is equal to E , and AG to F ; therefore as AB is to CD , so E is to F .

If therefore four strait lines be proportional, the Rectangle contained by the extremes is equal to the Rectangle contained by the means.

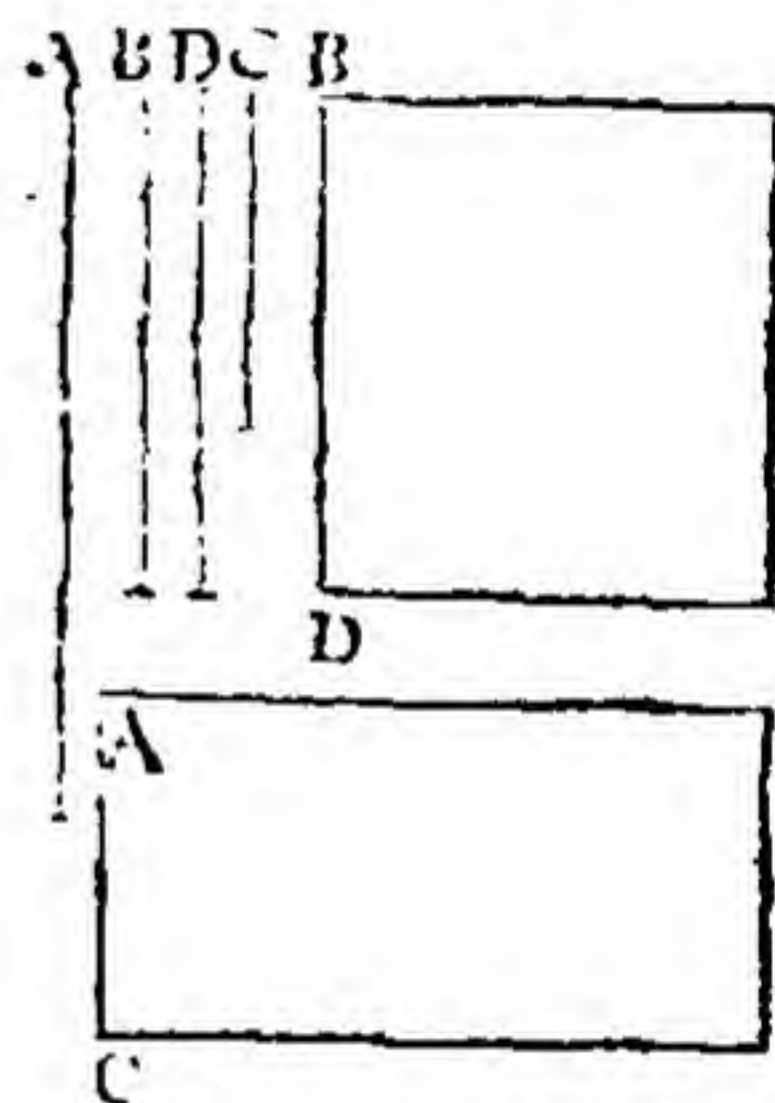
And if the Rectangle contained by the extremes be equal to the Rectangle contained by the means, the four strait lines shall be proportional. Which was to be demonstrated.

PROPOSITION XVII.

If three strait lines be proportional, the Rectangle contained by the extremes is equal to the square of the mean.

And if the Rectangle contained by the extremes be equal to the Square of the mean, the three strait lines shall be proportional.

Let three strait lines A, B, C , be proportional, as A to B , so B to C . I say, that the Rectangle contained by A, C , is equal to the Square of the mean B . Let D be put equal to B , and because as A is to B , so B is to C ; and D is equal to B , therefore as A is to B , so D is to C [Prop. 7. El. V.]. Now if four strait lines be proportional, the Rectangle contained by the extremes is equal to the Rectangle contained by the means; therefore the Rectangle under A, C , is equal to the Rectangle under B, D . But the Rectangle under B, D , is equal to the Square of B , for B is equal to D ; therefore the Rectangle contained by A, C , is equal to the Square of B .



But now let the Rectangle contained by the lines A, C , be equal to the Square of B . I say, that as A is to B , so B is to C . For the same Construction being made: because the Rectangle under A, C , is equal to the Square of B , and the Square of B is the Rectangle under B, D , for D is equal to B ; therefore the Rectangle under A, C , is equal to the Rectangle under B, D . But if the Rectangle under the extremes be equal to Rectangle under the means, the four strait lines are proportional [by Prop. 16. El. VI.]; therefore as A is to B , so D is to C . But B is equal to D ; therefore as A is to B , so B to C [Prop. 7. El. V.].

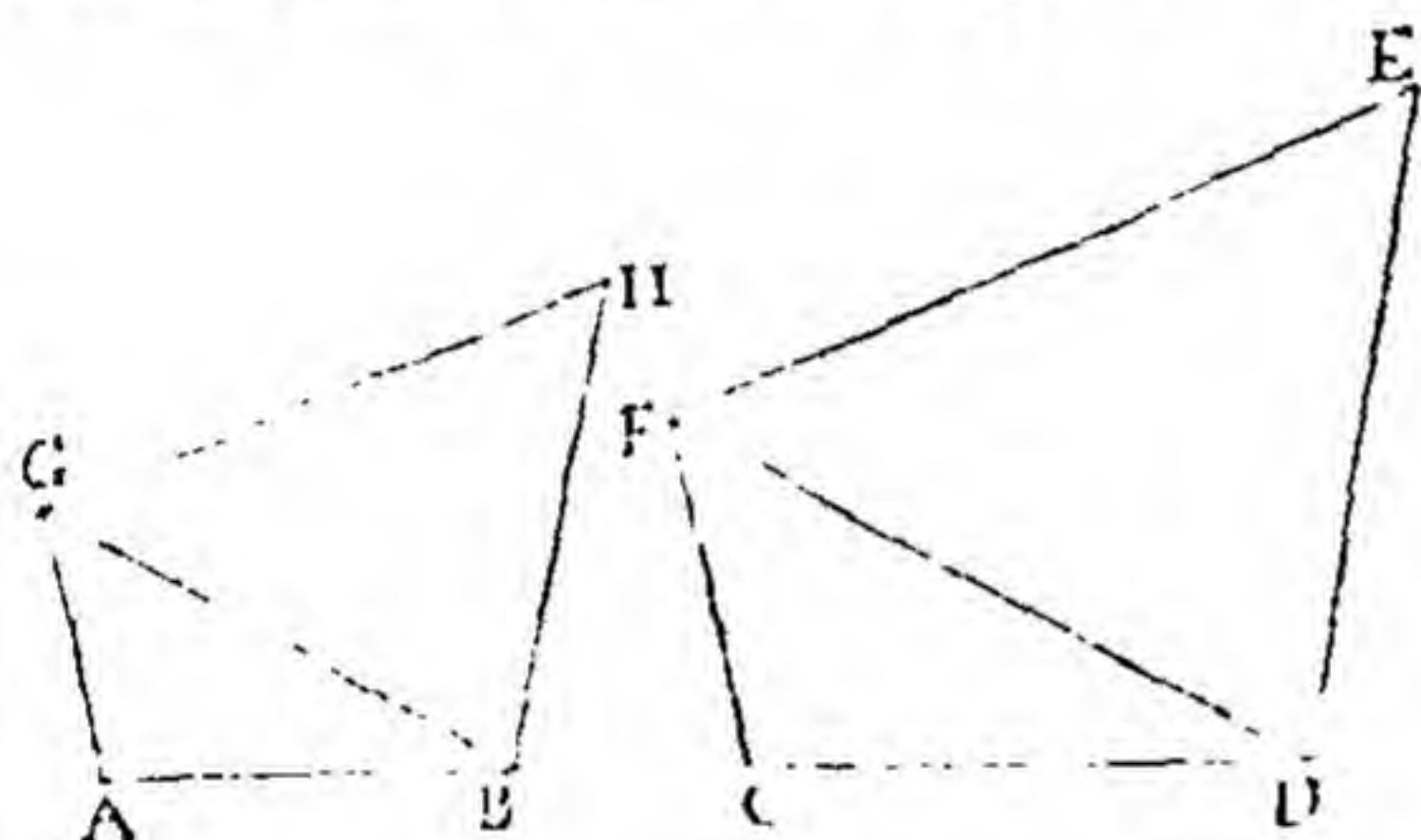
If therefore three strait lines be proportional, the Rectangle contained by the extremes is equal to the Square of the mean.

And if the Rectangle contained by the extremes be equal to the Square of the mean, the three strait lines shall be proportional. Which was to be demonstrated.

PROPOSITION XVIII.

UPon a given strait line, unto a given strait-lin'd Figure to describe a strait-lin'd Figure like and alike situated.

Let the given strait line be AB , and the given strait-lin'd Figure CE : it is required upon the strait line AB , unto the strait-lin'd Figure CE , to describe a strait-lin'd Figure like and alike situated. Let be joyned DF ; and to the strait line AB , and to the points in the same A, B , let be constituted the angle BAG , equal to the angle at C : and the angle ABG to the angle CDF [Prop. 23. El. I.]; therefore the remaining angle CFD is equal to the remaining angle AGB : wherefore the triangle FGD is equiangled to the triangle GAB ; it is therefore proportionally as FD to GB , so FC to GA , and CD to AB [Prop. 4. El. VI.]. Again, to the strait line BG , and to the points in the same B, G , let be constituted the angle BGH equal to the angle DFE , and the angle GBH to the angle FDE : therefore the remaining angle at E is equal to the remaining angle at H : wherefore the triangle FDE is equiangled to the triangle GBH ; it is therefore pro-



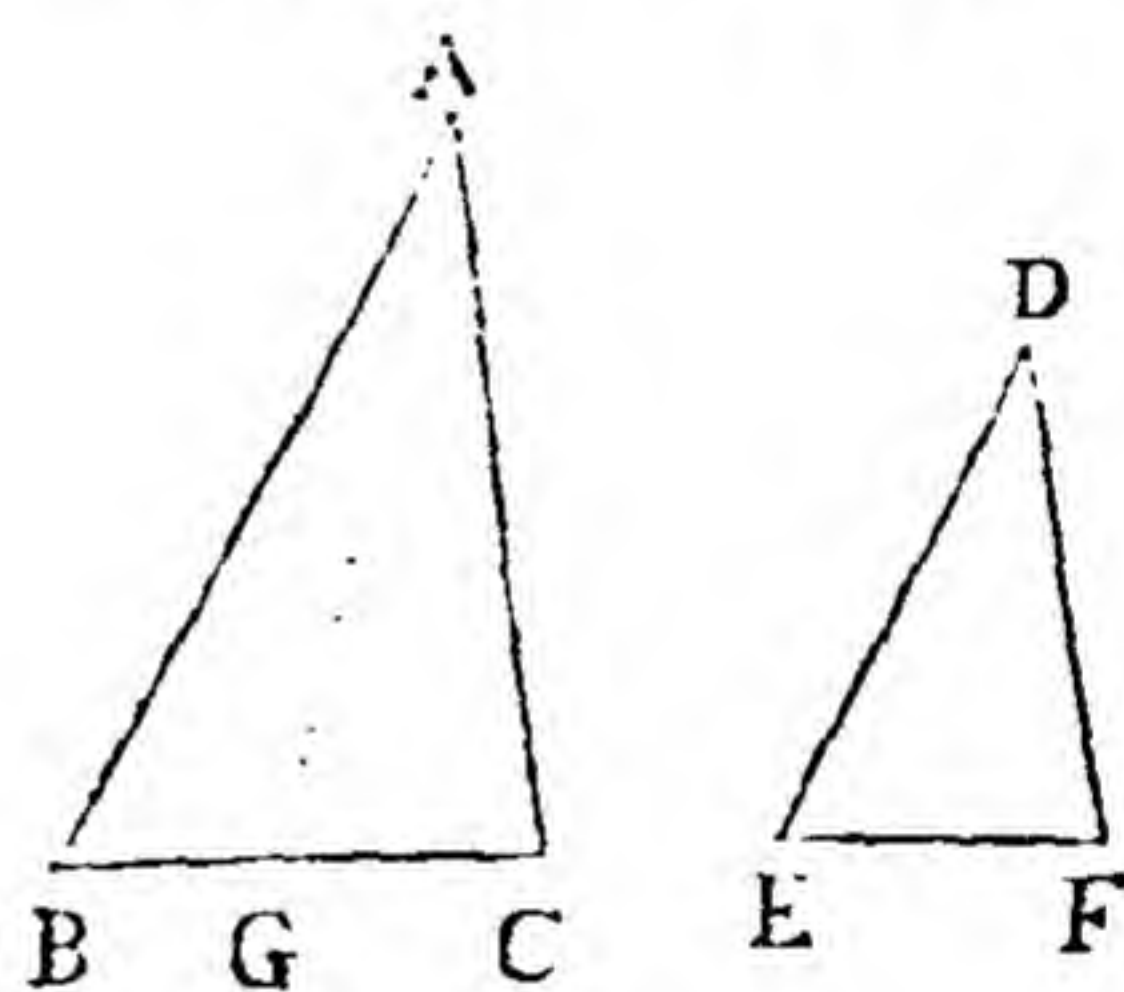
portionally, as FD to GB , so FE to GH , and ED to HB [Prop. 4. El. VI.]. And it has been prov'd, that also as FD to GB , so FC to GA , and CD to AB , therefore as FC to GA , so CD to AB , and FE to GH , and moreover ED to HB [Prop. 11. El. V.]. And because the angle CFD is equal to the angle AGB , and the angle DFE to the angle BGH ; therefore the whole angle CFE is equal to the whole angle AGH . By the same reason also the angle CDE is equal to the angle AHB . But also the angle at C is equal to the angle at A , and the angle at E to the angle at H : therefore AH is equiangled to CE ; and hath to the same about the equal angles the sides proportional. Wherefore the strait lin'd Figure AH , is like to the strait lin'd Figure CE [Def. 1. El. VI.].

Therefore upon the given strait line AB to the given strait-lin'd Figure CE , is described the strait-lin'd Figure AH , like and alike situated. Which was to be done.

PROPOSITION XIX.

Like triangles are to one another in a Duplicate proportion of their Homologal sides.

Let the like triangles be ABC , DEF , having the angle at B equal to the angle at E , and let it be as AB to BC , so DE to EF : so that BC is Homologal to EF . I say that the triangle ABC hath to the triangle DEF a duplicate proportion of that which BC hath to EF . For to BC , EF , let be taken a third proportional BG [Prop. 11. El. VI.]: so that it be as BC to EF , so EF to BG : and let be joyned GA . Now because it is as AB to BC , so DE to EF ; therefore it is alternately as AB to DE , so BC to EF : but as BC to EF , so EF to BG . And therefore as AB to DE , so EF to BG [Prop. 11. El. V.]: wherefore of the triangles ABG , DEF , the sides about the equal angles are reciprocally proportional. But triangles having one angle equal to one angle, and the sides about the equal angles reciprocally proportional, are equal to one another [Prop. 15. El. VI.]; therefore the triangle



ABG is equal to the triangle DEF . And because as BC is to EF , so EF is to BG , and that if three strait lines be proportional, the first to the third is said to have a duplicate proportion of that, which it hath to the second [Def. 10. El. V.]: therefore BC hath to BG a duplicate proportion of that, which BC hath to EF . But as BC is to BG , so the triangle ABC is to the triangle ABG [Prop. 1. El. VI.]; therefore the triangle ABC hath to the triangle ABG a duplicate proportion of that, which BC hath to EF . But the triangle ABG is equal to the triangle DEF : wherefore the triangle ABC hath to the triangle DEF a duplicate proportion of that, which BC hath to EF .

Therefore like triangles are to one another in a duplicate proportion of their Homologal sides. Which was to be demonstrated.

Corollary

From hence 'tis manifest, that if three strait lines be proportional; as the first is to the third, so the triangle upon the first is to the triangle upon the second, being like and alike described: for that it hath been prov'd, that as CB is to BG , so the triangle ABC is to the triangle ABG , that is, to DEF .

PROPOSITION XX.

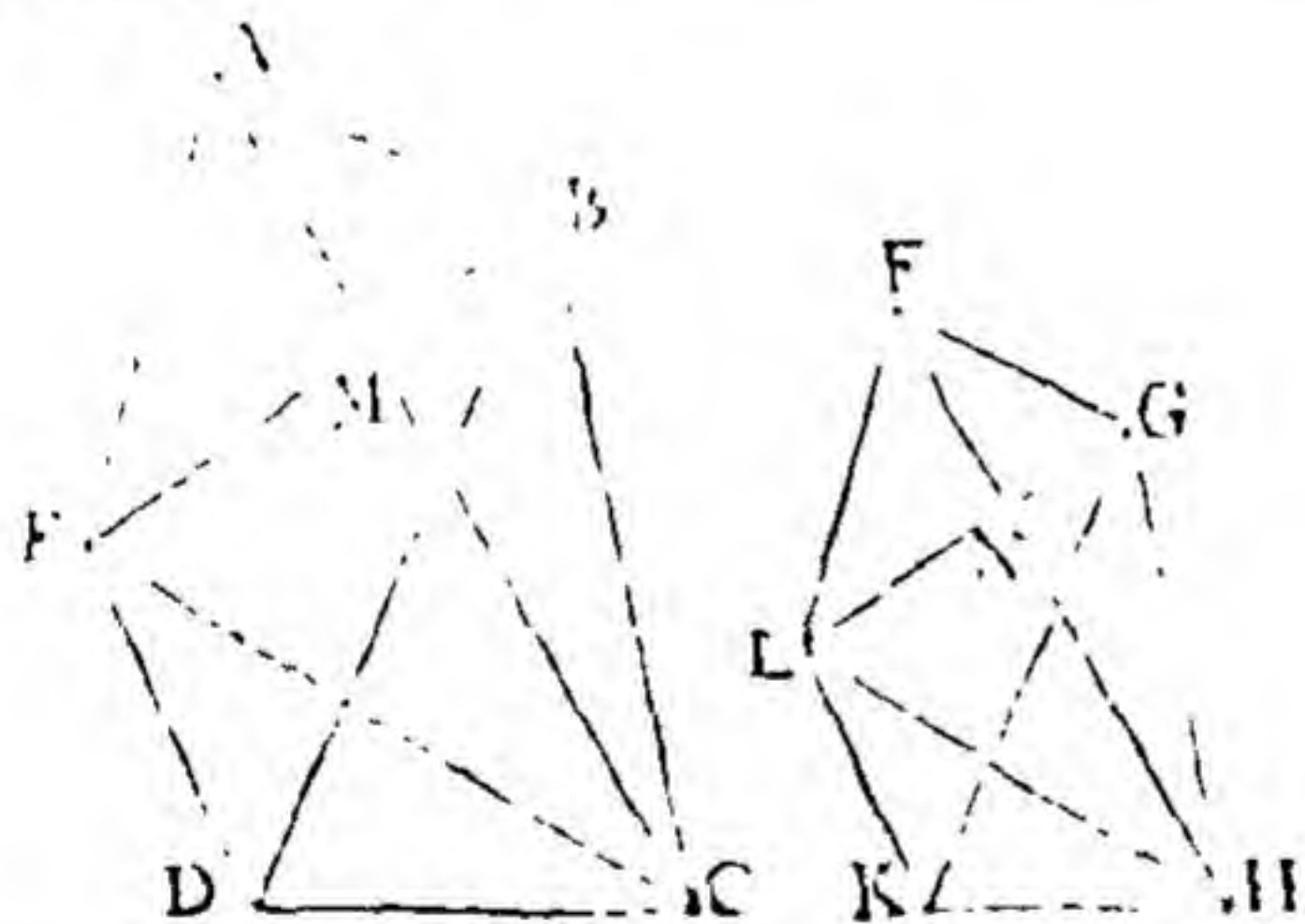
Like Polygons are divided into like triangles, equal in number:
And Homologal to the wholes.

And Polygon hath to Polygon a duplicate proportion of that, which an Homologal side hath to an Homologal side.

Let the like Polygons be $ABCDE$, $FGHKL$, and let AB be Homologal to FG . I say, that the Polygons $ABCDE$, $FGHKL$, are divided into like triangles, equal in number, and Homologal to the wholes. And the Polygon $ABCDE$ hath to the Polygon $FGHKL$ a duplicate proportion of that, which AB hath to FG .

Let be joyned BE , EC , GL , LH . Now because the Polygon $ABCDE$ is like to the Polygon $FGHKL$; therefore the angle BAE is equal to the angle GFL ; and as BA is to AE , so GF is to FL [Def. 1. El. VI.]. Forasmuch therefore as there are two triangles ABE , FGL , having one angle equal to one angle, and about the equal angles the sides proportional; therefore the triangle ABE is equiangled to the triangle FGL [Prop. 6. El. VI.], so that it is also like.

And because ABE is like to FGL , therefore the angle ABE is equal to the angle FGL : But also the whole angle ABC is equal to the whole angle FGH , for the likeness of the Polygons: therefore the remaining angle EBC is equal to the remaining angle LGH . And because for the likeness of the triangles ABE , FGL , it is as EB to BA , so LG to GF , and also for the likeness of the Polygons it is as AB to BC , so FG to GH : therefore by equality it is as EB to BC , so LG to GH ; and these proportional sides are about the equal angles EBC , LGH : therefore the triangle EBC is equiangled to the triangle LGH [Prop. 6. El. VI.]. And so also like.



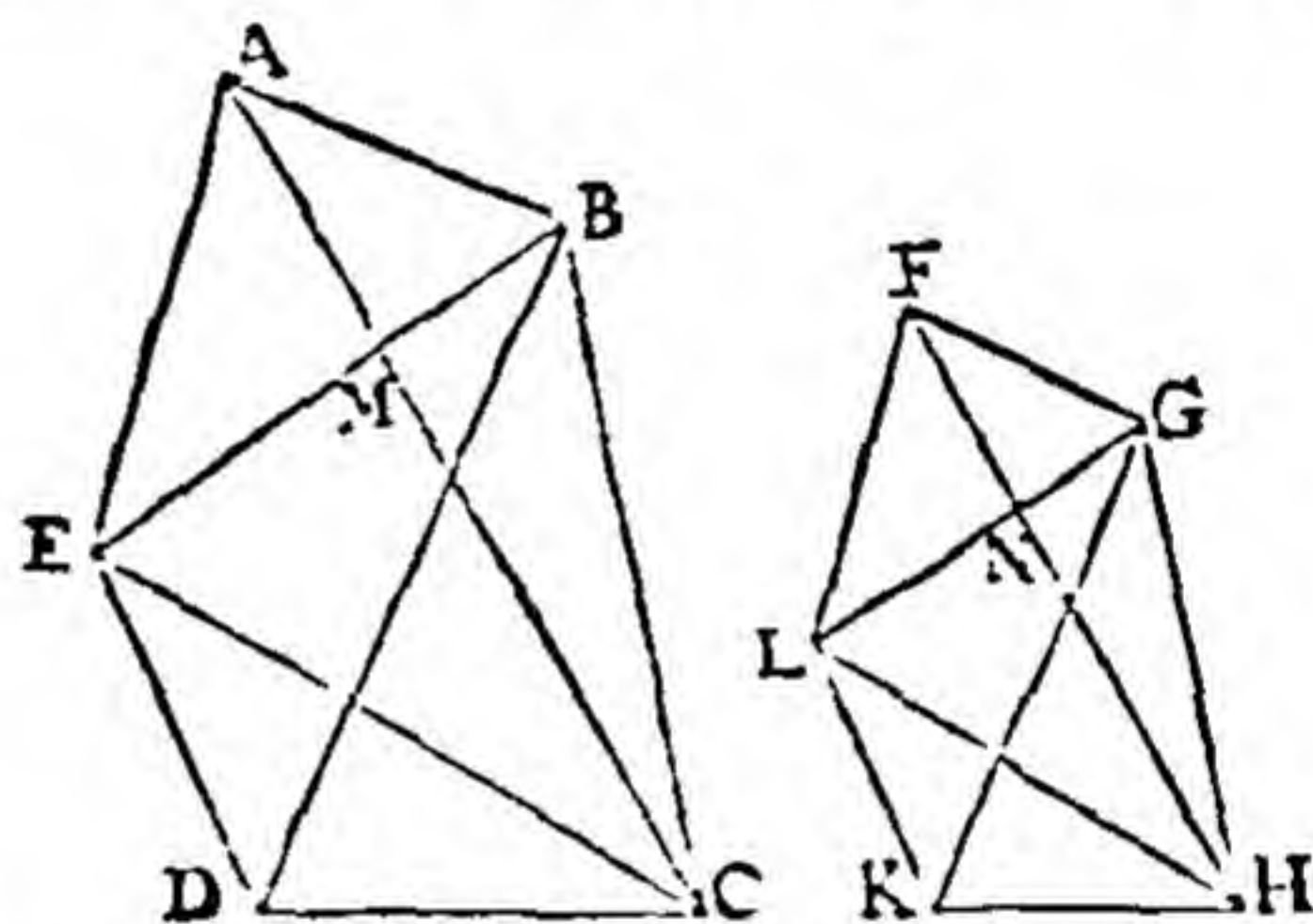
By the same reason also the triangle ECD is like to the triangle LHK ; therefore the Polygons $ABCDE$, $FGHKL$, are divided into like triangles, equal in number.

I say moreover, that the triangles are also Homologal to the wholes, that is, the triangles are proportional, and in the Polygon $ABCDE$, the Antecedents are ABE , EBC , ECD ; and in the Polygon $FGHKL$, their Consequents are FGL , LGH , LHK .

And also I say, that the Polygon $ABCDE$ hath to the Polygon $FGHKL$, a duplicate proportion of that which an Homologal side hath to an Homologal side; that is, which AB hath to FG . For let be joyned AC , FH . Now

Now because for the likeness of the Polygons, the angle ABC is equal to the angle FGH , and as AB is to BC , so FG is to GH ; therefore the triangle ABC is equiangled to the triangle FGH [Prop. 6. El. VI.]: wherefore the angle BAC is equal to the angle GFH , and the angle BCA to the angle GHF . And because the angle BAM is equal to the angle GFN : and it hath been prov'd, that the angle ABM is equal to the angle FGN ; therefore the remaining angle AMB is equal to the remaining angle FNG : wherefore the triangle AMB is equi-angled to the triangle FNG .

In like manner shall we prove, that the triangle BMC is equiangled to the triangle GNH : proportionally therefore it is, as AM to MB , so FN to NG , and as MB to MC , so NG to NH ; so that by equality as AM to MC , so FN to NH . But as AM to MC , so the triangle AMB is to the triangle BMC , and the triangle AME to the triangle EMC ; for they are to one another as their bases AM, MC [Prop. 1. El. VI.]. And as one of the Antecedents to one of the Consequents, so all the Antecedents to all the Consequents [Prop. 12. El. V.]; therefore as the triangle AMB to the triangle BMC , so the triangle ABE to the triangle EBC . But as AMB to BMC , so AM to MC ; and therefore as AM to MC , so the triangle ABE to the triangle EBC [Prop. 11. El. V.].



By the same reason also as FN to NH , so the triangle FGL to the triangle LGH . But as AM is to MC , so FN is to NH ; and therefore as the triangle ABE to the triangle EBC , so the triangle FGL to the triangle LGH ; and alternately, as the triangle ABE to the triangle FGL , so the triangle EBC to the triangle LGH .

In like manner shall we prove, BD, GK being joyned, that as the triangle EBC to the triangle LGH , so the triangle ECD to the triangle LHK .

And because as the triangle ABE is to the triangle FGL , so the triangle EBC is to the triangle LGH , and moreover the triangle ECD to the triangle LHK ; and as one of the Antecedents to one of the Consequents, so all the Antecedents to all the Consequents; therefore as the triangle ABE is to the triangle FGL , so the Polygon $ABCDE$ is to the Polygon $FGHKL$: wherefore the like triangles and equal in number, are also Homologal to the wholes.

And now I say again, that the Polygon $ABCDE$, hath to the Polygon $FGHKL$ a duplicate proportion of an Homologal side to an Homologal side.

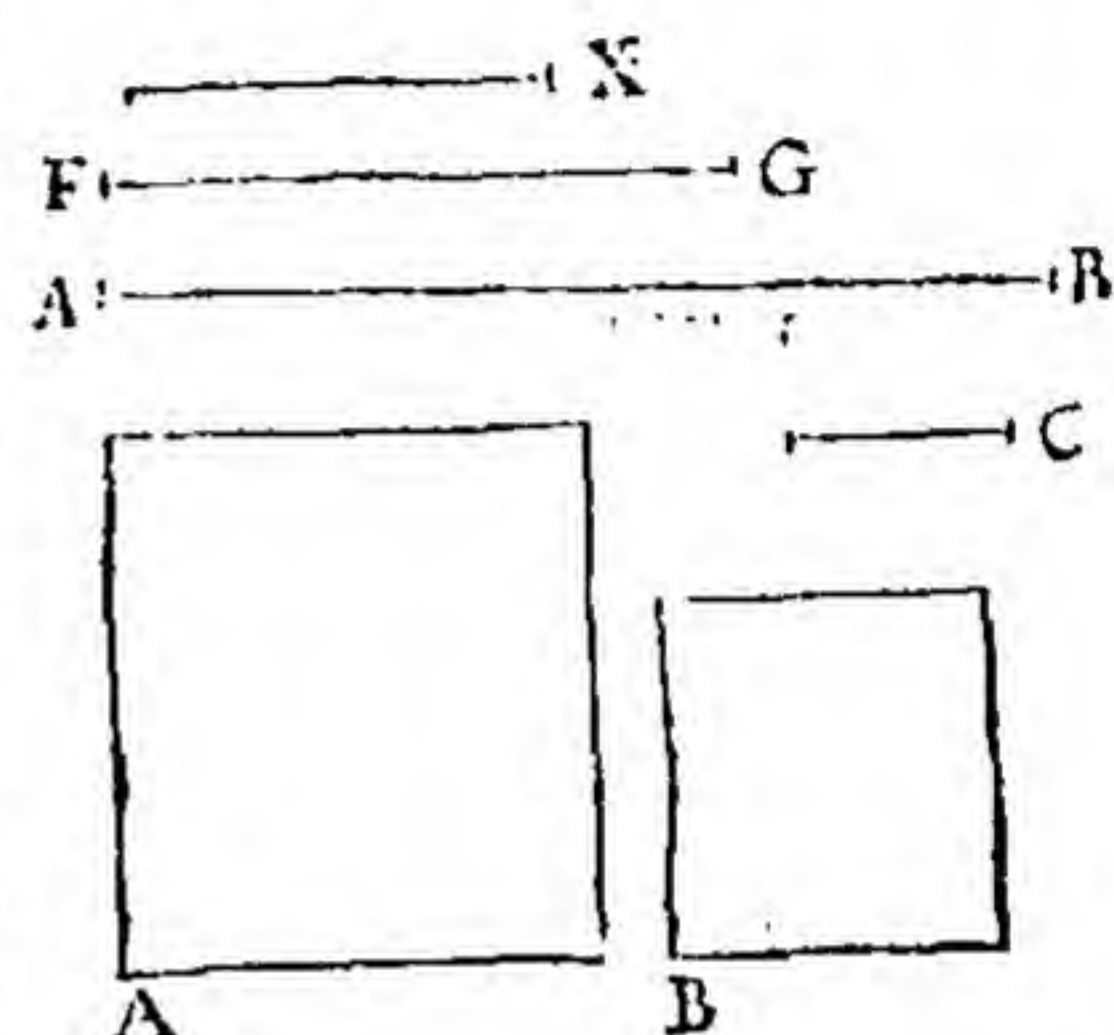
For

For the the triangle ABE hath to the triangle FGL a duplicate proportion of that which the Homologal side AB hath to the Homologal side FG . For like triangles are in a duplicate proportion of their Homologal sides [Prop. 19. El. VI.]. But as the triangle ABE is to the triangle FGL , so the Polygon $ABCDE$ is to the Polygon $FGHKL$; therefore also the Polygon $ABCDE$ hath to the Polygon $FGHKL$ a duplicate proportion of that, which the Homologal side AB hath to the Homologal side FG .

Therefore like Polygons are divided into like triangles, equal in number, and Homologal to the wholes, &c. Which was to be demonstrated.

Corollaries.

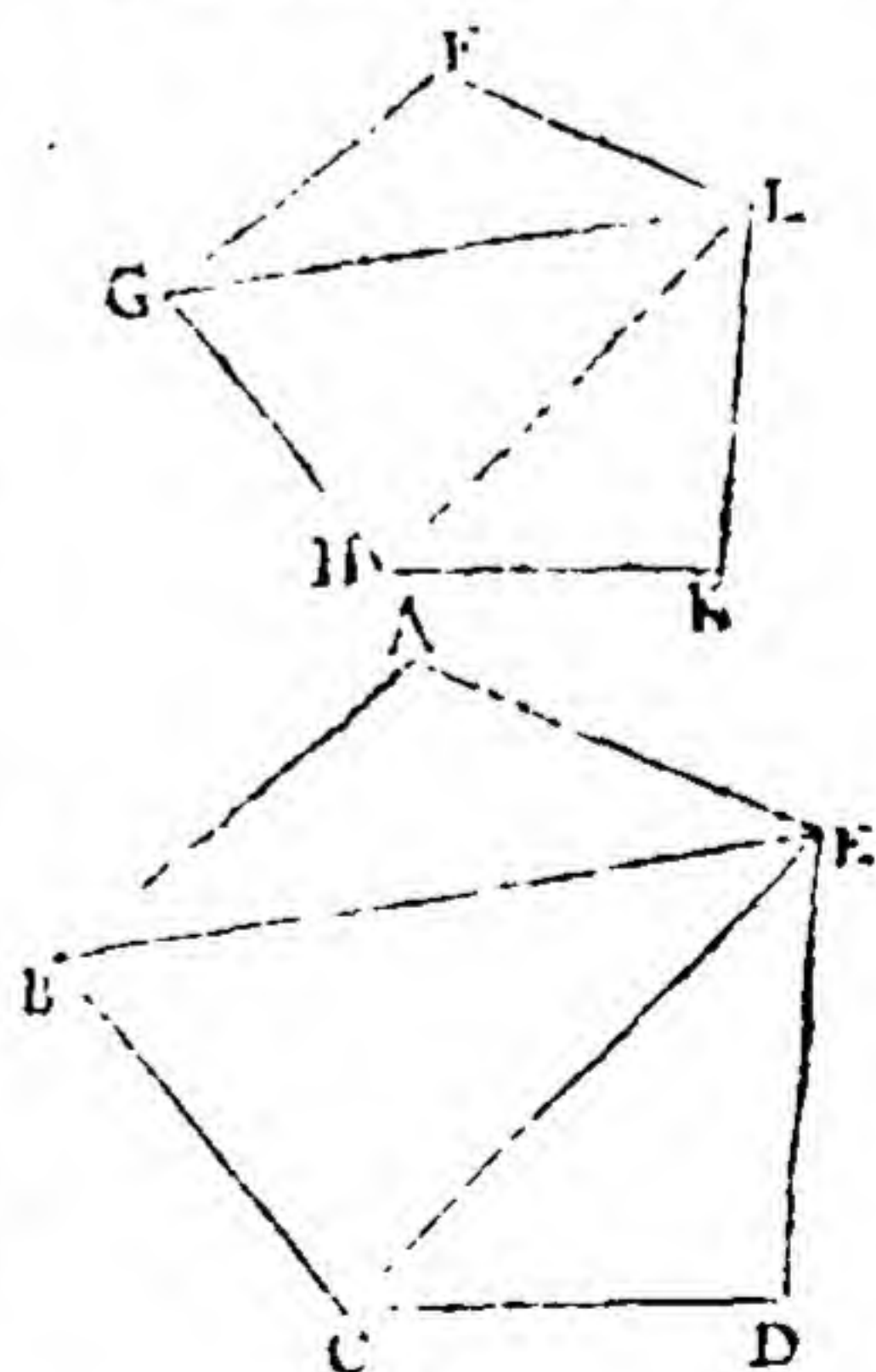
1. After the same manner also in like quadrilateral Figures shall be proved, that they are in a duplicate proportion of their Homologal sides. And the same hath been demonstrated in triangles. Therefore universally, like strait lin'd Figures are to one another in a duplicate proportion of their Homologal sides.



2. And if to AB, FG a third proportional x be taken, then AB is to x in a duplicate proportion of that, which AB hath to FG . But Polygon is to Polygon, and Quadrilateral figure to Quadrilateral figure in a duplicate proportion of their Homologal sides, that is, of AB to FG ; and the same hath been demonstrated in triangles. So that also in general it is manifest, that if three strait lines be proportional, it shall be as the first to the third, so the Figure upon the first, is to the Figure upon the second, like and alike described.

Otherwise.

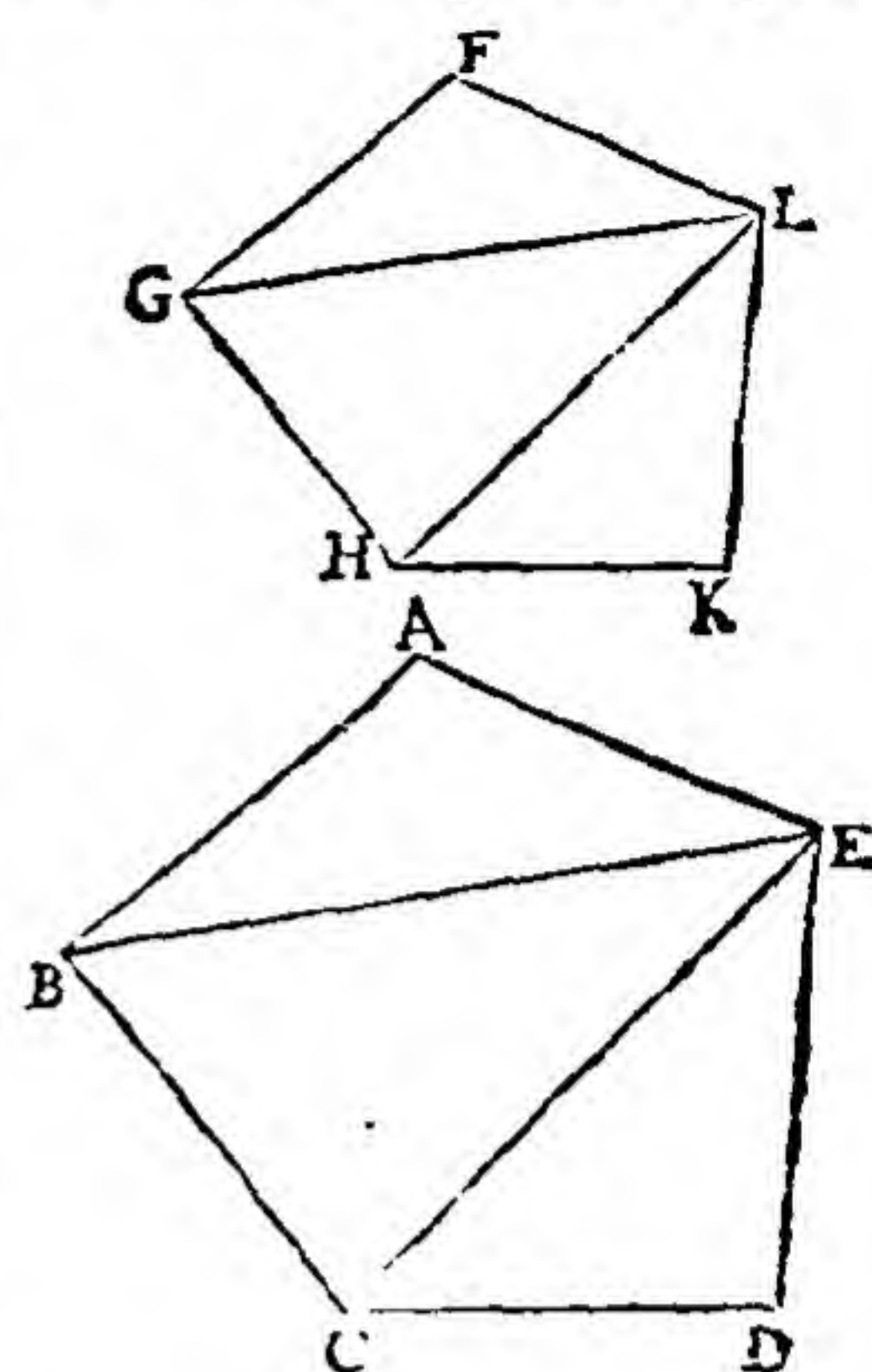
Now we will otherwise and shorter shew the triangles to be Homologal. For again, let the Polygons $ABCDE, FGHKL$, be put: and let be joyned $BE, EC; GL, LH$. I say, that as the triangle ABE is to the triangle FGL , so the triangle EBC is to the triangle LGH , and the triangle ECD to the triangle LHK . Now forasmuch as the triangle ABE is like to the triangle FGL ; therefore the triangle ABE hath to the triangle FGL a duplicate proportion of that, which BE hath to GL . By the same reason also the triangle EBC hath to the triangle LGH a duplicate proportion of that, which BE hath to GL ; therefore as the triangle ABE is to the triangle FGL , so the triangle EBC is to the trian-



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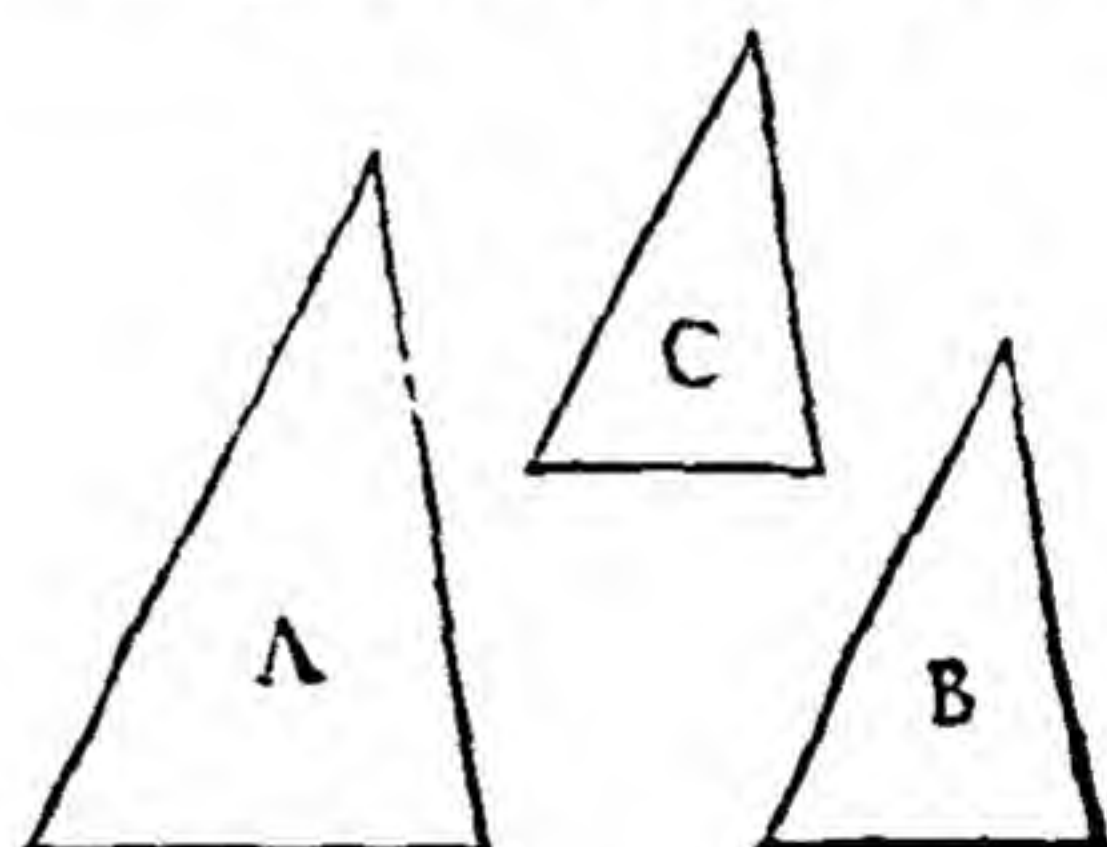
gle LGH. Again, because the triangle EBC is like to the triangle LGH; therefore the triangle EBC hath to the triangle LGH, a duplicate proportion of that which CE hath to HL. By the same reason also, the triangle ECD hath to the triangle LHK a duplicate proportion of that which CE hath to HL; therefore as the triangle EBC is to the triangle LGH, so the triangle ECD is to the triangle LHK. And it hath been proved, that as EBC is to LGH, so ABE to FGL; therefore as the triangle ABE is to the triangle FGL, so the triangle EBC is to the triangle LGH, and the triangle ECD to the triangle LHK: And as one of the Antecedents to one of the Consequents, so all the Antecedents to all the Consequents [Prop. 12. El. V.], and so forth, as in the former demonstration. Which was to be demonstrated.



PROPOSITION XXI.

S *Trait-lin'd Figures like to the same trait-lin'd Figure, are also like to one another.*

For let each of the trait-lin'd Figures A, B, be like to the trait-lin'd Figure c. I say, that A is also like to B, for because A is like to c; therefore it is both equiangled to the same, and also hath about the equal angles the sides proportional [Def. 1. El. VI.]. Again,



because B is like to c; therefore it is both equiangled to the same, and hath also about the equal angles the sides proportional [Def. 1. El. VI.]; therefore each of the trait-lin'd Figures A, B are both equiangled to c, and also have about the equal angles the sides proportional: so that A is also equiangled to B, and hath about the equal angles the sides proportional; therefore A is like to B. Which was to be demonstrated.

PRO.

PROPOSITION XXII.

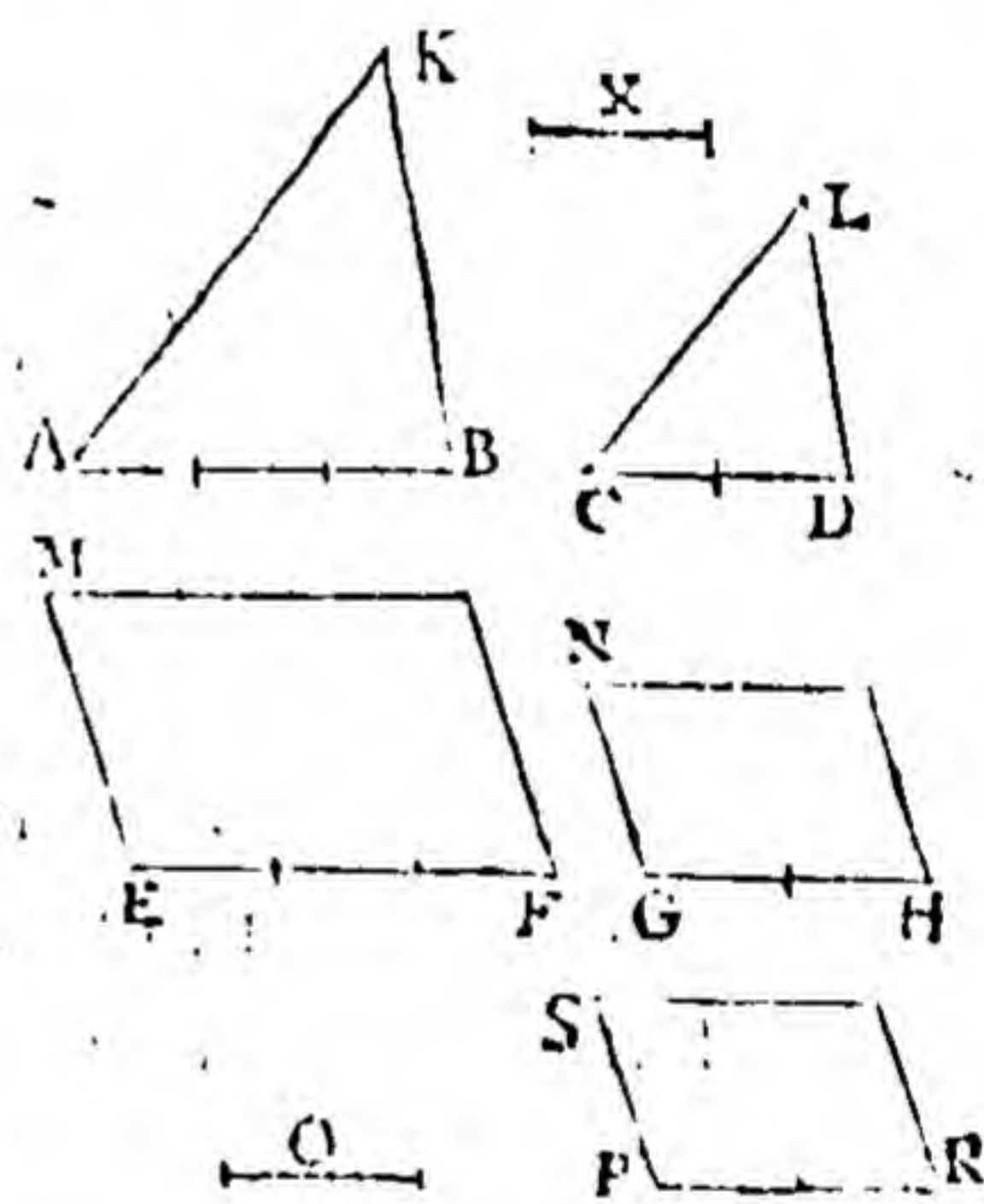
If four strait lines be proportional, the strait-lin'd Figures described upon them, like and alike situated, shall also be proportional. And if the strait-lin'd Figures described upon them, like and alike situated be proportional; also the strait lines shall be proportional.

Let four strait lines AB, CD, EF, GH , be proportional, as AB to CD so EF to GH , and let be described upon AB, CD , the strait-lin'd Figures KAB, LCD , like and alike situated [by Prop. 18. El. VI.]: and upon EF, GH , the strait-lin'd Figures MEF, NH , like and alike situated. I say, that it is as KAB to LCD , so MEF to NH . For to AB, CD , let be taken a third proportional X , and to EF, GH , a third proportional O [Prop. 11. El. VI.]. Now because it is as AB to CD , so EF to GH , and as CD to X , so GH to O ; therefore by equality as AB to X , so EF to O . But as AB to X , so KAB to LCD [Corol. 2. Prop. 20. El. VI.]: and as EF to O , so MEF to NH ; therefore as KAB is to LCD , so MEF is to NH .

But now let it be as KAB to LCD , so MEF to NH . I say, that it is as AB to CD , so EF to GH . For let it be made [Prop. 12. El. VI.], as AB to CD , so EF to PR : and upon PR let be described the strait-lin'd Figure SR , like and alike situated to either of the Figures MEF, NH . Now therefore, because it is as AB to CD so EF to PR : and upon AB, CD are described the strait-lin'd Figures KAB, LCD , like and alike situated: and upon EF, PR the strait-lin'd Figures MEF, SR , like and alike situated; therefore as KAB is to LCD , so MEF is to SR . But it is also supposed that as KAB is to LCD , so MEF is to NH ; therefore MEF hath the same proportion to each of the Figures NH, SR : wherefore NH is equal to SR [Prop. 9. El. V.]. And also it is like to the same and alike situated by the Lemma: therefore the line GH is equal to the line PR . And because as AB is to CD , so EF is to PR , and that PR is equal to GH ; therefore as AB is to CD , so EF is to GH .

If therefore four strait lines be proportional, also the strait-lin'd Figures described upon them, like and alike situated, shall be proportional.

And if the strait-lin'd Figures described upon them, like and alike situated be proportional; also the strait lines shall be proportional. Which was to be demonstrated.

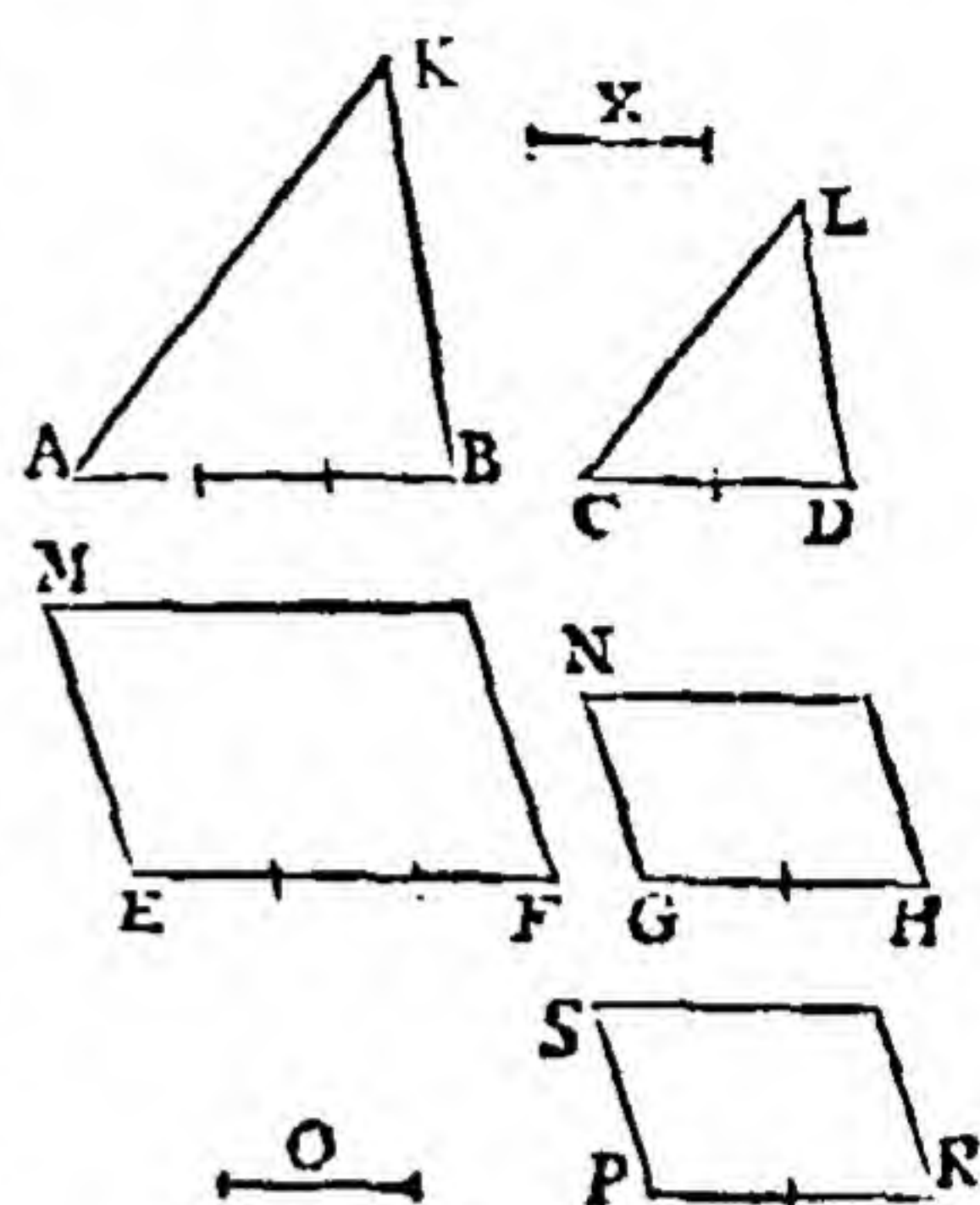


Lemma.

For if strait-lin'd Figures be equal, and like, that then their Homologal sides are equal to one another (Antecedents to Antecedents, and Consequents to Consequents, each to each respectively) we shall thus demonstrate.

Let the equal and like strait-lin'd Figures be NH , SR ; and let it be as HG , to GN , so RP to PS (so that HG and RP are Homologal). I say that RP is equal to HG .

For if they be unequal, one of them is the greater. Let RP be greater than HG ; and because as RP is to PS , so HG is to GN ; therefore alternately, as RP to HG , so PS to GN . But RP is greater than HG ; therefore PS is greater than GN ; so that also the Figure RS is greater than the Figure HN : but also it is equal, which is impossible: wherefore PR is not unequal to GH , equal therefore it is. Which was to be demonstrated.

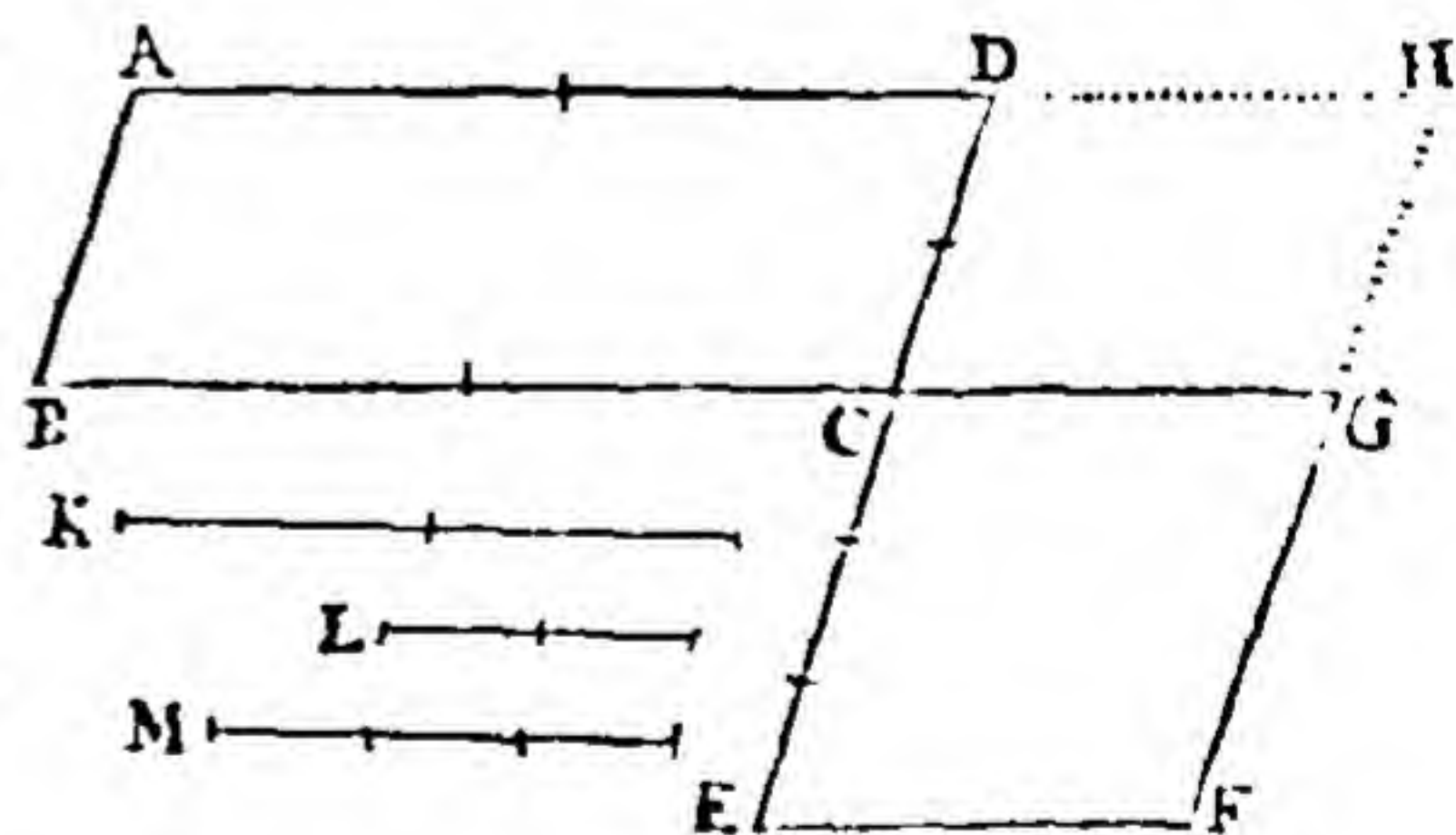


PROPOSITION XXIII.

Equiangled Parallelograms have to one another a proportion compounded of their sides.

Let the equiangled Parallelograms be AC , CF , having the angle BCD equal to the angle ECG . I say that the Parallelogram AC hath to the Parallelogram CF a proportion compounded of their sides, that is, compounded of the proportions which BC hath to CG , and which DC hath to CE , (in a direct proportion, the Antecedents BC , DC , being in one Parallelogram AC , and the Consequents CG , CE in the other Parallelogram CF .)

For let BC be put directly to CG (making one strait line BG); therefore DC is also directly to CE [by Prop. 13, and 14. El. I.], and let be completed the Parallelogram DG (by producing AD , FG till they meet in H).



Now let any strait line be put as K : then let it be made as BC to CG , so K to L ; and as DC to CE , so L to M [Prop. 12. El. VI.]; therefore the proportions of K to L , and of L to M , are the same with the proportions of the sides, of BC to CG ; and of DC to CE .

But the proportion of K to M is compounded of the proportions of

of K to L , and of L to M [according to Def. 10. El. V.]: so that also K hath to M the proportion compounded of the sides (of BC to CG , and of DC to CE).

And because it is as BC to CG , so the Parallelogram AC to the Parallelogram CH [Prop. 1. El. VI.], and as BC to CG , so is K to L ; therefore as K to L , so the Parallelogram AC is to the Parallelogram CH [Prop. 11. El. V.].

Again, because it is as DC to CE , so the Parallelogram CH to the Parallelogram CF , and as DC to CE , so is L to M ; therefore as L to M , so the Parallelogram CH is to the Parallelogram CF .

Now because it hath been proved, that as K to L , so the Parallelogram AC is to the Parallelogram CH : and as L to M , so the Parallelogram CH to the Parallelogram CF ; therefore *ex æquo* as K is to M , so the Parallelogram AC is to the Parallelogram CF : but K hath to M a proportion compounded of the sides: wherefore the Parallelogram AC hath to the Parallelogram CF , a proportion compounded of their sides.

Therefore equiangled Parallelograms have to one another a proportion compounded of their sides. Which was to be demonstrated.

ANNOTATIONS.

Therefore DC is directly to CE.] For the angles BCD , DCG , are equal to two Right [Prop. 13. El. I.] As likewise the angles BCE , ECG ; wherefore BCD , DCG , are equal to BCE , ECG ; and BCD is [by Supposition] equal to ECG ; therefore BCE is equal to DCG . Let the angle BCD be added in common; then shall BCE , BCD , be equal to BCD , DCG . But BCD , DCG , are equal to two Right; therefore BCE , BCD , are also equal to two Right: wherefore CD is directly seated to CE [by Prop. 14. El. I.].

But the proportion of K to M is compounded of the proportions of K to L, and of L to M.] By Def. 10. El. V. where in any three continued Terms the proportion of the first to the third, is said to be compounded of the first to the second, and of the second to the third; therefore in reference to that Definition *Euclide* reduces the two proportions of the four sides of the Parallelograms *viz.* of BC to CG , and of DC to CE into three continued Terms, into K to L and L to M , and then he proves the proportion of the Parallelograms to be to one another as the first Term K is to the third M : which [in Def. 10. El. V.] is said to be a proportion compounded of K to L , and of L to M . But these proportions were made the same with the proportions of the sides of those Parallelograms; and therefore the Parallelograms are to one another in a compound proportion of their sides.

The like demonstration *Euclide* uses also in Numbers, without any mention of the multiplication of the Quantities of proportions into one another.

For 1st. in Prop. 2^d. and 3^d. El. VII. he shews how to find the greatest common measure of any given numbers.

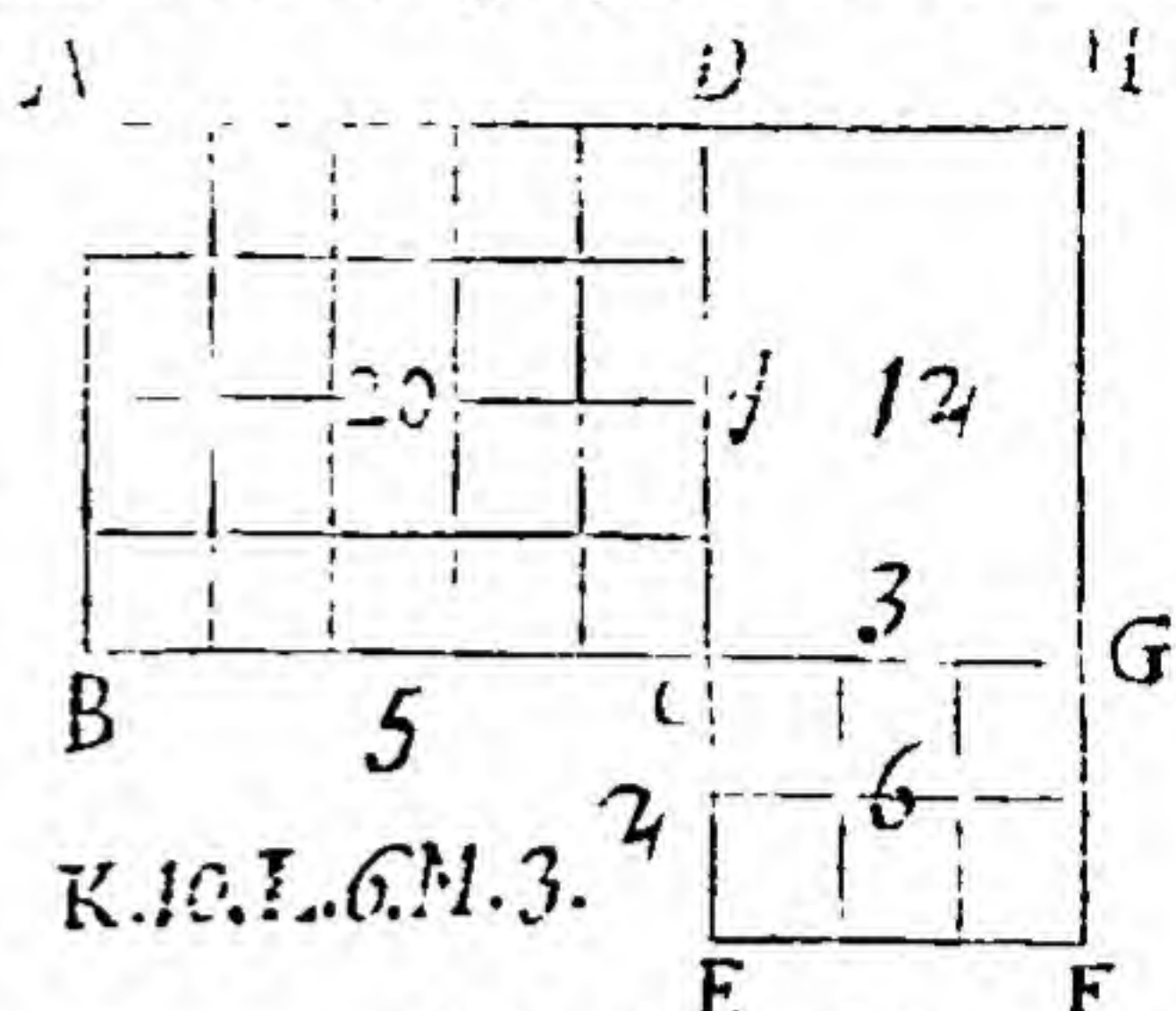
II. By the help of the greatest common measure he reduces any given numbers into the least numbers, having the same proportion with them. Prop. 37th. El. VII.

III. Proportions being given in their least numbers, he shews how to bring them into the least continued numbers in the same proportions. Prop. 4th. El. VIII.

IV. In the next following Prop. 5th. El. VIII. He demonstrates Plane numbers to have a proportion compounded of their sides, by bringing the proportions of the sides into the least continued numbers in the same proportions, the demon-

strations both in magnitudes and numbers proceeding upon the same grounds, and just after the same manner.

As the proportion of the Plane number 20 to the Plane number 6 (which is triple sesquialterial $3:1$) is proved to be compounded of the proportions of their sides, of 5 to 3, of BC to CG; and of 4 to 2, of DC to CE. Here therefore the proportions of the sides 5 to 3, and of 4 to 2, are first reduced into the least continued numbers in the same proportions, that is, into 10 to 6, and 6 to 3, by Prop. 4th. El. VIII. and then the Plane 20 is proved to be to the Plane 6, as 10 to 3, that is $3:1$. But 10 to 3 is compounded of 10 to 6, and of 6 to 3 [Def. 10th. El. V.], which proportions are the same with those of the sides 5 to 3, and 4 to 2, therefore the Plane number 20 is to the Plane number 6, in a proportion compounded of the sides 5 to 3, and 4 to 2. So that neither in magnitudes, nor numbers is there any use of Def. 5th. El. VI. in the demonstrations concerning compound proportions.



Moreover 'tis to be observed from the 19th. 20th. and 23^d. Propositions of this VI. Element, that compound proportions are exprest two manner of ways. For in this 23^d. Prop. because equiangled Parallelograms are not always like Figures, having their sides about the equal angles proportional; therefore they are proved in general to have to one another a proportion compounded of their sides, that is, in reality a proportion compounded of length to length, and of breadth to breadth, according to their two dimensions in length and breadth.

But because in Prop. 19th. and 20th. the Triangles, Parallelograms, and other Quadrilateral, and Multilateral Figures, are put for like Figures, where length is to length, and breadth to breadth in the same proportion; therefore they are proved to be to one another in a Duplicate proportion of their Homologal sides, which is in effect to say, that they have to one another a proportion compounded of two equal proportions, according to length and breadth.

Now whether the Figures be put like, as in Prop. 19th. and 20th. or unlike, as in Prop. 23^d. yet note, that one Plane is to an other Plane in a proportion ever compounded of length to length, and of breadth to breadth: and demonstrated to be so, by reducing these two proportions into three continued Terms, and proving Plane to be to Plane, as the first Term to the third; which by Def. 10th. El. 5. is said to be compounded of the proportions of the first to the second, and of the second to the third, and in equal proportions is called a Duplicate proportion of the first to the second.

In like manner a Solid is to a Solid in a proportion compounded of their three dimensions, according to length, breadth, and depth: And like Solids are one to an other in a Triplicate proportion of their Homologal sides. As from the eleventh Definition of the Fifth Element is demonstrated in Prop. 12th, 19th. El. VIII. Prop. 33^d. El. XI.

And thus much for a farther explication of the 10th. and 11th. Definitions of the Fifth Element: wherein we have had an occasion given from the supposititious fifth Definition El. VI. to shew what is Natural, what Artificial, and necessary for demonstration sake upon this Subject of compound proportions.

Moreover we are to observe in compound proportions, that in the 19th. and 20th. Propositions foregoing, *Euclid* demonstrates all like Figures, that is, all equi-angled Figures, which have their sides about their equal angles proportional [Def. 1. El. VI.] (whether Trilateral, Quadrilateral, or Multilateral) to have to one another a Duplicate proportion of their Homologal sides: and in this 23^d. Prop. all equiangled Parallelograms to have a proportion compounded of their sides: which is the first Proposition, where in express words *Euclid* names any proportion to be *compounded*. But the thing is the very same, and Composition of proportions alike in all these three Propositions.

For

For Planes are in quantity to be compared to Planes, according to their two dimensions of length and breadth: (One way of dimension being called length, and the Transverse called breadth) so that the proportion of Plane to Plane is truly compounded of two proportions, one of length to length, the other of breadth to breadth: and these two proportions are here in Prop. 19th, 20th, and 23^d. alike reduced (for demonstration sake) into *three continued Terms*, where the first compared to the second, and the second to the third, do answer to the two proportions of length to length, and of breadth to breadth; as upon this 23^d. Proposition *Commandinus*, and *Clavius* have demonstrated. And *Euclide* proves the proportion of these Planes to be to one another as the first Term is to the third, in a proportion compounded of the first to the second, and of the second to the third, that is, of length to length, and of breadth to breadth.

Now in the like Figures specified in the 19th, and 20th. Propositions, because length is to length as breadth to breadth in the same proportion; therefore these two equal proportions of length and breadth, being by *Euclide* reduced into *three continued Terms*, do make those continued Terms also *proportional* to one another, the first to the second, as the second to the third; and therefore when these like Figures are demonstrated to be to one another as the first Term is to the third, they are said to be in a Duplicate proportion of the first to the second [Def. 10th. El. V.], which Duplicate is a proportion compounded of the two equal proportions, that answer to their lengths and breadths, here now set forth in *three continued, and proportional Terms*. And like Parallelograms, as well as like Triangles, are to one another in a Duplicate proportion of their Homologal sides.

But because all *equiangled Parallelograms* are not always *like Figures* to have the proportions of their lengths and breadths, reducible into *three proportional Terms*; yet the proportions of their sides may be reduced into *three continued Terms*, representing the two proportions of their sides, which correspond with the proportions of their proper lengths and breadths; and the Parallelograms are proved to be to one another as the first Term is to the third: but can be only said in general to have a proportion compounded of the proportions of the first to the second, and of the second to the third; and not as in *like Parallelograms*, to have a Duplicate proportion of the first to the second.

For the meaning of this whole matter is, that if three, or more magnitudes be *continuedly* compared to one another, according to quantity (that is, how much the first contains of the second, then how much the second of the third, and how much the third of the fourth, &c.), and if these proportions be known, or prov'd to be all the same, making the Terms to be *successively proportionals* [as in Prop. 19th, and 20th. El. VI.] then the compound proportion of the first to the third, is said to be a Duplicate proportion of the first to the second, according to the express words of Def. 10th. El. V. But if the proportions be put as unknown, or as indifferent whether the same, or not the same (as in this 23^d. Proposition), then the proportion of the first to the third, or of the first to the fourth, &c. is in general said to be compounded of all the intermedial proportions.

Wherefore *Euclid's* demonstration of this 23^d. Proposition, as also of all the like compounded proportions is wholly grounded on the right understanding, and full extent of the 10th. Definition of the Fifth Element, as it hath been before explained.

Moreover we are to observe, that as in Def. 3^d. El. V. there is only defined proportion in magnitudes; yet the same stands for proportion in numbers, *mutatis mutandis*, without any new Definition given; so in the Definitions of Duplicate, and Triplicate proportions (names only proper to *equal proportions*, exposed in *continued Terms*) there is further to be understood *all other compound proportions* exposed in *continued Terms*, as plainly appears by *Euclid's* demonstrations, who nowhere owns any other Definition of a compound proportion, than what is comprised in Def. 10th. El. V.

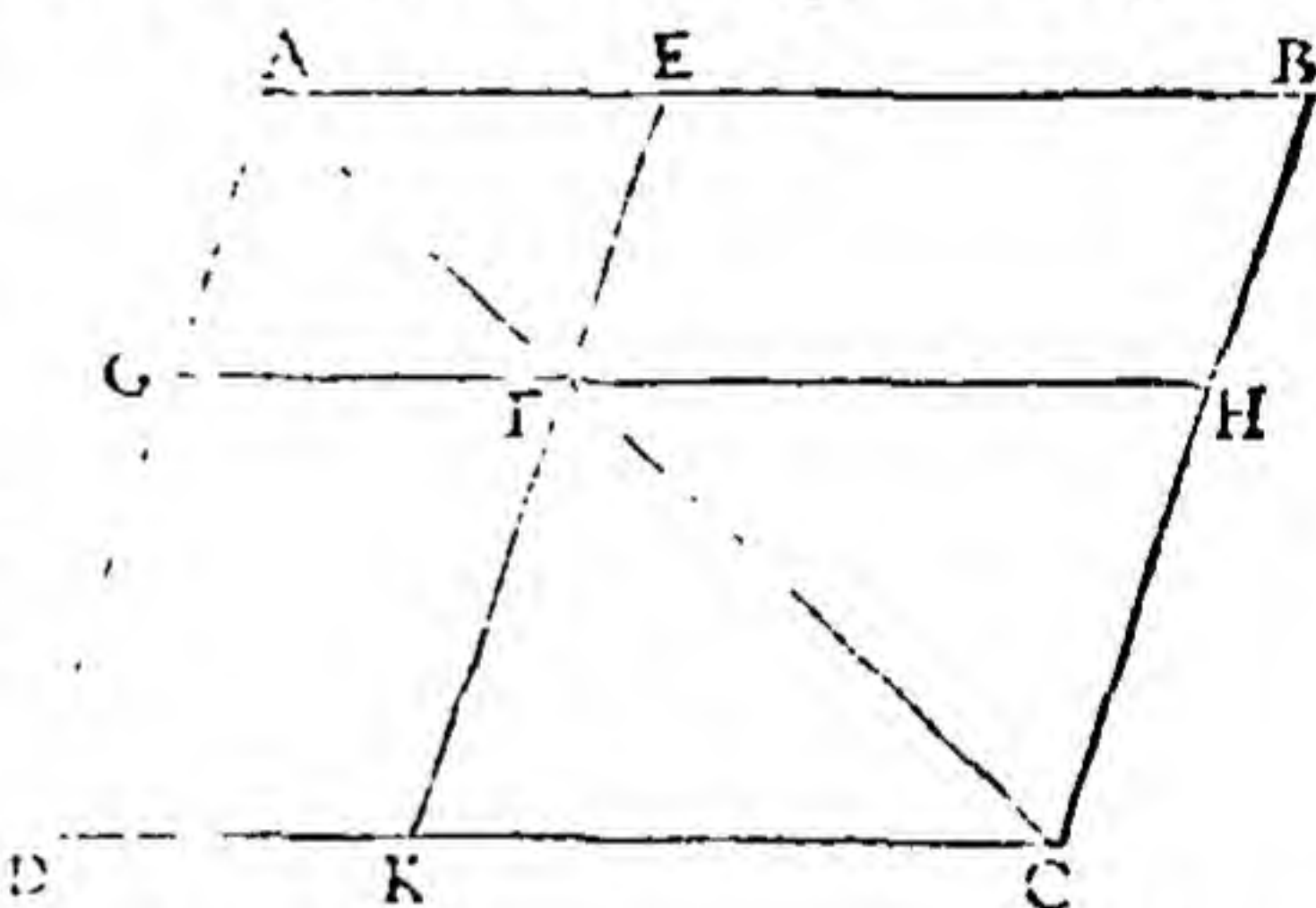
Thus the only four rugged passages met withal in these Elements are, I hope, made plain and smooth, even to the weakest Traveller in these Studies, and vindicated from the exceptions of some capricious Geometricians: Namely, *First*
Geometrical

Geometrical *Eplarmosis* used in Prop. 4th. El. I. Secondly, the 11th. Axiom, used in Prop. 29th. El. I. Thirdly and fourthly, the 5th, and 10th. Definitions of the Fifth Element concerning proportional magnitudes, and compounded proportions. If I be not censured for having on these matters too much enlarged.

PROPOSITION XXIV.

Of every Parallelogram, the Parallelograms about the Diameter are like to the whole, and to one another.

Let the Parallelogram be $ABCD$, and the Diameter AC : and about the Diameter AC , let the Parallelograms be EG , HK . I say, that each of the Parallelograms EG , HK , is like to the whole $ABCD$, and to one another. For because in the Triangle ABC to one of the sides BC is drawn EF parallel; therefore it is proportionally as BE to EA , so CF to FA [Prop. 2. El. VI.]. Again, because in the Triangle ACD to one of the sides CD , is drawn FG parallel; therefore it is proportionally as CF to FA , so DG to GA : but as CF to FA , so BE is prov'd to be unto EA : and therefore as BE to EA , so DG to GA [Prop. 11. El. V.]: and by Composition, as BA to EA , so DA to GA ; and alternately, as BA to DA , so EA to GA ; therefore of the Parallelograms $ABCD$, EG the sides about the common angle BAD , are proportional. And because GF is parallel to DC , therefore the angle AGF is equal to the angle ADC , and the angle GFA to the angle DCA [Prop. 29. El. I.]: also the angle DAC is common to the two Triangles ADC , AGF ; therefore the Triangle ADC , is equiangled to the Triangle AGF . By the same reason also the Triangle ABC , is equiangled to the Triangle AEF : wherefore the whole Parallelogram $ABCD$ is equiangled to the Parallelogram EG .



Proportionally therefore it is, as AD to DC , so AG to GF [Prop. 4. El. VI.], and as DC to CA , so GF to FA ; and as CA to CB , so FA to FE ; and moreover, as CB to BA , so FE to EA . And because it has been prov'd, that as DC to CA , so GF to FA ; and as CA to CB , so FA to FE : wherefore by equality, as DC to CB , so GF , to FE ; therefore of the Parallelograms $ABCD$, EG the sides about the equal angles are proportional: wherefore the Parallelogram $ABCD$, is like to the Parallelogram EG [Def. 1. El. VI.].

By the same reason also, the Parallelogram $ABCD$ is like to the Parallelogram HK ; therefore each of the Parallelograms EG , HK , is like to the Parallelogram $ABCD$. But Figures like to the same Figure, are like to one another [Prop. 21. El. VI.]; therefore the Parallelogram EG , is like to the Parallelogram HK .

Therefore of every Parallelogram, the Parallelograms about the Diameter,

Diameter, are like to the whole, and to one another. Which was to be demonstrated.

ANNOTATIONS.

From hence 'tis observable, that the Parallelograms about the Diameter, are like Figures having their sides to one another directly proportional, and the Complements are equal Parallelograms, having their sides reciprocally proportional to one another, Prop. 43^d. El. I. Prop. 14th. El. VI. Moreover either of the Complements is a mean proportional between the Parallelograms about the Diameter, by Prop. 1. El. VI. and Prop. 38th. El. I. which also are to one another in a Duplicate proportion of their Homologal sides. Prop. 20th. El. VI.

Peletarius hath very well advertised upon the excellency of this Diagram, *Hanc ego Figuram soleo vocare Mysticam. Ex ea enim, velut ex locupletissimo promptuario, innumerabiles exeunt demonstrationes, quod cum magna voluptate perspiciet, qui in re Geometrica serio se exercebit.*

PROPOSITION XXV.

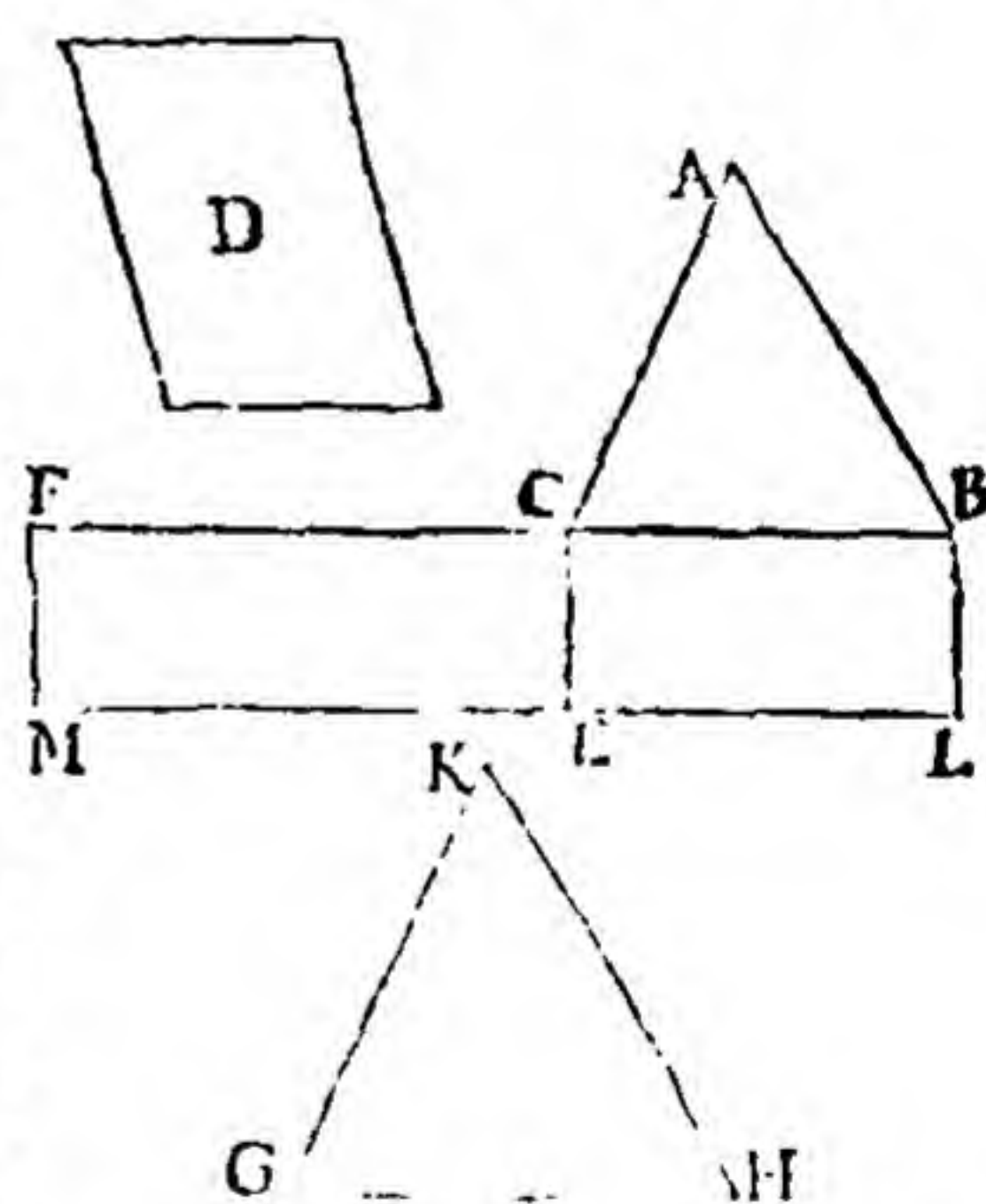
TO a given strait-lin'd Figure to constitute a like, and the same also equal to another given strait-lin'd Figure.

Let the given strait-lin'd Figure, to which a like is to be constituted be ABC , and that to which the same is to be equal let be D . It is required to ABC to constitute a like Figure, and the same equal to D . To the strait line BC let be apply'd a Parallelogram BE , equal to the Triangle ABC , and to CE the Parallelogram CM equal to D , in an angle FCE , which is equal to the angle CBL [Prop. 44. El. I.]; therefore BC is direct to CF [Prop. 14. El. I.], and LE to EM . And to BC , CF let be taken a mean proportional GH [Prop. 13. El. VI.], and upon GH let be described KGH like and alike situated to ABC [Prop. 18. El. VI.]. Now because it is as BC to GH , so GH to CF : and if three strait lines be proportional, it is as the first to the third, so the Figure upon the first to the Figure upon the second, like and alike describ'd; therefore as BC is to CF , so ABC is to KGH [Corol. 2. Prop. 20. El. VI.]. But also as BC is to CF , so the Parallelogram BE is to the Parallelogram EF ; therefore as ABC is to KGH , so the Parallelogram BE is to the Parallelogram EF [Prop. 11. El. V.]: wherefore alternately, as ABC is to the Parallelogram BE , so KGH is to the Parallelogram EF . But ABC is equal to the Parallelogram BE ; therefore also KGH is equal to the Parallelogram EF : and the Parallelogram EF is equal to D : wherefore also KGH is equal to D ; but KGH is like to ABC .

Therefore to the given strait-lin'd Figure ABC , there is constituted KGH like; and the same also equal to an other given strait-lin'd Figure. Which was to be done.

P p

PRO-



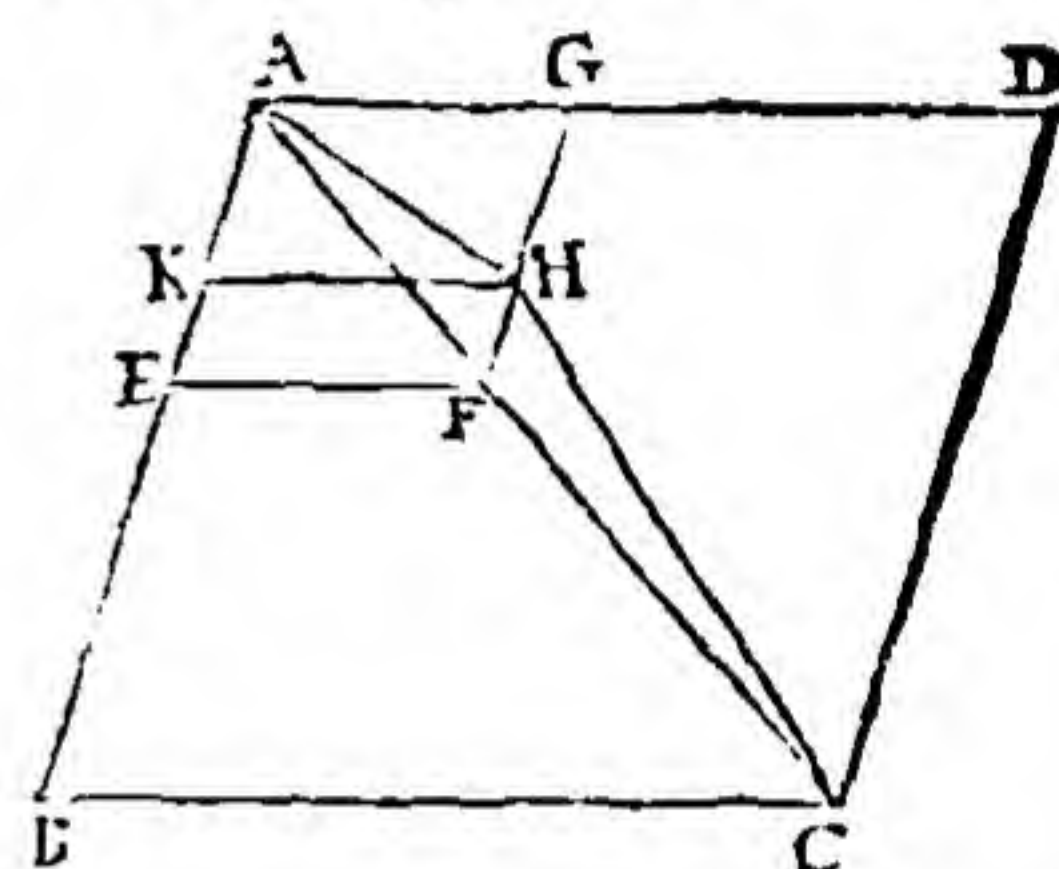
PROPOSITION XXVI.

IF from a Parallelogram be taken a Parallelogram like to the whole, and alike situated, having a common angle; it is about the same Diameter with the whole.

For from the Parallelogram $ABCD$, let be taken the Parallelogram AF , like to $ABCD$, and alike situated, having the angle DAB common. I say, that the Parallelogram $ABCD$ is about the same Diameter with the Parallelogram AF .

For if not, then if possible, let the Diameter of $ABCD$ be AHC , and by H to either of the lines AD, BC , let HK be drawn parallel.

Now forasmuch as the Parallelogram $ABCD$ is about the same Diameter with the Parallelogram KG [by Construction]; therefore $ABCD$ is like to KG [Prop. 24. El. VI.]: wherefore it is as DA to AB , so GA to AK [Def. 1. El. VI.]. But also for the likeness of the Parallelograms $ABCD, EG$ [by Supposition]; it is as DA to AB , so GA to AE ; therefore as GA to AE , so GA to AK [Prop. 11. El. V.]: wherefore GA hath the same proportion to each of the lines AK, AE ; therefore AK is equal to AE [Prop. 9. El. V.], the less to the greater, which is impossible; therefore $ABCD$ is not about the same Diameter with AH : wherefore the Parallelogram $ABCD$ is about the same Diameter with the Parallelogram AF .



If therefore from a Parallelogram be taken a Parallelogram like to the whole, and alike situated, having a common angle; it is about the same Diameter with the whole. Which was to be demonstrated.

PROPOSITION XXVII.

OF all Parallelogram spaces applyed to the same strait line, and deficient by Parallelogram Figures, like and alike situated to that which is described on the half line: The greatest is that Parallelogram, which is applyed to the half, being like to the defect (described on the other half.)

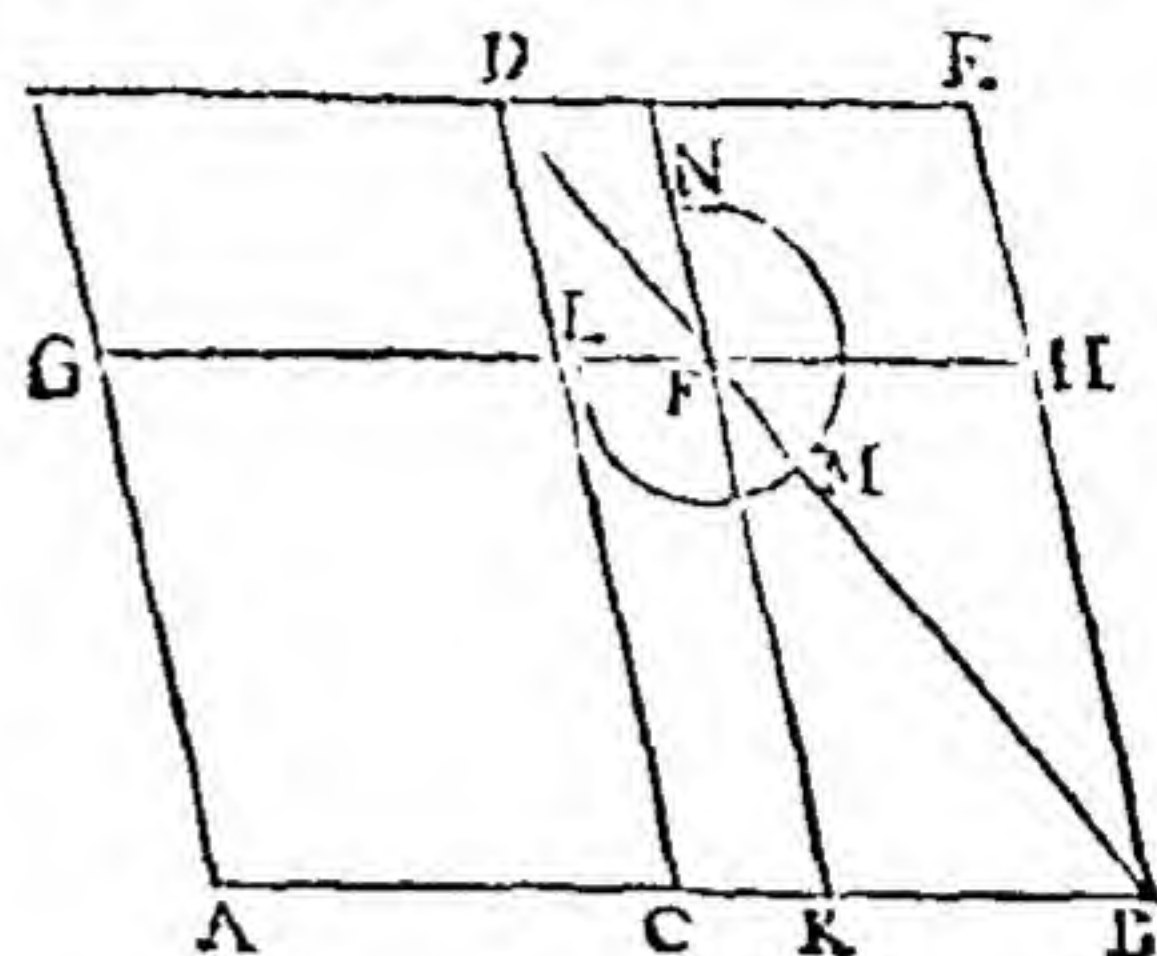
Let the strait line be AB , and let it be cut into halves in C ; and to AB let be applyed a Parallelogram AD , described on AC an half of AB , and deficient by the Parallelogram Figure DB , described on CB the other half of AB , like and alike situated to AD .

I say, that of all Parallelogram spaces applyed to AB , and deficient by Parallelogram Figures, like and alike situated to the defect DB , the greatest is AD .

For to the strait line AB , let be applied a Parallelogram AF , deficient by the Parallelogram Figure FB , like, and alike situated to DB . I say, that AD is greater than AF .

For because the Parallelogram DB is like to the Parallelogram FB ; therefore they are about the same Diameter [Prop. 26. El. VI.], let be drawn their Diameter DB : and the Scheme be completed.

Now forasmuch as CF is equal to FE [Prop. 41. El. I.], let FB be added in common; therefore the whole CH is equal to the whole KE : but CH is equal to CG , because AC is equal to CB [Prop. 36. El. I.]: and therefore CG is equal to KE . Let CF be added in common; therefore the whole AF is equal to the Gnomon LMN , so that also the Parallelogram DB , that is, AD is greater than the Parallelogram AF .



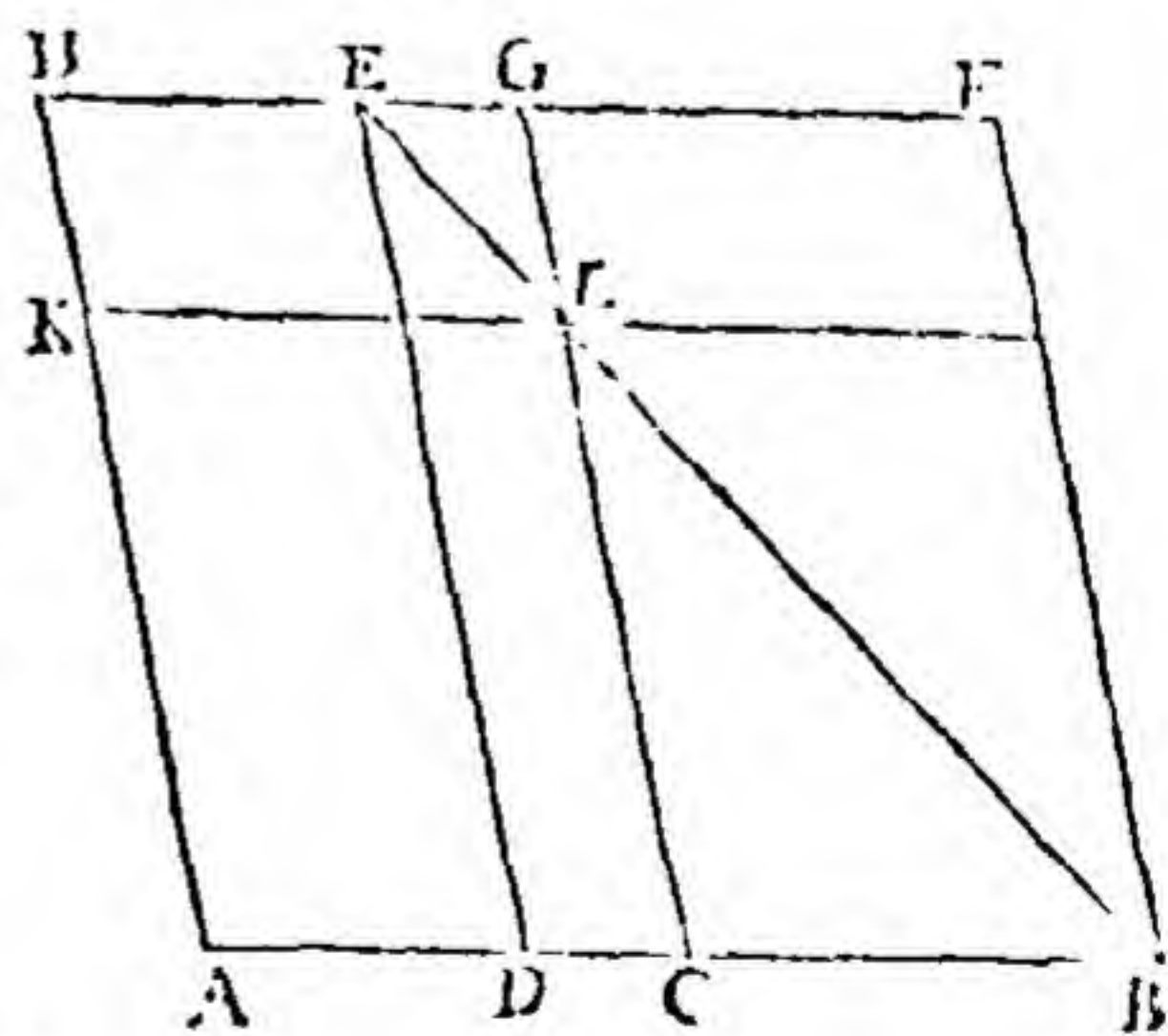
Therefore of all Parallelograms applied to the same strait line, and deficient by Parallelogram Figures, like and alike situated to that, which is described on the half line: The greatest applied Parallelogram is that described on the half, being like to the defect.

“By this full and general Conclusion *Euclide* seems to have “ended his demonstration of this Proposition: but because it admits of two Cases: and that only one of them is hitherto demonstrated, wherein the applied Parallelogram is described on such “a part of AB , as on AK , which is greater than the half AC ; therefore still there remains to consider the same, where the Parallelogram is described on a part less than the half AC , as here on AD , “which (tho’ not absolutely necessary) was likely supplied by *Theon*, “or some ancient *Scholiast*, as followeth.

Again, let AB be cut into halves in C ; and the applied Parallelogram be AL , described on the half AC , deficient by the Parallelogram LB . And again to the line AB , let be applied the Parallelogram AE , deficient by the Parallelogram EB , like, and alike situated to the Parallelogram LB , described on CB the half of AB .

I say, that the applied Parallelogram AE , described on the half of AB , is greater than the Parallelogram AL .

For because the Parallelogram EB is like to the Parallelogram LB ; therefore they are about the same Diameter. Let the Diameter be EB ; and the Scheme be described.

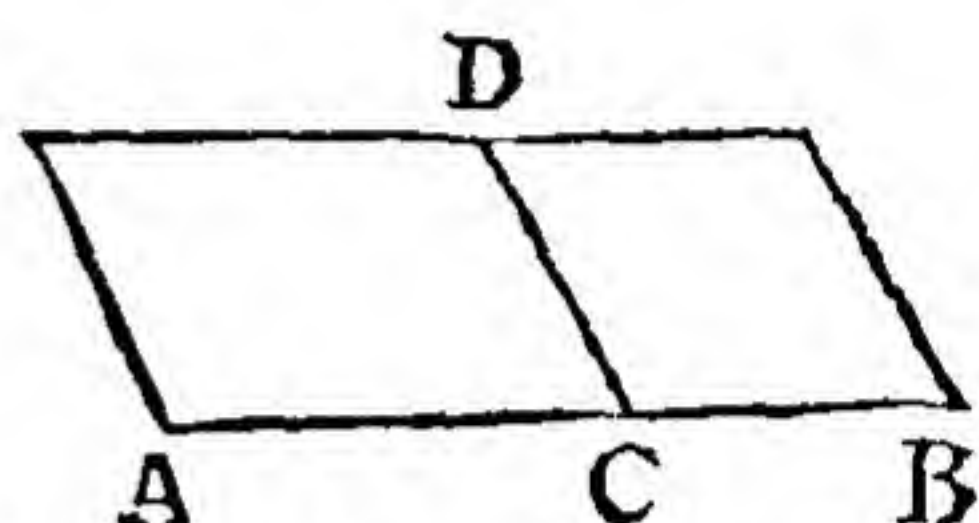
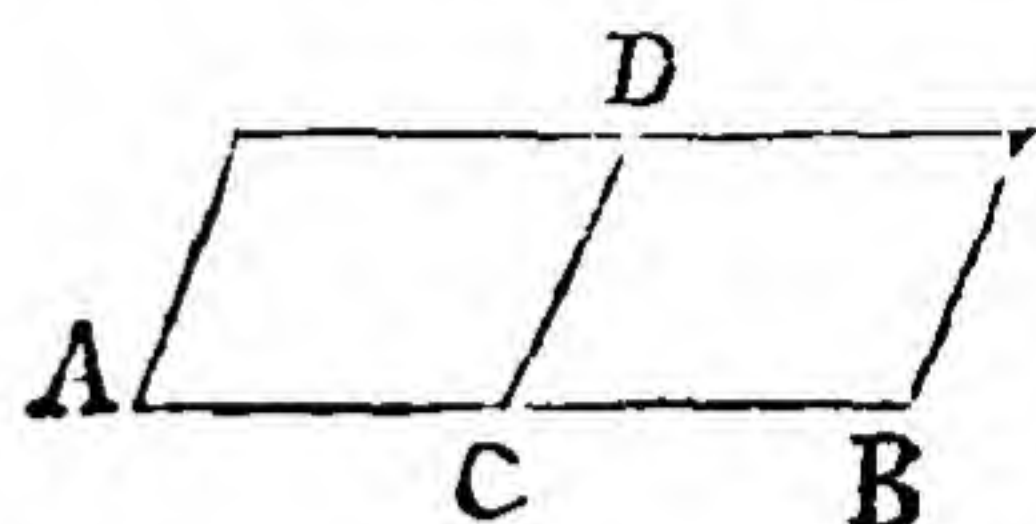


Now forasmuch as LF is equal to LH [Prop. 36. El. I.], for FG is equal to GH ; therefore FL is greater than EK . But FL is equal to LD [Prop. 43. El. I.]; therefore LD is greater than

than EK . Add in common KD ; therefore the whole AL is greater than the whole AE . Which was to be demonstrated.

ANNOTATIONS.

If a strait line be cut into two Segments equal, or unequal, as AB into AC , CB : and to either of them, as to AC be applyed any Parallelogram as AD ; then is AD said to be deficient by the Parallelogram DB , which in the same parallels is equiangled to it, and conjoynd by a common side, as DC , and described on the other Segment CB . Now this Parallelogram DB is called the defect, for that by so much the Parallelogram AD is deficient in completing the Parallelogram space on the whole line AB . And on the contrary, if to the Segment BC be applyed the Parallelogram BD , it is said to be deficient by the Parallelogram defect DA described on the other Segment AC , of the strait line AB . And note, that as in Prop. 43^d. El. I. and Prop. 24th. El. VI. a Parallelogram is divided into four Parallelograms, two whereof are said to be about the Diameter, and two are called Complements; so in this Proposition a Parallelogram is in a manner divided into two Parallelograms, and one of them is said to be deficient by the other, which is called the defect. And this distinction rightly observed, makes clear *Euclid's* expression of this Proposition, that hath seemed to some of our Modern Geometricians to be obscurely worded. But a Parallelogram *deficient*, and the *defect* is as properly here spoken, and as plainly to be apprehended, as Parallelograms about the Diameter, and Parallelograms the Complements.



PROPOSITION XXVIII.

Unto a given strait line to apply a Parallelogram equal to a given Rectilineal space, deficient by a Parallelogram Figure like to a given Parallelogram.

Now the given Rectilineal space, to which the applied Parallelogram is to be equal, ought not to be greater than that applied Parallelogram, which is described on the half line: both defects being like to the given Parallelogram, namely the defect of the Parallelogram described on the half line, and the defect of the Parallelogram required to be applied, equal to the given Rectilineal space.

Let the given strait line be AB , and the given Rectilineal space, equal to which a Parallelogram is to be applied to AB , let be c , the same being not greater than a Parallelogram applied to the half of AB : the defects of these Parallelograms being alike; and the Parallelogram to which the defect ought to be like, let be D .

It is required unto the given strait line AB to apply a Parallelogram equal to the given Rectilineal space c , deficient by a Parallelogram Figure like to D .

Let AB be cut into halves in the point E , and on EB let be described EFG , like and alike situated to D [Prop. 18. El. VI.]; and let be completed the Parallelogram AG .

Now AG is either equal to c , or greater than it, by the determination.

If AG be equal to c ; then that is done which was required. For to the given strait line AB is applied the Parallelogram AG , equal to the given Rectilineal space c , deficient by the Parallelogram Figure GB , being like to D .

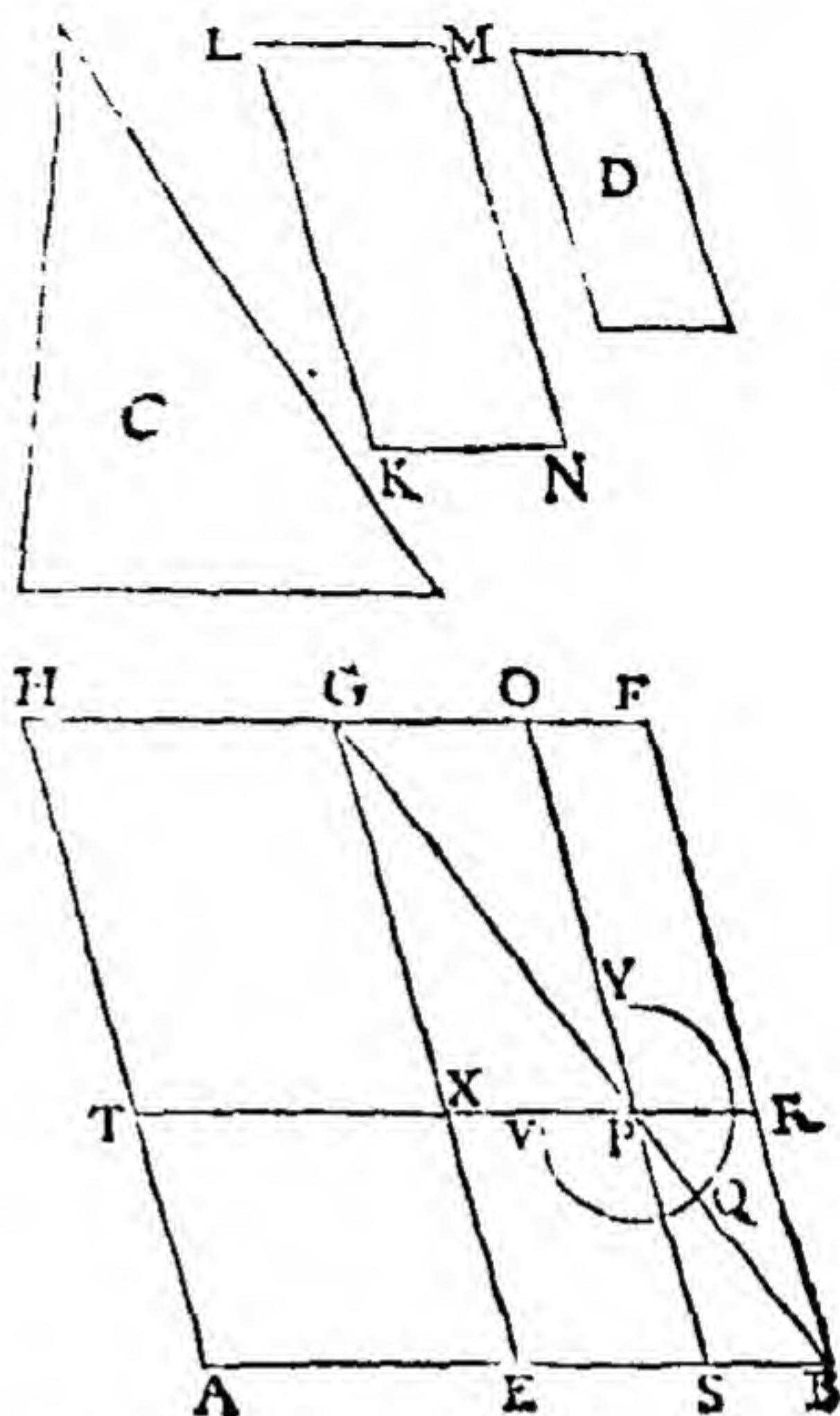
But if AG be not equal to c ; then it is greater: but AG is equal to GB [Prop. 36.El.I.]; therefore GB is greater than c .

Now by how much GB is greater than c , let a Parallelogram equal to that excess be constituted $KLMN$, the same also like and alike situated to D [Prop. 25.El.VI.]. But D is like to GB : wherefore also $KLMN$ is like to GB ; and here let KL be Homologal to EG , and LM to GF .

And forasmuch as GB is equal to c , and KM together; therefore GB is greater than KM ; and therefore the side EG is greater than the side KL , and GF than LM . Let now GK be put equal to KL , and GO to LM [by Prop. 3. El.I.]; and let be compleated the Parallelogram $XGOP$; therefore the Parallelogram GP is equal and like to the Parallelogram KM ; but KM is like to GB : wherefore also GP is like to GB [Prop. 21.El.VI.]; therefore the Parallelogram GP is about the same Diameter with the whole GB . Let their Diameter be GPB and the Scheme be described.

Now forasmuch as GB is equal to c , and KM together, and that GP is equal to KM ; therefore the remaining Gnomon VQY is equal to the remaining Rectilineal space c . And because OR is equal to XS [Prop. 43. El. I.], let PB be added in common; therefore the whole OB is equal to the whole XB . But XB is equal to TE , because the side AE is equal to the side EB [Prop. 36.El.I.]: wherefore also TE is equal to OB . Let XS be added in common; therefore the whole TS is equal to the whole Gnomon VQY : but the Gnomon VQY has been proved equal to c : therefore also TS is equal to c .

Wherefore to the given strait line AB is applied the Parallelogram TS equal to the given Rectilineal space c , deficient by the Parallelogram figure PB , which is like to the given Parallelogram D , for that PB is like to GP . Which was to be done.



logram figure PO , which is like to D ; for that EL is like to PO [Prop. 24. El. VI.]. Which was to be done.

ANNOTATIONS.

These two Propositions the 28th, and 29th, are of admirable use in the Tenth Element of *Euclide*, and the *Conic* Elements of *Apollonius*, where the deficient, or *Elliptical* application made in Prop. 28th, gave occasion for the name of that Conic Section called *ELLIPSIS* in Prop. 13th. El. I. of *Apollonius*, and the exceeding, or *Hyperbolical* application made in Prop. 29th, occasioned the name of that Section called *HYPERBOLA*, in Prop. the 12th, of the same Conic Element. When therefore *Tacquet* so rashly rejects these two Propositions, as of little, or no use; 'tis manifest that he had then made but a small progress in Geometry. And thirdly, the Conic Section called *PARABOLA*, we have before noted at Prop. 44th. El. I. to be so named from the exact application, or *Parabolism* of a given Rectilineal space to the entire, and whole given line precisely, neither deficient, or exceeding. As we find it in Prop. 11th. El. I. of *Apollonius*.

PROPOSITION XXX.

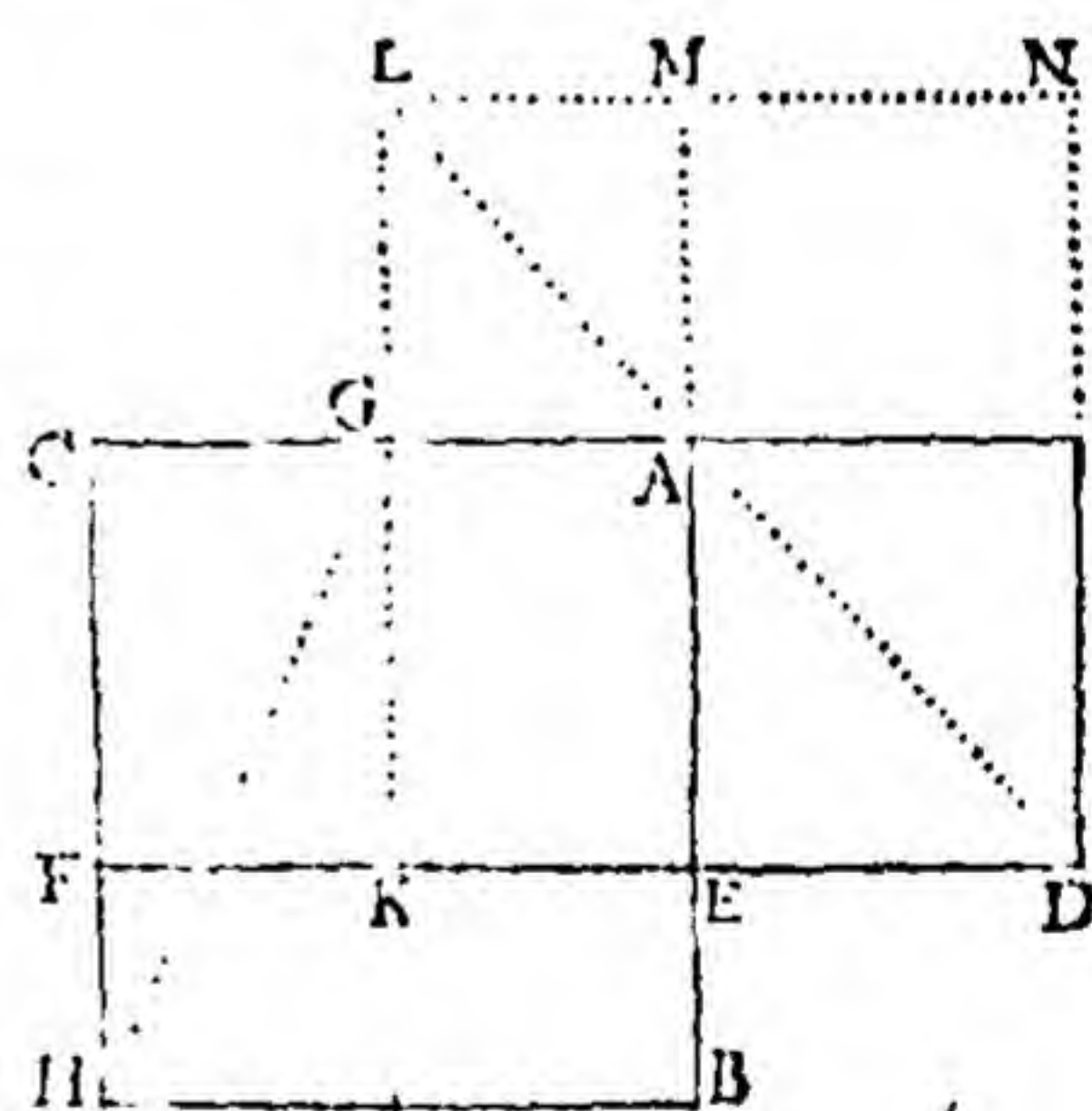
TO cut a given finite strait line in Extreme and Mean proportion.

Let the given finite strait line be AB . It is required to cut the strait line AB in extreme and mean proportion. On AB let be described the Square BC [by Prop. 46. El. I.]: and to AC let be applied the Parallelogram CD equal to BC , exceeding in Figure by AD , like to BC [Prop. 29. El. VI.]. Now BC is a Square; therefore also AD is a Square.

And forasmuch as BC is equal to CD ; let CE common be taken away; therefore the remainder BF is equal to the remainder AD . But it is also equiangular to the same; therefore the sides of BF , AD about the equal angles are reciprocally proportional [Prop. 14. El. VI.]: wherefore as FE to ED , so AE to EB . But FE is equal to AC [Prop. 34. El. I.], that is, to AB ; and ED to AE :

therefore as AB to AE , so AE to EB . But AB is greater than AE : wherefore also AE is greater than EB [Prop. 14. El. V.].

Therefore the strait line AB is cut in extreme and mean proportion in the point E ; and the greater Segment is AE . Which was to be done.



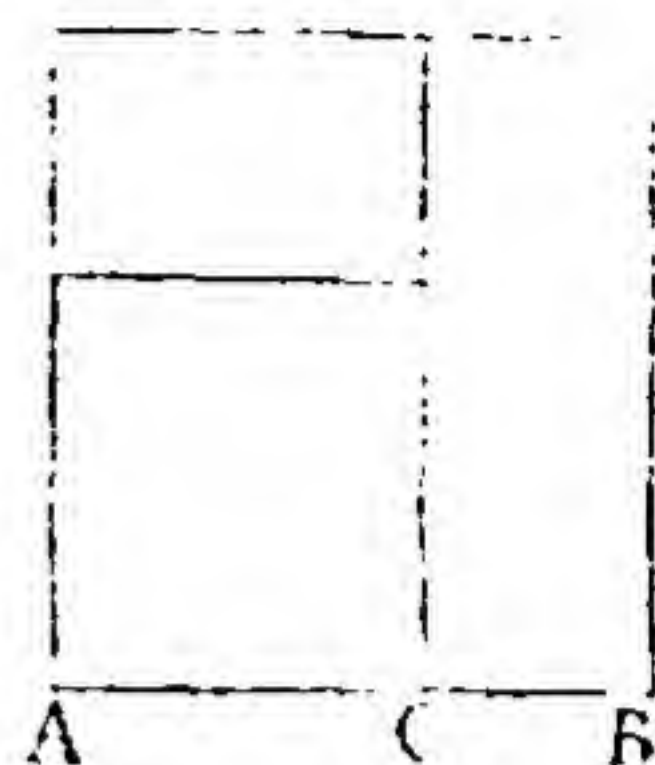
Other-

Otherwise.

Let the given strait line be AB . It is required to cut the strait line AB in extreme and mean proportion.

Let AB be cut in the point c , so that the Rectangle under AB, BC be equal to the Square of AC [by Prop. 11. El. II.].

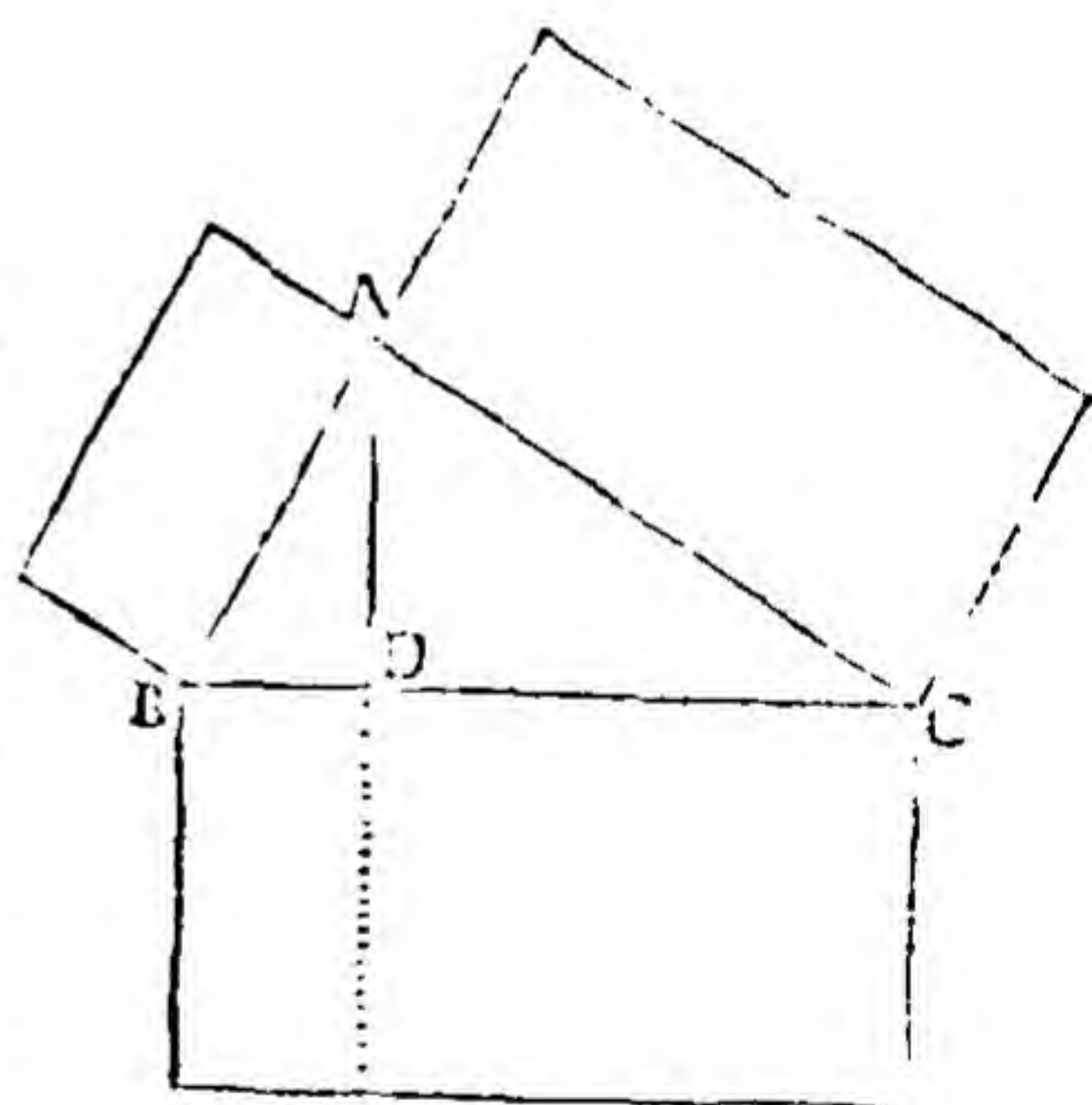
Now forasmuch as the Rectangle under AB, BC is equal to the Square of AC ; therefore as AB is to AC , so AC is to CB [Prop. 17. El. VI.]; AB therefore is cut in extreme and mean proportion in the point c . Which was to be done.



PROPOSITION XXXI.

IN Right-angled Triangles, the Figure constituted on the side subtending the Right angle is equal to the Figures constituted on the sides containing the Right angle, they being like and alike described.

Let the Right-angled Triangle be ABC , having the right angle BAC . I say, that the Figure upon BC is equal to the Figures on BA, AC , like and alike described. Let be drawn the perpendicular AD . Now forasmuch as in the Triangle ABC , from the right angle at A , is drawn to the base BC the perpendicular AD ; therefore the Triangles ABD, ADC , at the perpendicular are like to the whole ABC , and to one another [Prop. 8. El. VI.]. And because the Triangle ABC is like to the Triangle ABD ; therefore as CB is to BA , so AB is to BD : and because when three strait lines are proportional, it is as the first to the third, so the Figure on the first to the Figure on the second, like and alike described [Corol. 2. Prop. 20. El. VI.]; therefore as CB to BD , so the Figure on CB to the Figure on BA , like and alike described.



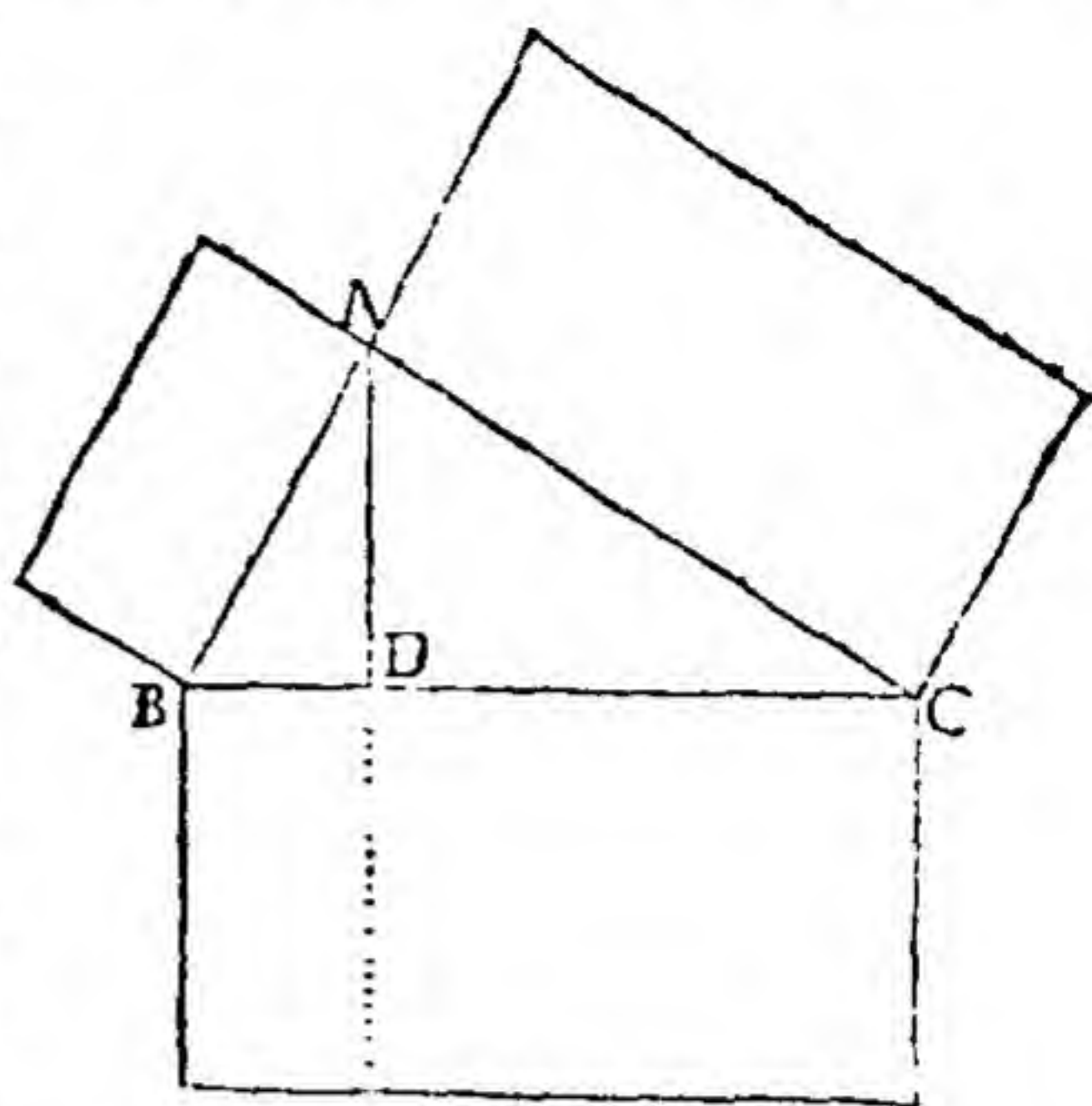
By the same reason also, as BC to CD , so the Figure on BC to the Figure on CA . So that also as BC to BD, DC , so the Figure on BC to the Figures on BA, AC , like and alike described. But BC is equal to BD, DC ; therefore the Figure on BC is equal to the Figures on BA, AC .

Therefore in Right-angled Triangles the Figure on the side subtending the right angle, is equal to the Figures on the sides containing the right angle, they being like and alike described. Which was to be demonstrated.

Other-

Otherwise.

Because like Figures are in a duplicate proportion of their Homologal sides [Prop. 20. El. VI.]; therefore the figure on BC hath to the Figure on BA , a duplicate proportion of that which BC hath to BA . But the Square of BC , hath to the Square of BA a duplicate proportion of that which BC hath to BA ; therefore also as the Figure on BC to the Figure on BA , so the Square of BC to the Square of BA [Prop. 11. El. V.]. By the same reason also as the Figure on BC to the Figure on CA , so the Square of BC to the Square of CA : so that also as the Figure on BC to the Figures on BA, AC , so the Square of BC to the Squares of BA, AC . But the Square of BC is equal to the Squares of BA, AC [Prop. 47. El. I.]; therefore the Figure on BC is equal to the Figures on BA, AC , like and alike described. Which was to be demonstrated.

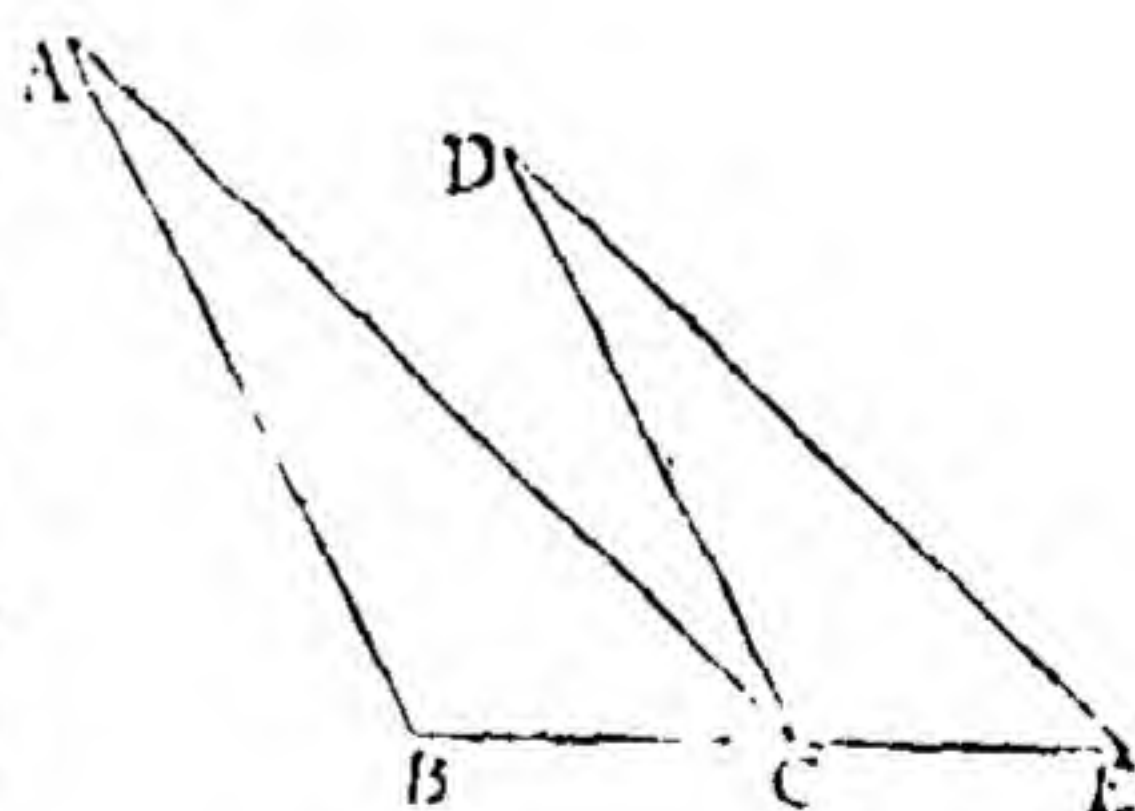


PROPOSITION XXXII.

If two Triangles having two sides proportional to two sides, be set together at one angle; so that also their Homologal sides be parallel: the remaining sides of the Triangles shall be directly situated to one another.

Let the two Triangles be ABC, DCE , having the two sides AB, AC , proportional to the two sides DC, DE , that is, as AB to AC , so DC to DE ; and let AB be parallel to DC , and AC to DE . I say, that BC is directly situated to CE .

Forasmuch as AB is parallel to DC , and on them falls the strait line AC ; therefore the alternate angles BAC, ACD , are equal to one another [Prop. 29. El. I.]. By the same reason also the angle CDE is equal to the angle ACD : so that also BAC is equal to CDE . And because there are two Triangles ABC, DCE , having one angle at A equal to one angle at D : and about the equal angles the sides proportional, as BA to AC , so CD to DE ; therefore the Triangle ABC is equiangled to the Triangle DCE [Prop. 6. El. VI.]: wherefore the angle ABC is equal to the angle DCE . But also the angle ACD has been prov'd equal to the angle BAC ; therefore the whole angle ACE is equal to the two angles ABC, BAC . Let the angle ACB be added in common; therefore the angles ACE, ACB ,

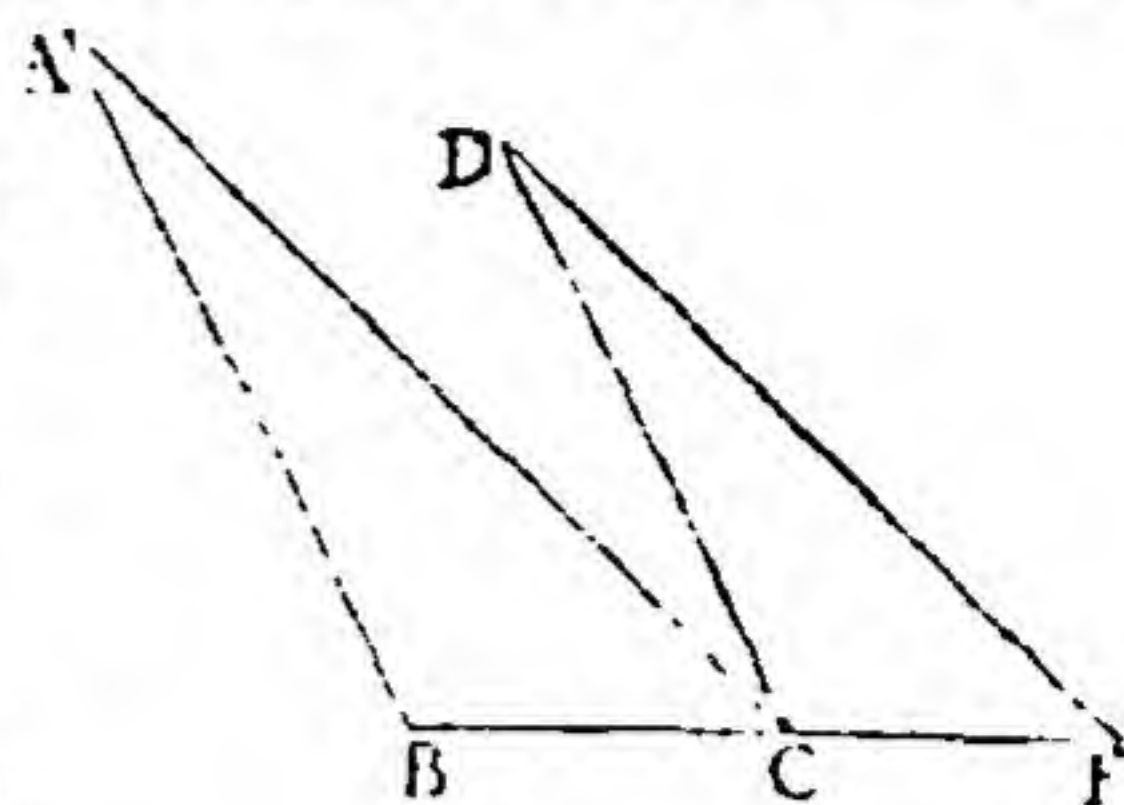


Qq

are

are equal to the angles BAC, ACB, CBA . But the angles BAC, ACB, CBA , are equal to two Right [Prop. 32. El. I.]; therefore also the angles ACE, ACB , are equal to two Right. Now to a certain strait line AC , and to a point in the same c , two strait lines BC, CE , not lying the same way make the consequent angles ACE, ACB , equal to two Right; therefore BC is directly to CE [Prop. 14. El. I.].

If therefore two Triangles having two sides proportional to two sides, be set together at one angle, so that their Homologal sides be parallel: the remaining sides of the Triangles shall be directly situated to one another. Which was to be demonstrated.



PROPOSITION XXXIII.

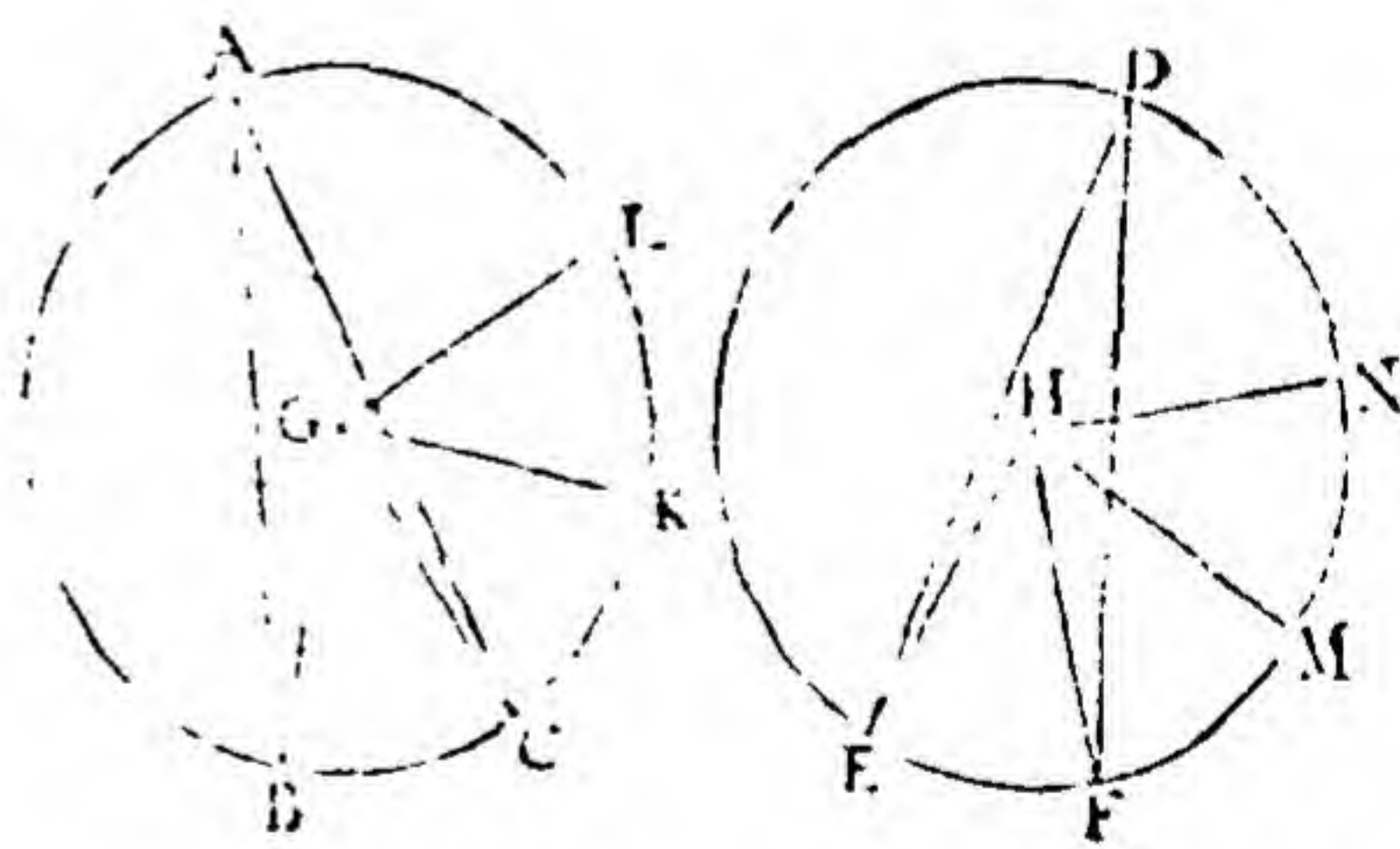
In equal Circles, the angles have the same proportion with the Circumferences on which they insist: whether the insisting angles be at the Centers, or at the Circumferences.

AND MOREOVER ALSO THE SECTORS (when constituted at the Centers).

Let the equal Circles be ABC, DEF , and at their Centers G, H , let the angles be BGC, EHF : and at the Circumferences, let the angles be BAC, EDF . I say, that as the circumference BC is to the circumference EF , so is the angle BGC to the angle EHF ; and the angle BAC to the angle EDF : And moreover the Sector BGC , to the Sector EHF .

For to the circumference BC , let in order be put equals how many soever, CK, KL : and to the circumference EF , let be put also equals how many soever, FM, MN , and let be joyned GK, GL, HM, HN .

Now forasmuch as the circumferences BC, CK, KL , are equal to one another; therefore the angles BGC, CGK, KGL , are also equal to one another [Prop. 27. El. III.]: wherefore Quotuple the circumference BL is of the circumference BC , Totuple is the angle BGL of the angle BGC . By the same reason Quotuple the circumference EN is of the circumference EF , Totuple is the angle EHN of the angle EHF .



If now the circumference BL be equal to the circumference EN , the angle BGL is also equal to the angle EHN [Prop. 27. El. III.]; and therefore if the circumference BL be greater than the circum-

5

ference

ference EN , the angle BGL is also greater than the angle EHN : and if less, 'tis less.

There being then four magnitudes, the two circumferences BC , EF , and the two angles BGC , EHF : and of the circumference BC , and of the angle BGC are taken equimultiples, the circumference BL , and the angle BGL . Also of the circumference EF , and of the angle EHF , are taken equimultiples, the circumference EN , and the angle EHN . And it is prov'd, that if the circumference BL exceeds the circumference EN , the angle BGL does also exceed the angle EHN ; and if equal, 'tis equal; and if less 'tis less: therefore as the circumference BC is to the circumference EF , so the angle BGC is to the angle EHF [Def. 5. El. V.]. But as the angle BGC is to the angle EHF , so the angle BAC is to the angle EDF [Prop. 15. El. V.]: for each is the double of each [Prop. 20. El. III.].

And therefore as the circumference BC is to the circumference EF , so the angle BGC is to the angle EHF , and the angle BAC to the angle EDF .

Therefore in equal Circles, the angles have the same proportion with the circumferences on which they insist, whether the insisting angles be at the Centers, or at the circumferences. Which was to be demonstrated.

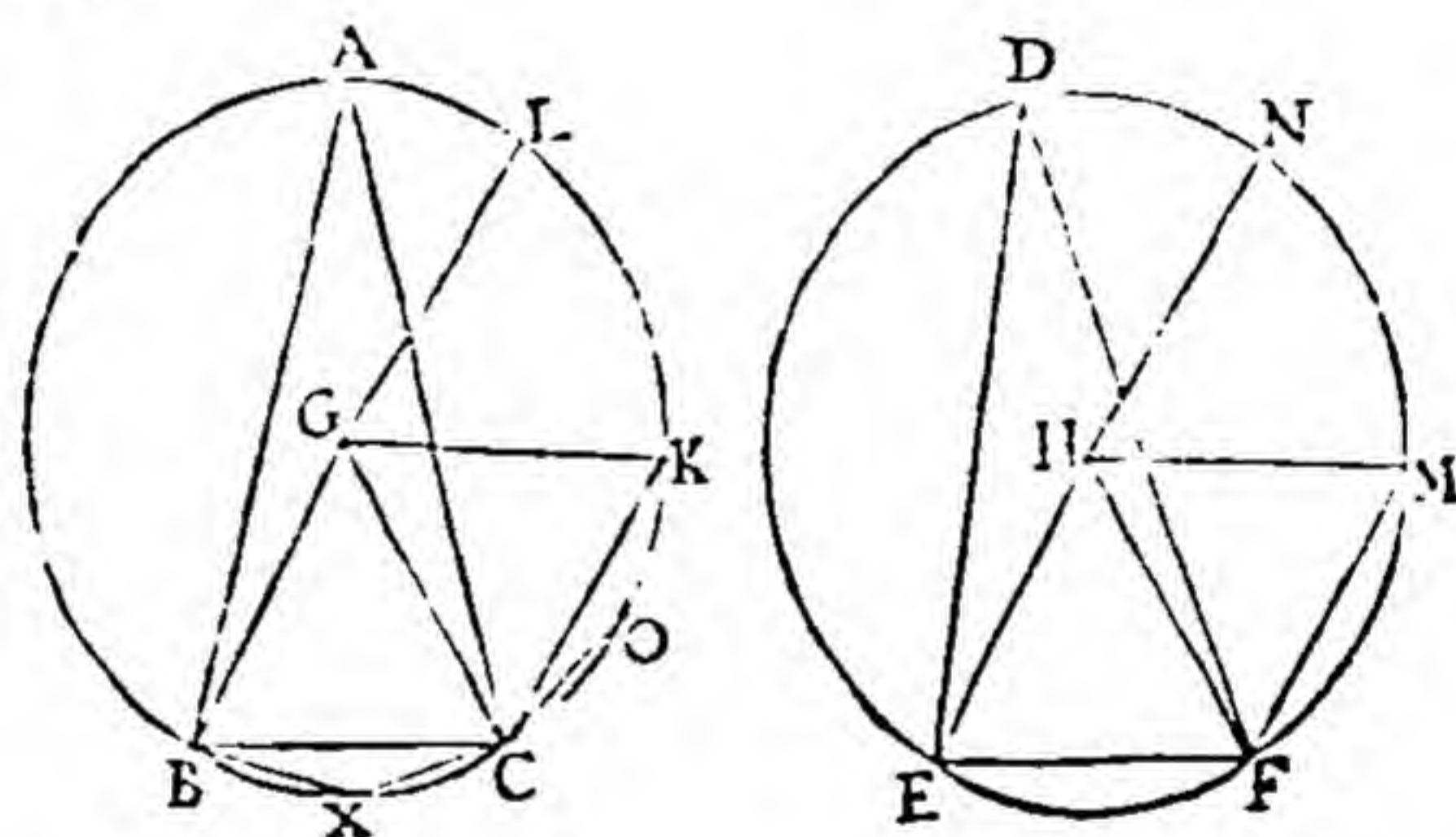
Here *Euclide* ends the last Proposition of his Sixth Element of Geometry.

T H E
A D D I T A M E N T
 O f *T H E O N*,
 And moreover also the Sectors,

That is, they have the same proportion with the Circumferences, on which they insist.

I say also, that as the circumference BC is to the circumference EF , so the Sector BGC is to the Sector EHF . For let be joyned BC , CK , and in the circumferences BC , CK , the points x , o , being taken, let Bx , xC , Co , oK , be also joyned.

Now forasmuch as the two lines BG , GC , are equal to the two lines CG , GK , and they contain equal angles; therefore the base BC is equal to the base CK , and the Triangle GBC to the Triangle GCK [Prop. 4. El. I.].



And because the circumference BC is equal to the circumference CK ; therefore the remaining circumference BAC compleating the whole Circle ABC , is equal to the remaining circumference KAC , compleating the same Circle: so that also the angle BxC is equal to the angle CoK [Prop. 27. El. III.]; therefore the Segment BxC is like to the Segment CoK [Def. 10. El. III.], and they are upon equal strait lines BC , CK : but like Segments of Circles upon equal strait lines are equal to one another [Prop. 24. El. III.]; therefore the Segment BxC is equal to the Segment CoK : and also the Triangle BGC is equal to the Triangle CGK ; therefore the whole Sector BGC is equal to the whole Sector CGK .

By the same reason also the Sector GKL is equal to each of the Sectors GKC , GCB ; therefore the three Sectors BGC , CGK , KGL , are equal to one another. And by the same reason the Sectors HEF , HFM , HMN , are equal to one another; therefore Quotuple the circumference LB is of the circumference BC , Totuple is the Sector GBL of the Sector GBC . By the same reason also Quotuple the circumference NE is of the circumference EF , Totuple is the Sector HEN of the Sector HEF .

Now if the circumference BL be equal to the circumference EN , the Sector BGL is also equal to the Sector HEN , and if the circumference BL exceeds the circumference EN , the Sector BGL does also exceed the Sector HEN ; and if less, 'tis less.

There being then four magnitudes, the two circumferences BC , EF , and the two Sectors GBC , HEF , and of the circumference BC , and of the Sector GBC are taken equimultiples, the circumference BL , and the Sector GBL : also of the circumference EF and of the Sector HEF , are taken equimultiples, the circumference EN and the Sector HEN . And it is prov'd, that if the circumference BL exceeds the circumference EN , the Sector BGL does also exceed the Sector HEN ; and if equal, 'tis equal; and if less, 'tis less; therefore as the circumference BC is to the circumference EF , so the Sector GBC is to the Sector HEF .

Corollary.

And it is manifest, that as the Sector is to the Sector, so also the angle is to the angle [Prop. II. El. V.].

ANNOTATIONS.

Theon in his Commentaries on *Ptolemy's Magna Syntaxis*, takes notice of this Additament he made to *Euclid's* last Proposition of the Sixth Element. Here therefore he says, ἐπὶ δὲ καὶ οἱ τοιαῦται, and moreover also the Sectors; which are the only words of *Theon* added to *Euclid's* Proposition. For what is subjoyned, ἀπὸ τοῦ τοῦ κέντρου συνιστάμενοι, when constituted at the Centers, must be some marginal note very absurdly put in, as supposing there were an other kind of Sectors, besides what are stated at the Center of a Circle, according to Def. 10. El. III. Indeed the Figures at the circumferences are not, as their angles are, in the same proportion with the Arches on which they insit: but these Figures are not called Sectors, neither have they any Note or Name in Geometry to give occasion for such a needless caution: An oversight too great for *Theon* to be guilty of.

Lastly, this Additament concerning Sectors mentioned by *Theon*, manifestly shews the error of *Zambertus* and others, who take the demonstrations of these Elements to have been all supplied by *Theon*. Yet it seems this mistake went so far, as that *Bovillus* an Eminent Geometrician of that time, writ a Select Treatise about it, to vindicate the Ancient Geometricians in this matter of confirming the Propositions by their own Demonstrations, and especially *Euclide* for his Elements, so methodically disposed, so plainly all along demonstrated, and in the end closed with those admirable Speculations on the five *Platonick* Bodies, thus celebrated in an Ancient *Greek* Epigram.

Σχήματα πάντα Πλάτων, à Πυθαγόρας σοφὸς ἔρε,
 Πυθαγόρας σοφὸς ἔρε, Πλάτων δ' ἀείδῃ ἐδίδαξε
 Εὐαλώς ἐπὶ τοῖσι κλέθ' ἀεικλὲς ἔπνευσε.

The five *Platonick* Bodies, so much fam'd,
Pythagoras first found, *Plato* explain'd:
Euclide on them Immortal Glory gain'd.

F I N I S.